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MATHEMATICAL MODELLING OF NETWORK TRAFFIC FLOW

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Abstract. The article describes mathematical models of traffic flows to initiate different traffic flow processes. Separate elements of traffic flow models are made in a way to be connected together to get a single complex model. A model of straight road with different boundary conditions is presented as a separate part of the network traffic flow model. First testing is conducted in case the final point of the whole modelled traffic line is closed and no output from that point is possible. The second test is performed when a constant value of traffic flow speed and traffic flow rate is entered. Mathematical simulation is carried out and the obtained results are listed.

Keywords: traffic flow, intersection, modelling, traffic flow regulation, vehicle.

1. Introduction

Modelling the process of traffic flow was previously studied from different points of view and different mathematical methods were used to describe the same process. It also encounters difficulties in choosing an appropriate method of deriving physical appearance we can notice on our streets and roads. Different authors have different views to the same phenomena and are focusing on different aspects of the same problem (Junevičius and Bogdevičius 2007; Junevičius *et al*. 2007; Berezhnoy *et al*. 2007; Akgüngör 2008a and 2008b; Daunoras *et al*. 2008; Yousefi and Fathy 2008; Gowri and Sivanandan 2008; Jakimavičius and Burinskienė 2007 and 2009; Antov *et al*. 2009, Knowles 2008; Gasser 2003; Helbing and Greiner; Knowles 2008 etc.).

All authors have an agreement on basic traffic flow parameters like, traffic flow density, traffic flow rate or the average speed of traffic flow. Besides, a lot of different investigations into the use of traffic flow models to deal with various problems of engineering are carried out, for example in Sivilevičius and Šukevičius (2007) paper.

A comparison of different continuum models has drawn that a number of scientific works were based on fluid dynamic theory and gas – kinetic traffic flow theory. The kinetic traffic flow theory is used for 'microscopic' or 'macroscopic' traffic flow models. The kinetic traffic flow theory is used in Flötteröd and Nagel (2007), Gning *et al.* (2008), Li and Xu (2008), Prigogine and Herman (1971) works where various approaches to the similar method are discussed. The equations of these models take different values to derive the same process. The kinetic theory was first used by Prigogine and Herman (1971) and co-workers. They suggested an equation analogous to Boltzmann equation. This theory was later criticized by many authors like Tampère (2004) etc. the papers of whose show the experience of Pavery-Fontna who noticed that Prigogine model had inaccuracies comparing the results of modelling and physical experiments. He suggested vehicle desired velocity towards which its actual velocity tends. Later, many authors mainly focused on a better statistical description of the traffic process.

The 'macroscopic' theory of traffic flows also can be developed as the hydrodynamic theory of fluids that was first introduced by Lighthill-Whitham and Richards model (Chalons and Goatin 2008; Kim and Keller 2002; Liu *et al.* 2008; Bonzani 2007; Nikolov 2008). They presented one dimensional model analogous to the fluid stream model. This theory was also criticized by such authors as Tampère (2004) and Daganzo and Nagatani (Liu *et al.* 2008) who proposed the lattice method. Nagatani and Nakanishi model took into account that all vehicles were moving at the same time-independent speed and in the same gap between vehicles. This method was improved later by considering the next-nearest neighbour interaction Liu *et al.* (2008).

Plenty of traffic flow models are based on car–following theories supported by the analogues to Newton's equation for each individual vehicle interacting in a system of vehicles on the road. Different forms of the equa-

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tion of motion give different versions of car-following models. Stimulus, from which response may occur, may be composed of the speed of a vehicle, difference in the speeds of leading and going after the vehicle, distance–headway etc.

Follow-the-leader and optimal-velocity theories are mostly known car-following theories and have been used by Tampère (2004), Kerner and Klenov (2006). Applying these methods, kinetic and fluid dynamic models could be extended to the critical points when the kinetic and fluid theory gives us inaccuracies comparing with experimental data. For example, the car-following theory could comprise the next-nearest neighbour effect in various lattice models, whereas optimal-velocity models give us an opportunity to model different situations, for example interacting vehicles having different characteristics (car and truck) or vehicles with different desired and optimal speeds. Nevertheless, all these improvements face the problems of properly working models or experience difficulties in achieving an appropriate solution.

Another point causing difficulties is the so called 'vehicular chaos' that is an analogue to 'molecular chaos' used in the kinetic theory of gases. The authors investigated such phenomena in their works (Chalons and Goatin 2008; Safanov *et al.* 2000; Kerner and Klenov 2006). Kerner and Klenov (2006) denotes unstable points on the fundamental diagram. These points indicate minimal density of growing infinite small fluctuations and express a zone for speed variation depending on vehicular density.

A similar zone for speed variation is presented in works by Chalons and Goatin (2008), Safanov *et al.* (2000). The authors derived alternate vehicle transition to the cases of unstable zones. These models clearly explain empirical data on the brake–down points of the fundamental diagram.

2. Description of Traffic Flow Mathematical Model

To model traffic flow, an equation system taking into account two parameters is used: traffic flow speed and traffic flow density. These parameters are calculated on each point of the road and information on the previous and next point of some road mesh is considered (Fig. 1).

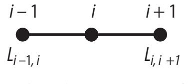

Fig. 1. A scheme for deriving traffic flow values at each traffic line point

At each point 'i' equations 1 and 2 are derived. Equation 1 derives variations in traffic flow speed and equation 2 derives variations in concentration at each point i.

$$\dot{v}_i = p_{in,i}(t)\cdot r_{v_i,in.,i}\cdot\left(\frac{v_{i-1}(t-\tau_{i-1,i})}{L_{i-1,i}}\right)\cdot\left(1-\frac{k_i(t)}{k_{\max,i}}\right)\cdot v_i(t)+ f_i(k_{i+1,i}) - p_{out,i}(t)\cdot r_{v_i,out}\cdot\left(\frac{1}{2}\frac{v_i(t)+v_{i+1}(t)}{L_{i+1,i}}\right)\cdot \left(1-\frac{k_{i+1}(t)}{k_{\max,i+1}}\right)^{m_1}\cdot v_i(t)-\left(\frac{v_i(t)}{v_{\max,i}}\right)\cdot e^{\left(\gamma_3\left(\frac{k_i(t)}{k_{\max,i}}\right)^{m_2}\right)\cdot\left(\frac{v_i(t)}{v_{\max,i}}\right)}; \quad (1)$$

$$\dot{k}_i = p_{in,i}(t)\cdot r_{k_i,in.,i}\cdot\left(1-\frac{k_i(t)}{k_{\max,i}}\right)\cdot\left(\frac{q_{i-1}(t-\tau_{i-1})}{q_{\max,i-1}}\right)\cdot k_i(t) - p_{out,i}(t)\cdot r_{k_i,out,i}\cdot\left(1-\frac{k_{i+1}(t)}{k_{\max,i+1}}\right)\cdot\left(\frac{q_i(t)}{q_{\max,i}}\right)\cdot k_i(t), \quad (2)$$

where: $r_{v_i,in}$, $r_{k_i,in}$, $r_{k_i,out}$ and $r_{v_i,out}$ – parameters are taken from empirical data; $v_{\max}$ – the maximal possible value of traffic flow speed at each point; L_i – road segment depending on point 'i'; $k_{\max}$ – the maximal possible value of traffic flow density at point 'i'; $q_{\max}$ – the maximal possible traffic flow rate at point 'i', q_i – the calculated traffic flow rate; $q_{in,i}$, $q_{out,i}$ – the probability of flow splitting or connecting at some traffic line intersecting point (It means that traffic flow could split between several traffic lines or be diverted to some exact traffic line or connected to one from several separate traffic lines. Depending on time, this parameter could be a constant or a function. It could be used as a control function to model traffic flow intersections, traffic accidents and other perturbations that could occur on the road network); $f_i(k_{i+1,i})$ – is some function depending on parameter k:

$$f_i = \begin{cases} e^{\gamma_2\left(1-\frac{\varepsilon_{i,i+1}}{\varepsilon_{i,i}}\right)\cdot\varepsilon_{i,i}\cdot sign\left(p_{out,i}(t)\right)\cdot sign\left(\left(1-\frac{\varepsilon_{i,i+1}}{\varepsilon_{i,i}}\right)\right)}, & \varepsilon_{i,i}>\varepsilon_{i,i+1}, and\ \varepsilon_{i,i}>0 \\ 0, \text{ otherwise.} & \end{cases} \quad (3)$$

This function takes into account the state of the road segment in front of point 'i'.

Other coefficients are $\gamma_3 = 5.5$, $\gamma_2 = 2.5$, $m_1 = 6$, $m_2 = 10$.

$$\varepsilon = \frac{k_i}{k_{\max}}. \quad (4)$$

Some explanations about the members of equations (1) and (2) are given below.

These are the members of equation 1:

- Member $\left(\frac{v_{i-1}(t-\tau_{i-1,i})}{L_{i-1,i}}\right)\cdot v_i(t)$ describes traffic flow acceleration at point 'i – 1' and member $\left(\frac{1}{2}\frac{v_i(t)+v_{i+1}(t)}{L_{i+1,i}}\right)\cdot v_i(t)$ specifies the average traffic flow acceleration.
- Member $\left(1-\frac{k_{i+1}(t)}{k_{\max,i+1}}\right)^{m_1}$ shows variations in the acceleration of traffic flow between points 'i' and 'i + 1'.
- Member $1-\frac{k_i(t)}{k_{\max,i}}$ characterizes traffic flow fulfilling point 'i'.

- Member $\left(\dfrac{v_i(t)}{v_{\max,i}}\right)\cdot e^{\left[\gamma_3\left(\frac{k_i(t)}{k_{\max,i}}\right)^{m_2}\right]\cdot\left(\frac{v_i(t)}{v_{\max,i}}\right)}$ takes into account the amount of vehicles at point '*i*' and is subject to concentration value at point '*i*'.

These are the members of equation 2:

- Member $1-\dfrac{k_i(t)}{k_{\max,i}}$ considers traffic concentration at point '*i*' and member $1-\dfrac{k_{i+1}(t)}{k_{\max,i+1}}$ considers traffic flow concentration at point '*i* + 1'. It means that in case a road in front of point '*i*' is occupied, there is no possibility of entering the road segment in front of point '*i*'.
- Member $\dfrac{q_{i-1}(t-\tau_{i-1})}{q_{\max,i-1}}$ takes into account traffic flow rate at point '*i* – 1' which means that at point '*i* – 1', there should be some quantity of vehicles that can enter point '*i*'; otherwise the value of traffic flow rate becomes equal to 0. The delay of traffic flow rate that comes from point '*i* – 1' to point '*i*' is also regarded.
- Member $\dfrac{q_i(t)}{q_{\max,i}}$ shows outgoing traffic flow rate from point '*i*' to point '*i* + 1'.

The quantity of vehicles at each road segment could be calculated by the equation:

$$N_e = \int_{x_i}^{x_j} k(x)dx, \tag{5}$$

where: $x_{i,j}$ – traffic line segment boundary points; $k_{i,j}$ – concentration values at boundary points.

Variance in the quantity of vehicles at each road segment could be derived by the equation:

$$N_i(t) = N_i(t) + \int_{t_i}^{\Delta t + t_i} q_i(t)dt\,. \tag{6}$$

3. Model Description. Numerical Experimental Study

Two cases of mathematical experiment are presented.

Case 1.

To model traffic flows in this paper, the following considerations are required. First, it is acknowledged that the part of the road between two intersections is divided into some intervals e_i (Fig. 2).

Each element has two points at the ends of the interval. Two elements are connected at the same point, so each element has two points that belong to two different elements.

An exception is the first and the last point of the road part that is under investigation as these points are road input and road output respectively.

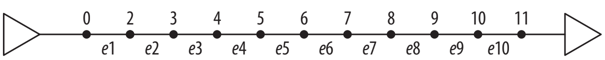

Fig. 2. The structure of creating a part of one way road

The number of points is 11 (10 elements); the length of the road is L = 1 km (Fig. 2).

Boundary conditions at the first and final points are:

- Traffic flow rate:
  $q(x=0,t)=q_1=0.5$ veh/s;
  $q(x=L,t)=q_{11}=0$ veh/s;
- Traffic velocity:
  $v(x=0,t)=v_1=13.888$ m/s $=50$ km/h;
  $v(x=L,t)=v_{11}=0$ m/s.
- Initial conditions:
  $v_i(t=0)=10^{-4}$ m/s; $k_i(t=0)=10^{-4}$ veh/m;
  $i=2,\ldots,10$.

Velocity, traffic flow rate and flow density rate are shown in Fig. 3, 4 and 5.

The dependency of a total number of vehicles on the road on time is shown in Fig. 6.

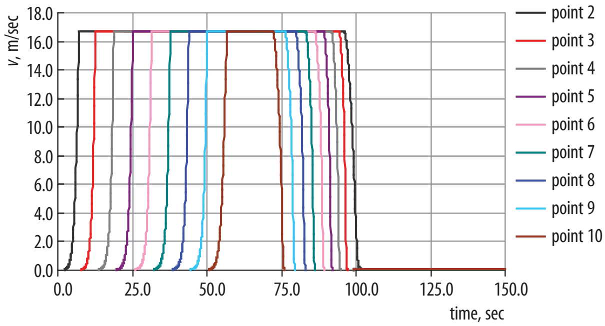

Fig. 3. The dependency of flow velocity on time at each point '*i*'

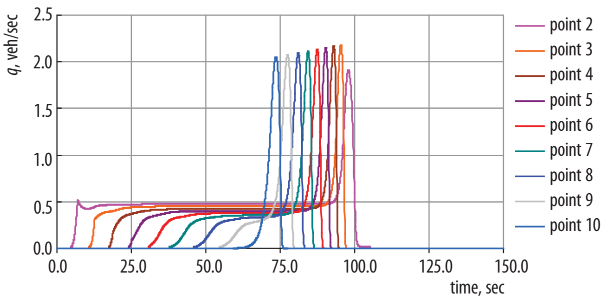

Fig. 4. The dependency of the traffic flow rate on time at each point '*i*'

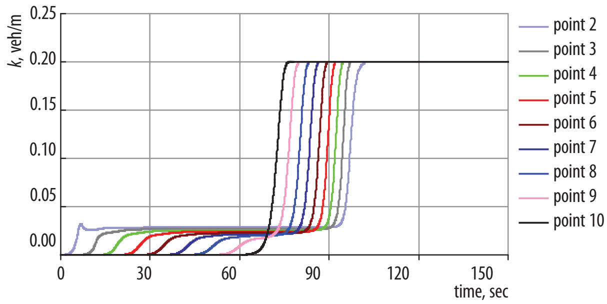

Fig. 5. The dependency of traffic concentration on time at each point '*i*'

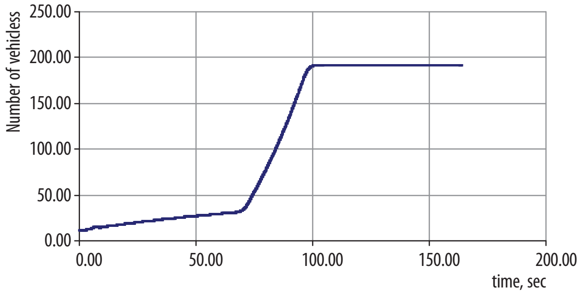

Fig. 6. The dependency of a total number of vehicles on the road on time

The end of the road is closed so the vehicles enter the road but do not leave it. Estimating the result of simulation shows that the road should be overfilled. The investigated part of the road was empty at the start, so speed at the beginning should grow. At a later stage, speed should reach a maximum value. When the road is overfilled, speed should decline to 0.

The data of the conducted mathematical experiment point to the expected results. The empty traffic line was filed with vehicles and the maximum 0.2 veh/m concentration was reached. First, the end of the traffic line was filed up, and then the entire road was filed. Traffic flow speed reaches the maximum value at the beginning of simulation and when concentration starts growing, the speed value reduces to zero.

Traffic flow rate reaches the maximum value and starts declining from the end point of the road. The maximum value of vehicles on the road at peak moment is almost 200 which is the maximum value that could appear on the road when vehicles are bumper to bumper.

Case 2.

Boundary conditions at the first and final points are (Fig. 2):

- Traffic flow rate:
  $q\left(x=0,t\right)=q_1=0.5\ \text{veh/s}$;
  $q\left(x=L,t\right)=q_{11}=0.575\ \text{veh/s}$.
- Traffic velocity:
  $v(x=0,t)=v_1=13.888\ m/s=50\ \text{km/h}$;
  $v(x=L,t)=v_{11}=10\ \text{m/s}$.
- Initial conditions: $v_i\left(t=0\right)=10^{-4}\ \text{m/s}$;
  $k_i\left(t=0\right)=10^{-4}\ \text{veh/m}$; $i=2,\ldots,10$.

Velocity, traffic flow rate and flow density rate are shown in Fig. 7, 8 and 9. The dependency of a total number of vehicles on the road on time is shown in Fig. 10.

This test has come up with similar results. This time, the end of the road is open, so all vehicles entering the traffic line could leave it. Speed at the first point is lower than incoming speed in the first case. Traffic flow rate at the end point is higher than that in the first case. Thus, in general, traffic flow rate and concentration decline at each point of the road coming from the first point to the last one and this is due to difference in traffic flow rate under the boundary conditions of the traffic line. Also at the end of simulation, the constant value of vehicles on the road is received. The quantity of vehicles on the road becomes constant at the end of simulation (Fig. 10).

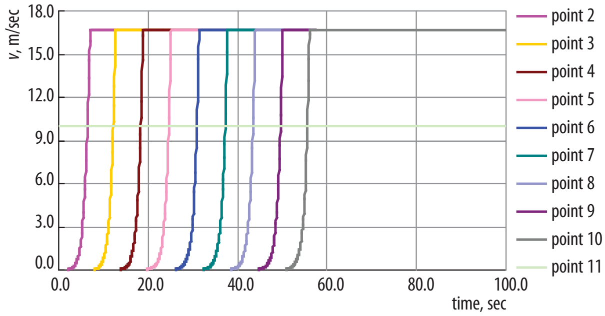

Fig. 7. The dependency of flow velocity on time at each point '*i*'

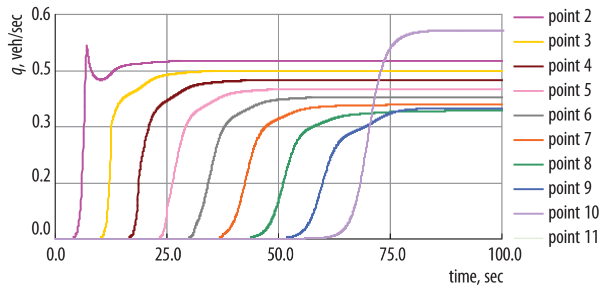

Fig. 8. The dependency of the traffic flow rate on time at each point '*i*'

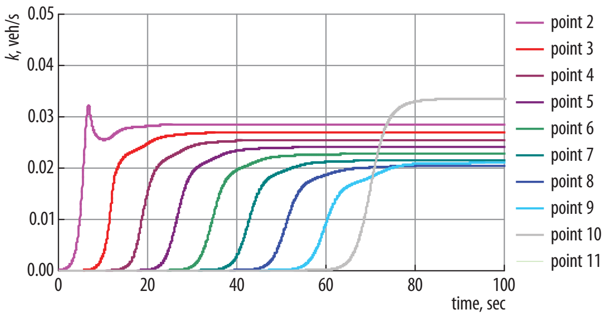

Fig. 9. The dependency of traffic concentration on time at each point '*i*'

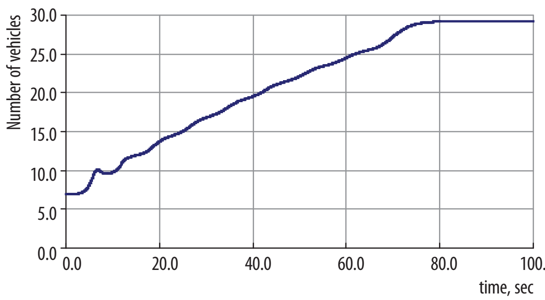

Fig. 10. The dependency of a total number of vehicles on the road on time

4. Conclusions

1. The presented traffic flow model gives theoretically expected results. In each case of simulation, the results are related to boundary conditions. In the first case, the end of the road is closed, $q(x=L,t)=q_{11}=0$ veh/s, so the number of vehicles on the road increases and reaches the max possible quantity of almost 200 vehicles. In the second case, the end of the road is opened, $q(x=L,t)=q_{11}=0.575$ veh/s, so after some time, the maximum quantity of almost 30 vehicles is reached.
2. At the beginning of the simulation process, the road was empty. After some time, all segments on the road were filled. First, some concentration and flow values were received. For a while, those values were almost constant. Overfilling the last point starts at a time step of 60 sec. Then, all cells were filed in equal time steps (see Fig. 4 and Fig. 5). Fig. 5 shows that concentration reaches a maximum possible value because the road is closed. Fig. 9 indicates that concentration values are different at all points due to boundary conditions.
3. Traffic flow speed is maximal at all points when concentration is low and begins to increase when concentration starts growing
4. The process of road filling starts from the end point in Case 1 which means that the last road segment was filled first. At a later stage, road segment before him was filed. Thus, the process of filling the entire road starts from the last segment and reaches the first one. Fig. 3, Fig. 4 and Fig. 5 clearly indicate that traffic flow rate and traffic flow concentration change in the same order.

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