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A THEORY OF MULTIPLE VEHICLE TYPE DYNAMIC MARGINAL COST CONSIDERING DEPARTURE TIME CHOICES

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Abstract. The analysis of single vehicle type dynamic marginal cost is extended to multiple vehicle type dynamic one based on time-dependent multiple vehicle type queue analysis at a bottleneck. First, a dynamic link model to represent the interactions between cars and trucks is provided. Then, the analytic expression of a multiple vehicle type dynamic marginal cost function considering departure time choices is deduced under congested and un-congested conditions and consequently, a dynamic toll function is given. A heuristic algorithm is introduced to solve multiple vehicle type dynamic queues and toll under system optimum and user equilibrium conditions taking into account traveler's departure time. A numerical example shows that a dynamic congestion toll can diminish queues and improve system conditions when traffic demand is not changed.

Keywords: multiple vehicle types, dynamic marginal cost function, point queue.

1. Introduction

The method of congestion pricing to ease traffic congestion has aroused the attention of many countries. Many congestion pricing projects have been implemented in Singapore and more recently, in London. The previous studies (Vickrey 1969; Yang and Huang 1997; Kuwahara 2007) have paid attention to dynamic congestion pricing of single class. However, differences in multiple user types cause different externalities in terms of congestion, road damage and pollution emissions and an important question of dynamic congestion pricing is estimating tolls for different user classes. The method of multiple user dynamic congestion pricing can be classified into two classes: one class considers multiple user types with distinct values of time (VOT) and model their travel behavior using the bottleneck model (Arnott *et al.* 1992), whereas another study takes into account both congestion effects of different user classes with different vehicle characteristics such as vehicle length, speed etc. Mathematical expressions of the dynamic multiple vehicle type toll function with speed difference is given (Verhoef *et al.* 1999). However, this model is applicable only to one lane with no overtaking. Holguín-Veras and Cetin (2009) developed an analytical formulation to compute optimal tolls for multi-class traffic and used micro-simulation in modeling the interactions with different vehicle types. However, the previous studies neglected the departure time choice of a different vehicle type.

In this paper, first, we give a multiple vehicle type dynamic bottleneck model while considering dynamic queues. Then, the analytical multiple vehicle type dynamic marginal cost function considering departure time choices is given. Finally, a simple heuristic algorithm is presented to calculate multiple vehicle type dynamic tolls.

2. Multiple Vehicle Type Dynamic Bottleneck Models

Let us consider a single bottleneck link with fixed capacity, *s*, pcu (passenger car unit)/h. We assume there are two vehicle types, including a car and a truck. Li and Su (2005) proposed a multimode stochastic dynamic simultaneous route/departure time equilibrium model and proved that the multimode deterministic point model within each mode-user met FIFO discipline and the speeds of different modes approach consistently during congestion.

Link flow propagation on a bottleneck link is shown in Fig. 1a where, s_i^1 and s_i^2 are the times at which the first and last car / truck traveler respectively depart home. Furthermore, t_0^a and t_1^a represent the times at which transition from uncongested condition to congested edge condition occurs at the early arrival

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stage and the time at which transition from congested edge condition to un-congested condition occurs at the late arrival stage respectively. t_0 and t_1 respectively describe the starting and ending time of the total congested queue. $\overline{s_i}$ is the departure time of a car / truck traveler who incurs no scheduling cost. $\tilde{s}$ is the desired time interval for arrival at destination *s*. Cars and trucks have their own characteristics such as free flow speed, vehicle size, a dynamic link function. To model the interactions between cars and trucks on the bottleneck, a link is presented given that each link consists of two distinct segments and the first segment is the running segment of the bottleneck link on which cars / trucks run at their free-flow speeds. For example, the velocity of a car is higher than that of a truck under un-congested traffic conditions. In other words, cars and trucks have no impact on each other except for difference in velocity on the running segment. The second segment is the exit queue segment (a car or a truck is assumed to be a point without length). Class-specific pcu_m parameter is used to transform the effect of a truck into passenger car equivalents. Then, the exit flow of cars and trucks can be calculated according to the exit capacity (in pcu) of the bottleneck link. In other words, class-specific pcu_m parameter is used to model the interactions between cars and buses on the exit queue segment. Further, Fig.1b gives changes in the total cumulative trips (in pcu) with respect to time. The following parts give a link flow model considering the interaction of cars and trucks at the bottleneck link.

Link Dynamic Function

The flow of car / truck travelers entering link *a* at time $t-t_i$, $u_i\,(t-t_i)$ experiences constant running time t_i to arrive at the exit queue of the bottleneck link and the arrival flow rate of car / truck travelers to the exit the queue at time *t* is $u_i\,(t-t_i)$ while the flow rate of car / truck travelers exiting from the exit queue is $v_i\,(t)$.

Thus, link dynamic functions can be formulated as follows:

$$\frac{dq_i(t)}{dt} = u(t-t_i) - v_i\,(t), \forall i \in (\text{car, truck}), t. \quad (1)$$

Let $q_i(t)$ be the number of car / truck travelers waiting in the queue at time *t* on the bottleneck link. It is to say that the queue length marginal change of car / truck travelers is equal to the difference between the arrival flow rate of car / truck travelers to exit queue part at time *t* and the departure flow rate of car / truck travelers from exit queue part at time *t* on the bottleneck link.

Further, according to the integral of Eq. (1), one can see that:

$$\int_0^t \frac{dq_i(x)}{dt} dx = \int_0^t (u(t-t_i) - v_i\,(t))dt \Rightarrow$$
$$q_i(t) = U(t-t_i) - V_i\,(t), \forall i \in (\text{c,t}), t, \quad (2)$$

where: $i \in (\text{c,t})$ – expresses the abbreviated description of $i \in (\text{car, truck})$ as follow.

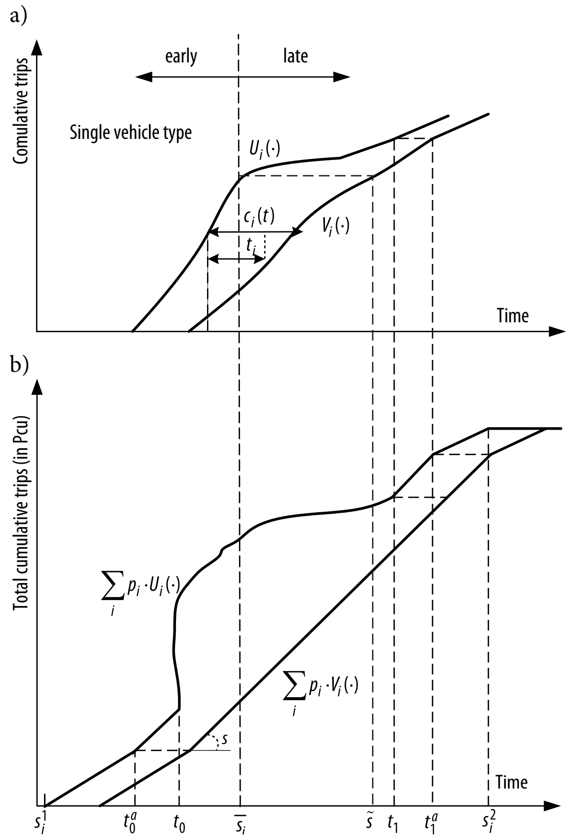

Fig. 1. Cumulative arrival and departure at the bottleneck

Link Exit Flow Function

The following assumptions are used for deriving the function of multiple vehicle type link exit flow:

– class-specific pcu_m parameter that transforms the effect of a car / truck into passenger car equivalents is fixed under all traffic conditions;
– the mixture of travelers by cars and trucks is homogenous on the bottleneck link.

The temporal and spatial interactions of travelers by cars and trucks mainly appear in the exit queue part of the bottleneck link. If the sum of the queue length of the travelers at bottleneck is $\sum_{i\in(\text{c,t})} p_i \cdot q_i(t) > 0$ (in pcu) or the sum of the arrival flow rate of travelers by cars and trucks to exit queue is $\sum_{i\in(\text{c,t})} p_i \cdot u(t-t_i) \geq s$ (in pcu/h), then, arrival flow to the exit queue must not wait and departure flow rate exiting the bottleneck link exit queue is s where p_i is a passenger car equivalent parameter of a car / truck.

According to assumption (2), the link outflow rate of car / truck travelers from the exit queue can be calculated as:

$$v_i\,(t) = \frac{u_i\,(t-t_i)}{\sum_i p_i \cdot u_i\,(t-t_i)} \cdot s, \forall i \in [\text{c,t}]. \quad (3)$$

If $\sum_{i\in[\text{c,t}]} p_i \cdot u(t-t_i) < s$, cars and buses will pass the exit queue without delay. Thus, the link exit function can be formulated as:

$$v_i(t)=\begin{cases}\dfrac{u_i\,(t-t_i\,)}{\sum_i p_i\cdot u_i\,(t-t_i\,)}\cdot s \\ if\quad \sum_i p_i\cdot q_i\,(t)>0 \\ or\quad \sum_i p_i\cdot u_i\,(t-t_i\,)\geq s_a \\ \hline u_i\,(t-t_i\,)\quad otherwise\end{cases}\quad \forall i\in[\text{c,t}]. \tag{4}$$

Link Travel Time Function

The queue delay of car / truck travelers on the bottleneck link can be determined only by the total queue length of a car / truck in the exit queue of the bottleneck link. Car / truck travelers entering the bottleneck link at time t will spend travel time t_i to arrive at the exit queue. On the other hand, they will enter into the exit queue of the bottleneck link at time $t+t_i$ and the number of the total queue length (in pcu) in the exit queue at time $t+t_i$ is $\sum_{m\in(\text{c,t})} p_m\cdot q_m(t+t_i)$.

Thus, queue delay on link a for car / truck travelers entering the bottleneck link at time t can be given as:

$$c_i(t)=t_i+\frac{\sum_{m\in(\text{c,t})} p_m\cdot q_m(t+t_i)}{s}\quad \forall i\in[c,t]. \tag{5}$$

The interested readers can further refer to work by Li and Su (2005) and Li and Xu (2007).

3. Multiple Vehicle Type Dynamic Marginal Cost Function Considering Departure Time Choices

The work of Mun (1999) is extended to the multiple vehicle type dynamic marginal cost function considering departure time choices. The travelers of each vehicle type choose departure time so as to minimize travel cost consisting of travel time cost and scheduling cost.

Travel cost for a car / truck traveler departing home at t, $w_i(t)$ can be formulated as:

$$w_i(t)=\begin{cases}w_i^1(t)=\alpha\cdot c_i(t)+\beta(\tilde{s}-t-c_i(t)), \\ if\quad \tilde{s}>t+c_i(t), \\ \hline w_i^2(t)=\alpha\cdot c_i(t)+\gamma(t+c_i(t)-\tilde{s}), \\ if\quad \tilde{s}<t+c_i(t).\end{cases}\forall i\in[c,t]. \tag{6}$$

Denote $\tilde{s}$ as the desired time interval for arrival at destination s. α is the monetary value of time for travelling, β and γ are the values of time early and late respectively. Considering that $t+c_i(t)$ is the time at which an individual departing home at t arrives at the office, $w_i^1(t)$ represents travel cost for an early arrival while $w_i^2(t)$ represents that for the late one.

User equilibrium is obtained when no car / truck traveler has an incentive to change his/her departure time. Since the travel cost of car / truck travelers must be the same at all times when departures occur, hence, equilibrium conditions can be stated as follows:

$$\begin{cases}D_i^{-1}(N_i)=\alpha\cdot c_i(t)+\beta(\overline{s}-t-c_i(t)), \\ if\quad s_i^1<t<\overline{s}-c_i(t), \\ \hline D_i^{-1}(N_i)=\alpha\cdot c_i(t)+\gamma(t+c_i(t)-\overline{s}), \\ if\quad \overline{s}-c_i(t)<t<s_i^2,\end{cases}\forall i\in[c,t], \tag{7}$$

where: s_i^1 and s_i^2 are the times at which the first and last car and truck traveler respectively depart home.

System optimal is obtained when the dynamic marginal travel cost of car / truck travelers must be the same at all times when departures occur. Hence, system optimal conditions can be described as follows:

$$\begin{cases}D_i^{-1}(N_i)=toll_i(t)+\alpha\cdot c_i(t)+ \\ \beta(\overline{s}-t-c_i(t)), \\ if\quad s_i^1<t<\overline{s}-c_i(t), \\ \hline D_i^{-1}(N_i)=toll_i(t)+\alpha\cdot c_i(t)+ \\ \gamma(t+c_i(t)-\overline{s}), \\ if\quad \overline{s}-c_i(t)<t<s_i^2,\end{cases}\forall i\in[c,t], \tag{8}$$

where: $toll_i(t)$ is the optimal toll of a car / truck; $D_i^{-1}(N_i)$ expresses the variable demand function of a car / truck.

One can see that:

$$D_i\,(w_{\min}^i)=N_i, w_{\min}^i=D_i^{-1}(N_i), i=[\text{c,t}], \tag{9}$$

where: $w_{\min}^i$ is minimum travel cost under user equilibrium and system optimal conditions; N_i represents demand for a car / truck.

Three cases of traffic conditions at the bottleneck link can be expressed as follows. Case 1 supposes that a traffic jam occur. Case 2 is the un-congested condition. Case 3 applies when traffic volume maintains the level equal to bottleneck capacity. Dynamic marginal travel cost under the three introduced cases of traffic conditions can be stated as the one below.

Since the total travel cost is written as $\sum_i(\int_0^{\overline{s}_i} u_i(x)\cdot w_i^1(x)\mathrm{d}x+\int_{\overline{s}_i}^{T} u_i(x)\cdot w_i^2(x)\mathrm{d}x)$, the dynamic marginal cost of a car / truck at time t is defined by taking a derivative with respect to demand at time t, $u_i(t)\cdot dt$. $\overline{s}_i+c(\overline{s}_i)=\tilde{s}, i=[\text{c,t}]$ where $\overline{s}_i$ is the departure time of a car / truck traveler who incurs no scheduling cost.

Case 1 (congested and early cost): when a car / truck traveler entering the bottleneck link experiences the congested queue at the exit segment, i.e. $c_c(t)>t_c\,(c_t(t)>t_t\,)$ and incurs early cost, $t<\overline{s}_c\,\,(t<\overline{s}_t\,\,)$.

The dynamic marginal travel cost of a car / truck can be equal to the derivative of the total travel cost with respect to car demand considering the queuing property of a car / truck (t_1 is the total queue vanishing time):

$$MC_c(t)=\frac{d(\sum_i(\int_0^{\bar{s}_i} w_i^1(x)\cdot u_i(x)dx+\int_{\bar{s}_i}^{T} w_i^2(x)\cdot u_i(x)dx))}{du_c(t)\cdot dt}=$$
$$\int_0^{\bar{s}_c} w_c^1(x)\cdot\frac{u_c(x)dx}{u_c(t)dt}+\int_0^{\bar{s}_c}\frac{dw_c^1(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_c(x)dx+$$
$$\int_0^{\bar{s}_t}\frac{dw_t^1(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_t(x)dx+\int_{\bar{s}_c}^{t_1}\frac{dw_c^2(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_c(x)dx+$$
$$\int_{\bar{s}_t}^{t_1}\frac{dw_t^2(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_t(x)dx=w_c^1(t)+(\alpha-\beta)\times$$
$$\int_0^{\bar{s}_c}\frac{dc_c(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_c(x)dx-\beta\int_0^{\bar{s}_c}\frac{dx}{du_c(t)}\cdot\frac{1}{dt}\cdot u_c(x)dx+$$
$$(\alpha-\beta)\int_0^{\bar{s}_t}\frac{dc_t(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_t(x)dx-\beta\times$$
$$\int_0^{\bar{s}_c}\frac{dx}{du_c(t)}\cdot\frac{1}{dt}\cdot u_t(x)dx+(\alpha+\gamma)\times$$
$$\int_{\bar{s}_c}^{t_1}\frac{dc_c(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_c(x)dx+(\alpha+\gamma)\int_{\bar{s}_t}^{t_1}\frac{dc_t(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_t(x)dx. \qquad (10)$$

When substituting Eq. (2) with Eq. (5), we obtain:

$$c_c(t)=t_c+\frac{p_c\cdot q_c(t+t_c)+p_t\cdot q_t(t+t_c)}{s}=t_c+$$
$$\frac{p_c\cdot(U_c(t)-V_c(t+t_c))+p_t\cdot(U_t(t+t_c-t_t)-V_t(t+t_c))}{s}; \qquad (11a)$$
$$c_t(t)=t_t+$$
$$\frac{p_t\cdot(U_t(t)-V_t(t+t_t))+p_c\cdot(U_c(t+t_t-t_c)-V_c(t+t_t))}{s}. \qquad (11b)$$

Then, substituting Eqs. (11 a and b) by Eq. (10):

$$MC_c(t)=w_c^1(t)+\frac{p_c}{s}(\alpha-\beta)\int_t^{\bar{s}_c}u_c(x)dx+$$
$$\frac{p_c}{s}(\alpha-\beta)\int_{t+t_c-t_t}^{\bar{s}_t}u_t(x)dx+\frac{p_c}{s}(\alpha+\gamma)\int_{\bar{s}_c}^{t_1}u_c(x)dx+$$
$$\frac{p_c}{s}(\alpha+\gamma)\int_{\bar{s}_t}^{t_1}u_t(x)dx-\beta\frac{u_c(t)+u_t(t)}{du_c(t)/dt}=$$
$$w_c^1(t)+\frac{p_c}{s}(\alpha-\beta)(U_c(\bar{s}_c)-U_c(t))+$$
$$\frac{p_c}{s}(\alpha-\beta)(U_t(\bar{s}_t)-U_t(t+t_c-t_t))+$$
$$\frac{p_c}{s}(\alpha+\gamma)(U_c(t_1)-U_c(\bar{s}_c))+\frac{p_c}{s}(\alpha+\gamma)\times$$
$$(U_t(t_1)-U_t(\bar{s}_t))-\beta\frac{u_c(t)+u_t(t)}{du_c(t)/dt}. \qquad (12)$$

Similarly, the dynamic marginal travel cost of a truck can be expressed as:

$$MC_t(t)=w_t^1(t)+\frac{p_t}{s}(\alpha-\beta)(U_t(\bar{s}_t)-U_t(t))+$$
$$\frac{p_t}{s}(\alpha-\beta)(U_c(\bar{s}_c)-U_c(t+t_t-t_c))+$$
$$\frac{p_t}{s}(\alpha+\gamma)(U_t(t_1)-U_t(\bar{s}_t))+\frac{p_t}{s}(\alpha+\gamma)\times$$
$$(U_c(t_1)-U_c(\bar{s}_c))-\beta\frac{u_c(t)+u_t(t)}{du_t(t)/dt}. \qquad (13)$$

According to Eqs. (12–13), the toll function of a car / truck can be stated as:

$$toll_c^1(t)=\frac{p_c}{s}(\alpha-\beta)(U_c(\bar{s}_c)-U_c(t))+$$
$$\frac{p_c}{s}(\alpha-\beta)(U_t(\bar{s}_t)-U_t(t+t_c-t_t))+$$
$$\frac{p_c}{s}(\alpha+\gamma)(U_c(t_1)-U_c(\bar{s}_c))+$$
$$\frac{p_c}{s}(\alpha+\gamma)(U_t(t_1)-U_t(\bar{s}_t))-\beta\frac{u_c(t)+u_t(t)}{du_c(t)/dt}; \qquad (14)$$

$$toll_t^1(t)=\frac{p_t}{s}(\alpha-\beta)(U_t(\bar{s}_t)-U_t(t))+$$
$$\frac{p_t}{s}(\alpha-\beta)(U_c(\bar{s}_c)-U_c(t+t_t-t_c))+$$
$$\frac{p_t}{s}(\alpha+\gamma)(U_t(t_1)-U_t(\bar{s}_t))+$$
$$\frac{p_t}{s}(\alpha+\gamma)(U_c(t_1)-U_c(\bar{s}_c))-\beta\frac{u_c(t)+u_t(t)}{du_t(t)/dt}. \qquad (15)$$

Case 2 (congested and late cost): when a car (truck) traveler entering the bottleneck link experiences the congested queue at the exit segment, i.e. $c_c(t)>t_c$ ($c_t(t)>t_t$) and incurs late cost, $t>\bar{s}_c$ ($t>\bar{s}_t$).

The dynamic marginal travel cost of a car can be expressed as:

$$MC_c(t)=\int_{\bar{s}_c}^{t_1} w_c^2(x)\cdot\frac{u_c(x)dx}{u_c(t)dt}+$$
$$\int_{\bar{s}_c}^{t_1}\frac{d(\alpha\cdot c_c(x)+\gamma(x+c_c(x)-\tilde{s}))}{du_c(t)}\cdot\frac{1}{dt}\cdot u_c(x)dx+$$
$$\int_{\bar{s}_t}^{t_1}\frac{d(\alpha\cdot c_t(x)+\gamma(x+c_t(x)-\tilde{s}))}{du_c(t)}\cdot\frac{1}{dt}\cdot u_t(x)dx=$$
$$w_c^2(t)+(\alpha+\gamma)\int_{\bar{s}_c}^{t_1}\frac{dc_c(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_c(x)dx+\gamma$$
$$\int_{\bar{s}_c}^{t_1}\frac{dx}{du_c(t)}\cdot\frac{1}{dt}\cdot u_c(x)dx+(\alpha+\gamma)\times$$
$$\int_{\bar{s}_t}^{t_1}\frac{dc_t(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_t(x)dx+\gamma\int_{\bar{s}_c}^{t_1}\frac{dx}{du_c(t)}\cdot\frac{1}{dt}\cdot u_t(x)dx=$$
$$w_c^2(t)+(\alpha+\gamma)\int_{\bar{s}_c}^{t_1}\frac{dc_c(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_c(x)dx+$$
$$\gamma\int_{\bar{s}_c}^{t_1}\frac{du_c(x)dx}{du_c(t)dt}\cdot\frac{u_c(x)}{du_c(x)}dx+(\alpha+\gamma)\times$$
$$\int_{\bar{s}_t}^{t_1}\frac{dc_t(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_t(x)dx+\gamma\int_{\bar{s}_c}^{t_1}\frac{du_c(x)dx}{du_c(t)dt}\cdot\frac{u_t(x)}{du_c(x)}dx=$$
$$w_c^2(t)+(\alpha+\gamma)\int_{\bar{s}_c}^{t_1}\frac{dc_c(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_c(x)dx+\gamma\frac{u_c(t)}{du_c(t)}dt+$$
$$(\alpha+\gamma)\int_{\bar{s}_t}^{t_1}\frac{dc_t(x)}{du_c(t)}\cdot\frac{1}{dt}\cdot u_t(x)dx+\gamma\frac{u_t(t)}{du_c(t)}dt. \qquad (16)$$

Then, substituting Eqs. (11 a and b) with Eq. (16) we receive:

$$MC_c(t) = w_c^2(t) + \frac{p_c}{s}(\alpha+\gamma)\int_t^{t_1} u_c(x)\mathrm{d}x + \frac{p_c}{s}(\alpha+\gamma)\int_{t+t_c-t_t}^{t_1} u_t(x)\mathrm{d}x + \gamma\frac{u_c(t)+u_t(t)}{\mathrm{d}u_c(t)/\mathrm{d}t} =$$

$$w_c^2(t) + \frac{p_c}{s}(\alpha+\gamma)(U_c(t_1)-U_c(t)) + \frac{p_c}{s}(\alpha+\gamma)(U_t(t_1)-U_t(t+t_c-t_t)) + \gamma\frac{u_c(t)+u_t(t)}{\mathrm{d}u_c(t)/\mathrm{d}t}. \quad (17)$$

Similarly, the dynamic marginal travel cost of a truck can be expressed as:

$$MC_t(t)=w_t^2(t)+\frac{p_t}{s}(\alpha+\gamma)(U_t(t_1)-U_t(t))+\frac{p_t}{s}(\alpha+\gamma)(U_c(t_1)-U_c(t+t_t-t_c))+\gamma\frac{u_t(t)+u_c(t)}{\mathrm{d}u_t(t)/\mathrm{d}t}. \quad (18)$$

The toll function of a car / truck is:

$$toll_c^2(t) = \frac{p_c}{s}(\alpha+\gamma)(U_c(t_1)-U_c(t)) + \frac{p_c}{s}(\alpha+\gamma)(U_t(t_1)-U_t(t+t_c-t_t)) + \gamma\frac{u_c(t)+u_t(t)}{\mathrm{d}u_c(t)/\mathrm{d}t}; \quad (19)$$

$$toll_t^2(t) = \frac{p_t}{s}(\alpha+\gamma)(U_t(t_1)-U_t(t)) + \frac{p_t}{s}(\alpha+\gamma)(U_c(t_1)-U_c(t+t_t-t_c)) + \gamma\frac{u_t(t)+u_c(t)}{\mathrm{d}u_t(t)/\mathrm{d}t}. \quad (20)$$

Case 3 (un-congested condition): when a car / truck traveler entering the bottleneck link does not experience the congested queue at the exit segment, i.e. $c_i(t)=t_i$ and $\sum_m p_m \cdot q_m(t+t_i) < s \quad \forall i \in [c,t]$.

According to Eqs. (12–13) and (17–18), the dynamic marginal travel cost of a car / truck can be expressed as:

$$MC_i(t) = \begin{cases} w_i(t) - \beta\frac{u_c(t)+u_t(t)}{\mathrm{d}u_i(t)/\mathrm{d}t}, t < \overline{s_i}, \\ w_i(t) + \gamma\frac{u_c(t)+u_t(t)}{\mathrm{d}u_i(t)/\mathrm{d}t}, t > \overline{s_i}, \end{cases} , \forall i = [c,t]. \quad (21)$$

Case 4 (congested edge condition): when a car / truck traveler entering the bottleneck link experiences the congested edge condition at the exit segment, i.e. $c_i(t)=t_i$ and $\sum_m p_m \cdot q_m(t+t_i) = s \quad \forall i \in [c,t]$. In this case, if the demand rate of a car / truck is increased at time *t*, a queue would start. If you decrease demand for a car / truck, $\sum_m p_m \cdot q_m(t+t_i)$ would become smaller than s. When $t < \overline{s_i}$, the dynamic marginal travel cost of a car / truck can be calculated as presented below.

First, differentiating Eq. (8) with respect to time *t*:

$$\frac{\mathrm{d}toll_i(t)}{\mathrm{d}t} = \frac{\mathrm{d}D_i^{-1}(N_i)}{\mathrm{d}t} + \beta. \quad (22)$$

Further, taking into account the integral of the above equation:

$$toll_i(t) = D_i^{-1}(N_i) + \beta t + q_0, \quad (23)$$

where: q_0 is a constant.

Next, when time *t* is equal to the starting time of congested queue t_0 and $t < \overline{s_i}$, toll value is $toll_i^1(t_0)$, and thus we obtain:

$$q_0 = toll_i^1(t_0) - D_i^{-1}(N_i) - \beta t_0. \quad (24)$$

Substituting Eq. (24) by Eq. (23) we have:

$$toll_i(t) = toll_i^1(t_0) + \beta(t-t_0). \quad (25)$$

Following the above derivation when $t > \overline{s_i}$, the dynamic marginal travel cost function can be written as:

$$toll_i(t) = toll_c^2(t_1) + \gamma(t_1 - t). \quad (26)$$

4. Multiple Vehicle Type Tolls

According to the above analysis, the toll of a car / truck can be written as:

$$toll_i(t) = MC_i(t) - w_i(t) = \begin{cases} -\beta(u_c(t)+u_t(t))/du_i(t)/dt,\ s_i^1 < t < t_0^a; \\ toll_i^1(t_0) + \beta(t-t_0),\ t_0^a < t < t_0; \\ toll_i^1(t),\ t_0 < t < \overline{s_c}; \\ toll_i^2(t),\ \overline{s_c} < t < t_1; \\ toll_i^2(t_1) + \gamma(t_1 - t),\ t_1 < t < t_1^a; \\ \gamma(u_c(t)+u_t(t))/du_i(t)/dt, s_{i\,c}^2 < t < t_1^a, \end{cases} \quad (27)$$

where: s_i^1 and s_i^2 are the times at which the first and last car and truck traveler respectively depart home. Furthermore, t_0^a and t_1^a are the times at which transition from Case 3 to 4 occurs at the early arrival stage and the time at which transition from Case 4 to 3 occurs at the late arrival stage respectively. t_0 and t_1 describe respectively the starting and ending time of the total congested queue. The above toll function cannot guarantee a non-negative toll value, and thus non-negative transformation is given as follows:

$$toll_i^*(t) = \max\{toll_i(t), 0\}. \quad (28)$$

5. Algorithm

In order to solve dynamic traffic states and dynamic marginal travel cost under the above given system optimization and user equilibrium conditions, a similar Newton steepest descent algorithm is given. The detailed procedures are explained stepwise below.

Step 1: Set the initial distribution of car / truck inflow rates: $u_i^0(t)$, $k = 0$.

Step 2: According to the inflow rate of a car / truck and using Eqs. (1–5) calculate travel time, queue length. Then, queue vanishing time, t_1, is found and further, the travel cost and dynamic marginal cost of a car / truck under system optimization and user equilibrium conditions are calculated.

Step 3: flow update (SO using Eq.2 0 and UE using Eq. 21):

$$u_i^k(t) = u_i^{k-1}(t) - \eta_k \cdot (D_i^{-1}(N_i) - MC_i(t)); \quad (29)$$

$$u_i^k(t) = u_i^{k-1}(t) - \eta_k \cdot (D_i^{-1}(N_i) - w_i(t)). \quad (30)$$

Set $\eta_k = 1/k$.

Step 4: Convergence computing if $\sum_{i,t} \left| u_i^k(t) - u_i^{k-1}(t) \right| < \varepsilon$ stop; otherwise, set k=k+1, then go to Step 2.

6. Simulation Example

Let us consider a bottleneck with its capacity of 500 pcu/h. The passenger equivalent parameters of a car / truck are 1 and 2 respectively.

The free flow times of a car / truck are 0.6 and 1 (unit time).

The inverse demand function is given as $D_i^{-1}(t) = (a_i / M_i) \cdot (M_i - c_i(t)), i = \text{c,t}$, where M_i means the maximum potential demand rate of a car / truck when potential demand for a car / truck are 2000 and 1000 respectively; a_i is the parameter of a car / truck.

The other parameters are $\alpha = 8$, $\beta = 7$ and $\gamma = 16.8$.

The total time interval is 240, time interval length is 0.1 (unit times) and work starting time is 120 which is time interval number.

Fig. 2 shows inflow rates and travel costs for car / truck travelers under user equilibrium conditions. Thus, we can see that the equalization of travel costs of the actually chosen departure time can be obtained. In other words, dynamic user equilibrium for car / truck travelers has been reached.

Also, Fig. 3 provides inflow rates and dynamic marginal travel costs for car / truck travelers under system optimal conditions.

We can find a similar equilibrium condition between dynamic marginal cost and inflow rates.

Demands for a car / truck under system optimal conditions are 1970 and 903 respectively, whereas demands for car and truck under user equilibrium condition are 1970 and 978 accordingly. Actual demand under these conditions is similar.

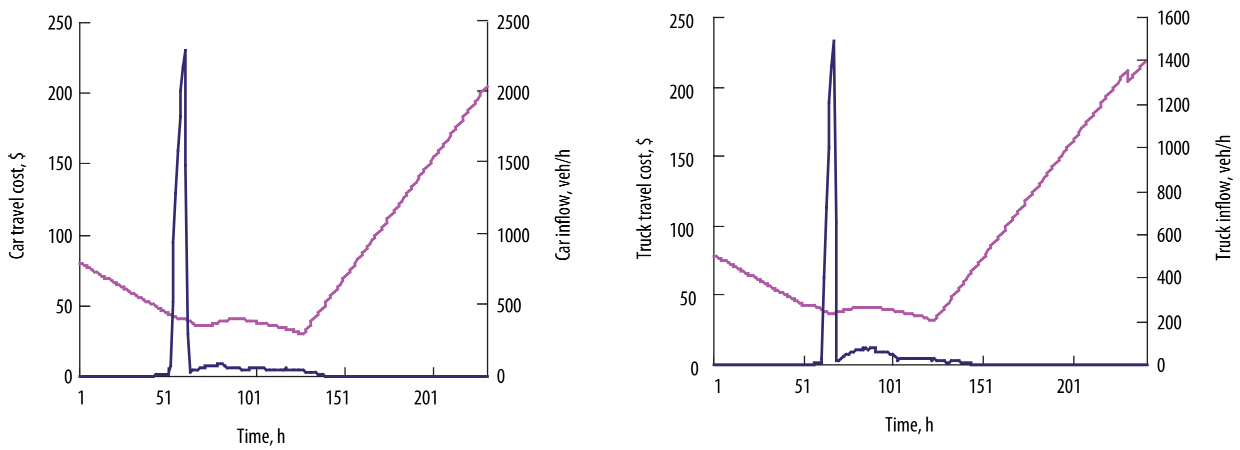

Fig. 2. Flows and costs of user equilibrium

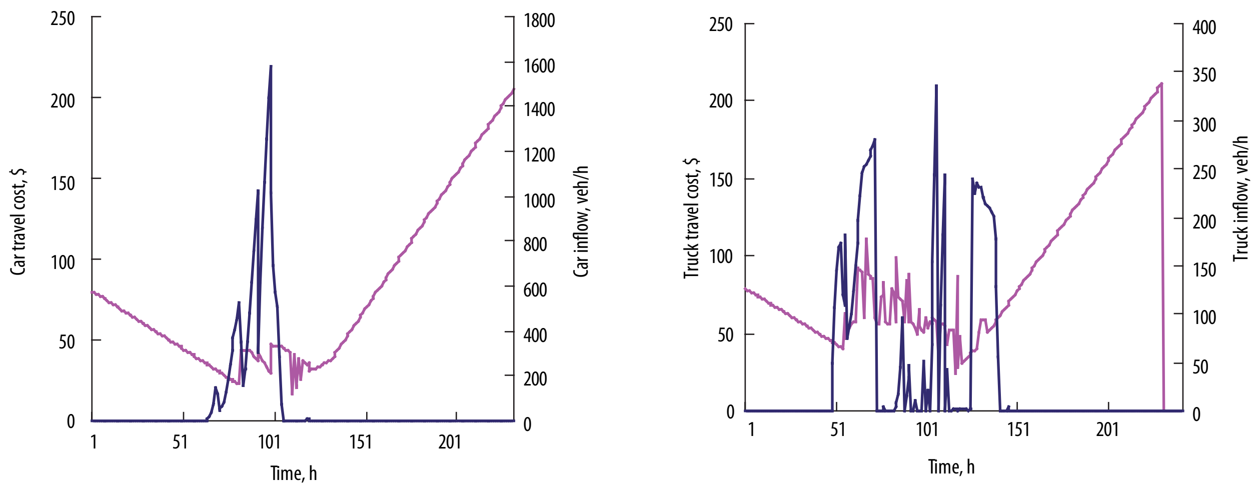

Fig. 3. Flows and costs of system optimal

A change in queue length is given in Fig. 4 which obviously indicates that the queue length of SO condition is smaller than that of UE condition.

Fig. 5 gives a toll of a car / truck and shows that the toll of the truck is evidently larger than that of the car due to higher road space occupied by the truck.

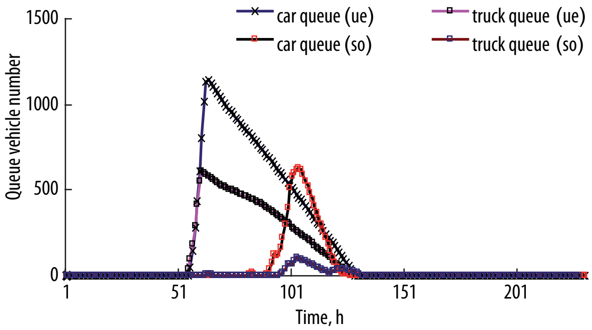

Fig. 4. Queues

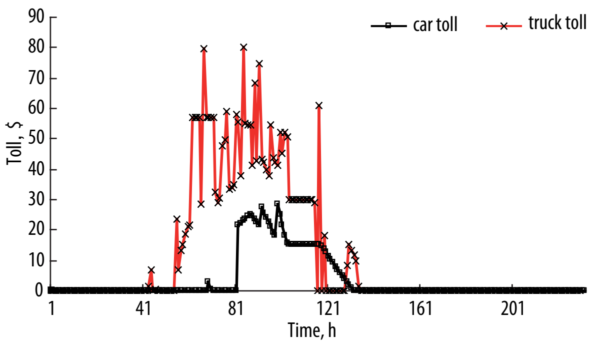

Fig. 5. Toll

7. Conclusions

A multiple vehicle type dynamic system optimal model considering departure time choices at a bottleneck is presented.

The obtained results are as follows:

- a multiple vehicle type point queue model of the bottleneck link is described; the interaction of a car / truck is modeled using free flow time and passenger car equivalents of a car / truck;
- the multiple vehicle type dynamic marginal travel cost function is presented considering choice about departure time;
- the presented simulation shows that the queue of a car / truck can be reduced and peak time is relaxed under system optimal conditions.

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