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MODELLING OF NON-MARKOVIAN QUEUING SYSTEMS

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Abstract. The purpose of this paper is to suggest a method and software for evaluating queuing approximations. A numerical queuing model with priorities is used to explore the behaviour of exponential phase-type approximation of service-time distribution. The performance of queuing systems described in the event language is used for generating the set of states and transition matrix between them. Two examples of numerical models are presented – a queuing system model with priorities and a queuing system model with quality control.

Keywords: queuing approximation, phase-type distributions, distribution fitting, Markov chains, numerical model.

1. Introduction

Queuing models are important tools for studying the performance of complex systems, but despite the substantial queuing theory literature, it is often necessary to use approximations in case a system is non-markovian. Use of phase-type (PH) distributions is a common means of obtaining tractable queuing models [1–4]. The approach of this investigation is to begin with service-time distribution to be approximated. A simple three-moment approximation, along with a more refined approximation taking into account distribution shape, is presented for an original input distribution. Then both the original and approximating distributions are used in modelling the queue with simple priorities.

It is known that creation of analytical models requires large efforts. Use of numerical methods permits to create models for a wider class of systems. The process of creating numerical models for systems described by Markov chains consists of the following stages: 1) definition of the state of a system; 2) creating equations describing a Markov chain; 3) computation of stationary probabilities of a Markov chain; 4) computation characteristics of the system performance. The most difficult stages are obtaining the set of all the possible states of a system and transition matrix between them. A method for automatic construction of numerical models for systems described by Markov chains with a countable space of states and continuous time is used in the work.

2. Approximation of service-time distribution

Let us consider the *M*/*G*/c queue. In this multi-server model with *c* servers the arrival process of customers is a Poisson process with rate l and the service time *S* of a customer has a general probability distribution function *G*(*t*). It is assumed that r = l*E* (*S*)/*c* is smaller than 1. The *M*/*G*/*c* queue with general service times permits no simple analytical solution, not even for an average waiting time. Useful approximation can be obtained by the mixture and convolutions of exponential (phase-type) distributions. Then a Markov chain with a countable space of states and continuous time can represent the evolution of the system. Suppose we let $m_k,\ k=\overline{1,3}$ denote the *k*th non-central moment (i e $E\,[\,S^k\,]$, where *S* is a random variable of service time). Construct a new random variable *Y* which can be represented as

$$Y=\begin{cases} Y_1 \text{ with prob. } p_2; \\ Y_1+Y_2 \text{ with prob. } p_1 p_2; \\ \qquad\dots \\ Y_1+Y_2+\ldots+Y_2 \text{ with prob. } p_1^{n-1} p_2, \\ \qquad\dots \end{cases} \tag{1}$$

where $Y_i,\ \ i=1,\,2,$ are independent random variables having exponential distributions with means $1/\mu_1$ and $1/\mu_2$, respectively; $p_1+p_2=1$. The random variable *Y* equals the sum of independent variables with random number *N* of summands. N is non-negative, integer-valued random variable with $E(N) < \infty$ having geometrical distribution. Its probability density is the following:

$$f(y) = p_2\mu_1\left(\frac{\mu_2-\mu_1}{\mu_2 p_2-\mu_1}e^{-\mu_1 y} - \frac{\mu_2 p_1}{\mu_2 p_2-\mu_1}e^{-\mu_2 p_2 y}\right). \quad (2)$$

If $\mu_1 = \mu_2$, then probability density of Y is given by $f(y) = p_2\mu_2 e^{-\mu_2 p_2 y}$.

Define a random variable X in this way:

$$X = \begin{cases} X_1 & \text{with prob. } p_2; \\ X_1 + X_2 & \text{with prob. } p_1, \end{cases} \quad (3)$$

where X_1 and X_2 are independent random variables having exponential distribution with parameters μ_1 and $p_2\mu_2$, respectively; $p_1 + p_2 = 1$.

Statement*. Random variables X and Y determined by (3) and (1) are equivalent.*

Proof. We will show that probability density functions of X and Y are equal. It is easy to verify that the density function of X is given by

$$f(x) = \mu_1 e^{-\mu_1 x} + \frac{p_1\mu_1}{p_1\mu_2-\mu_1}\left(\mu_1 e^{-\mu_1 x} - p_2\mu_2 e^{-p_2\mu_2 x}\right)$$

or

$$f(x) = p_2\mu_1\left(\frac{\mu_2-\mu_1}{\mu_2 p_2-\mu_1}e^{-\mu_1 x} - \frac{\mu_2 p_1}{\mu_2 p_2-\mu_1}e^{-p_2\mu_2 x}\right). \quad (4)$$

Expressions (2) and (4) are identical.

Note. Duration of service time as a random variable given by (3) allows us to apply a method for automatic construction of numerical models for queuing systems described by Markov chains. It would be impossible to do so with expression (1).

Moment matching is a common method for approximating distributions, especially in the area of queuing approximations. Though two-moment queuing approximations are common, they may lead to a serious error when the coefficient of variation, ν (the standard deviation divided by the mean), is high [1, 5]. The first three moments of any non-degenerate distribution with support on $[0, \infty)$ can be matched by the distribution (4).

To obtain the values of the parameters μ_1, μ_2, p_1 and p_2 of approximation, a complex system of non-linear equations needs to be solved:

$$\begin{cases} \dfrac{1!\, p_2\mu_1}{\mu_2 p_2-\mu_1}\left(\dfrac{\mu_2-\mu_1}{\mu_1^2} - \dfrac{\mu_2 p_1}{\mu_2^2 p_2^2}\right) = m_1; \\ \dfrac{2!\, p_2\mu_1}{\mu_2 p_2-\mu_1}\left(\dfrac{\mu_2-\mu_1}{\mu_1^3} - \dfrac{\mu_2 p_1}{\mu_2^3 p_2^3}\right) = m_2; \\ \dfrac{3!\, p_2\mu_1}{\mu_2 p_2-\mu_1}\left(\dfrac{\mu_2-\mu_1}{\mu_1^4} - \dfrac{\mu_2 p_1}{\mu_2^4 p_2^4}\right) = m_3. \\ p_1 + p_2 = 1. \end{cases} \quad (5)$$

The solution of the system is the following [6]:

$$\mu_2 = \frac{g_2 - g_1^2}{g_1^3 - 2g_1 g_2 + g_3}, \; g_k = \frac{m_k}{k!}, \; k = \overline{1,3}; \quad (6)$$

$$\mu_1 = \frac{1 + \mu_2 g_1 \pm \sqrt{(1-\mu_2 g_1)^2 + 4a^2\left(g_2 - g_1^2\right)}}{2g_1 - 2\mu_2\left(g_2 - g_1^2\right)};$$

$$p_1 = \frac{\mu_2(\mu_1 g_1 - 1)}{\mu_2(\mu_1 g_1 - 1) + \mu_1};$$

$$p_2 = \frac{\mu_1}{\mu_2(\mu_1 g_1 - 1) + \mu_1}.$$

3. Approximation of lognormal distribution

Suppose that service-time is distributed according to lognormal distribution with probability density

$$g(x) = \frac{1}{\alpha x\sqrt{2\pi}}\exp\left[-(\ln x - \lambda)^2 / 2\alpha^2\right], \quad x > 0 \quad (7)$$

and parameters $\alpha = 0{,}9$, $\lambda = -0{,}05$. The first three non-central moments are

$$m_1 = 1{,}42618, \; m_2 = 4{,}57225 \text{ and } m_3 = 28{,}3606. \quad (8)$$

The approximating density parameters are the following:

$$\mu_1 = 0{,}75935, \; \mu_2 = 0{,}141605,$$
$$p_1 = 0{,}00559, \quad p_2 = 0{,}99441. \quad (9)$$

The original and approximating densities are shown in Fig 1.

4. Numerical model of queuing system with simple priority

First of all the created software is tested with a simple system that has analytical formulas to calculate system characteristics.

Suppose that there are two classes of customers in a queuing system. Their service times follow a lognormal distribution. The arrival process is Poisson with the parame-

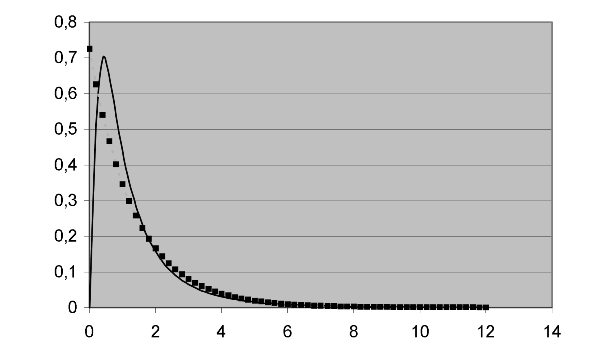

Fig 1. Densities of lognormal distribution (continuous line) and approximation (dot line)

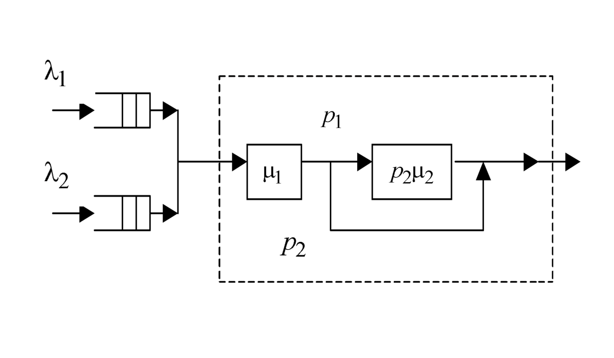

Fig 2. Queuing system with simple priority

ters λ_1 and λ_2, respectively. We shall suppose that class 1 has a higher simple priority than class 2. This queuing model has l_1 and l_2 waiting positions for each class of customers to await service, respectively. Let us calculate the mean number of customers in the queue and the mean waiting time of a customer in each class.

Let us assume that service-time distribution is approximated by expression (3). The scheme of considering a system is represented in Fig 2.

A new customer cannot be accepted for servicing while a previous one has not passed throughout all the phases of service.

A Markov chain with the countable space of states and continuous time can describe the functioning of such a system. To construct a numerical model of the system the approach proposed in [7] will be applied.

The set of events in the system is:

$$E = \{e_1, e_2, e_3, e_4, e_5\},$$

where

e_1 – a customer arrived from class 1;
e_2 – a customer arrived from class 2;
e_3 – completed service in the first phase with probability p_2;
e_4 – completed service in the first phase with probability p_1;
e_5 – completed service in the second phase.

The set of all the feasible states of the system is:

$$N = \{(n_1, n_2, n_3, n_4)\}, n_1 = \overline{0, l_1}; n_2 = \overline{0, l_2},$$

where

n_1 – number of customers from class 1 present in the system;
n_2 – number of customers from class 2 present in the system;

$$n_3 = \begin{cases} 0, \text{if the system is empty;} \\ 1, \text{ if a customer from class 1 is being served;} \\ 2, \text{ if a customer from class 2 is being served;} \end{cases}$$

$$n_4 = \begin{cases} 0, \text{if the system is empty;} \\ 1, \text{ if a customer is being served in the first phase;} \\ 2, \text{if a customer is being served in the second phase.} \end{cases}$$

The mean number of customers $L^{(1)}$ and $L^{(2)}$ in the queue and the mean waiting time $W^{(1)}$ and $W^{(2)}$ of a customer in each class are given by the following formulas:

$$L^{(j)} = \sum_{n_j=1}^{l_j} \sum_{n_1, n_2, n_3, n_4} n_1 \pi(n_1, n_2, n_3, n_4),$$

$$W^{(j)} = \frac{L^{(j)}}{\lambda_j}, \quad j = 1, 2,$$

where $\pi(n_1, n_2, n_3, n_4)$ is the stationary probability of the system state. As an example describe the event e_3 in the event language.

```
e3 :  IF  n4 = 1
        if  n3 = 1        then n1 ← 0
                          else n2 ← 0
        end if
        if  n1 > 0        then n3 ← 1
                          else if  n2 > 0
                          then n3 ← 2
                          else n3 ← 0   n4 ← 0
                          end if
        end if
      end IF
Return Intens ← μ1 p2
END e3
```

The created software, using the description of events, generates the set of feasible states of the system, the matrix of transition rates between them and the stationary probabilities of the states. Applying the obtained probabilities, it is possible to compute the desired characteristics of the system performance.

4.1. Results

If the number of waiting positions for service in each class of customers is unlimited, e g $l_1 = \infty$ and $l_2 = \infty$, then the values $W^{(i)}$ and $L^{(i)}$ can be calculated by the analytical formulas:

$$L_q^{(1)} = \lambda_1 \cdot W_q^{(1)} = \lambda_1 \cdot \frac{(\lambda_1 + \lambda_2) E^2(X)\left(1+\nu_X^2\right)}{2\left(1-\lambda_1 E(X)\right)};$$

$$L_q^{(2)} = \lambda_2 \cdot W_q^{(2)} = \lambda_2 \cdot \frac{(\lambda_1 + \lambda_2) E^2(X)\left(1+\nu_X^2\right)}{2\left(1-\lambda_1 E(X)\right)\left(1-(\lambda_1+\lambda_2) E(X)\right)};$$

$$\nu_X = \frac{\sigma(X)}{E(X)},$$

where $E(x)$ and $\sigma(X)$ are the mean and the standard deviation of the service time.

The results of the analytical model with the parameters $\lambda = -0{,}05$, $\alpha = 0.9$, $m_1 = 1{,}42618$, $m_2 = 4{,}57225$, $m_3 = 28{,}3606$, $\lambda_1 = 0{,}2$, $\lambda_2 = 0{,}09$, $l_1 = \infty$, $l_2 = \infty$, are the following:

$$L_q^{(1)} = 0{,}1855, \quad L_q^{(2)} = 0{,}1423.$$

The results of the numerical model with the following values of parameters

$$\lambda_1 = 0{,}2, \lambda_2 = 0{,}9,$$

$$\mu_1 = 0{,}74437, \quad \mu_2 = 0{,}227796,$$

$$p_1 = 0{,}018503, \quad l_1 = 25, \quad l_2 = 25$$

are:

$$L_q^{(1)} = 0{,}1855, \quad L_q^{(2)} = 0{,}1424.$$

As it is seen from the results, software calibration is successful, and we can move forward with analysing queuing systems that do not have analytical formulas to calculate various system characteristics.

5. Numerical model of queuing system with quality control

Suppose that there are one flow of customers and two queuing systems. Their service time follows a lognormal distribution. The arrival time process is Poisson with the parameter λ. This queuing model has l_1 and l_2 waiting positions before each queuing system to await service. This system holds quality control that redirects already processed customers to go through both queuing systems again. Let us calculate the mean number of customers in both queues and the mean waiting time of a customer in each queue.

A new customer cannot be accepted for servicing while a previous one has not passed throughout both phases of a queuing system.

The set of events in the system is:

$$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\},$$

where

e_1 – a customer arrived to the first queuing system;

e_2 – completed service of the first queuing system in the first phase with probability p_1;

e_3 – completed service of the first queuing system in the first phase with probability p_2;

e_4 – completed service of the first queuing system in the second phase;

e_5 – completed service of the second queuing system in the first phase with probability p_1;

e_6 – completed service of the second queuing system in the first phase with probability p_2 and a customer has passed quality control;

e_7 – completed service of the second queuing system in the first phase with probability p_2 and a customer has failed quality control;

e_8 – completed service of the second queuing system in the second phase, and a customer has passed quality control;

e_9 – completed service of the second queuing system in the second phase, and a customer has failed quality control.

The set of all the feasible states of the system is:

$$N = \{(n_1, n_2, n_3, n_4, n_5, n_6)\}, \quad n_1 = \overline{0, l_1}; \quad n_4 = \overline{0, l_2},$$

where

n_1 – number of customers in the first queue;

$$n_2 = \begin{cases} 0, & \text{if the first phase of the first queuing system is empty;} \\ 1, & \text{if a customer is being served in the first phase of the first queuing system.} \end{cases}$$

$$n_3 = \begin{cases} 0, & \text{if the second phase of the first queuing system is empty;} \\ 1, & \text{if a customer is being served in the second phase of the first queuing system.} \end{cases}$$

n_4 – number of customers in the second queue;

$$n_5 = \begin{cases} 0, & \text{if the first phase of the second queuing system is empty;} \\ 1, & \text{if a customer is being served in the first phase of the second queuing system.} \end{cases}$$

$$n_6 = \begin{cases} 0, & \text{if the second phase of the second queuing system is empty;} \\ 1, & \text{if a customer is being served in the second phase of the second queuing system.} \end{cases}$$

Let us assume that service-time distribution is approxi-

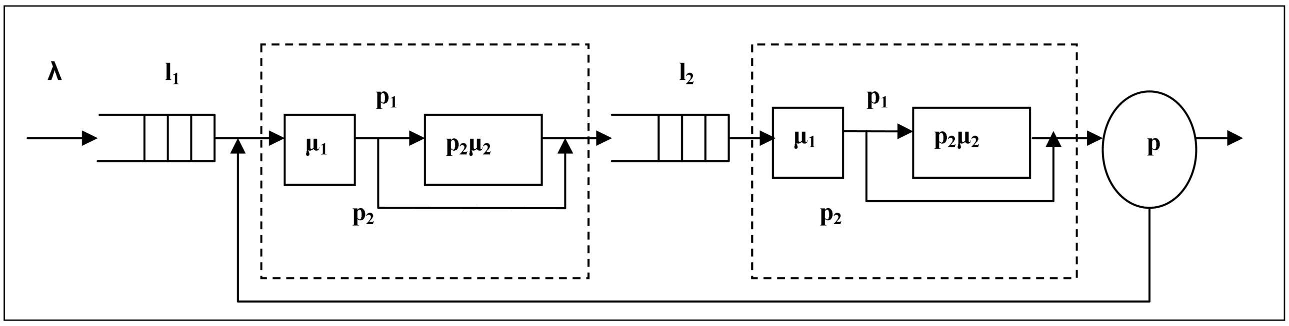

Fig 3. Queuing system with quality control

mated by expression (3). The scheme of considering the system is represented in Fig 3.

The mean number of customers $L^{(1)}$ and the mean waiting time $W^{(1)}$ of a customer in the first queue are given by the following formulas:

$$L^{(j)} = \sum_{m_1=1}^{l_1} \sum_{n_1,n_2,n_3,n_4,n_5,n_6} m_1\pi(n_1,n_2,n_3,n_4,n_5,n_6);$$

$$W^{(j)} = \frac{L^{(1)}}{\lambda_1}, \quad j = 1, 2,$$

where $\pi(n_1,n_2,n_3,n_4,n_5,n_6)$ is the stationary probability of the system state. As an example describe the event e_3 in the event language.

e_3 : **IF** $n_3 > 0$
 if $n_5 < 1$ *and* $n_6 < 1$
 then $n_5 \leftarrow n_5 + 1$
 else $n_4 \leftarrow n_4 + 1$
 end if
 if $n_1 > 0$ **then**
 $n_1 \leftarrow n_1 - 1;\quad n_2 \leftarrow n_2 + 1;\quad n_3 \leftarrow n_3 - 1;$
 else $n_3 \leftarrow n_3 - 1$
 end if
 End IF
Return $Intens \leftarrow \mu_2 p_2$
END e_3

The created software, using the description of events, generates the set of feasible states of the system, the matrix of transition rates between them and the stationary probabilities of the states. Applying the obtained probabilities, it is possible to compute the desired characteristics of the system performance.

5.1. Results

The results of the numerical model with the following values of parameters

$$\lambda = 1,$$

$$\mu_1 = 0{,}74437,\ \ \mu_2 = 0{,}227796,$$
$$p_1 = 0{,}018503,\ p = 0{,}9, l_1 + l_2 = 20$$

are:

$$L_q^{(1)} = 8.9849,\ \ L_q^{(2)} = 7.6322,$$

$$W_q^{(1)} = 12.0706,\ \ W_q^{(1)} = 33.4796.$$

6. Conclusions

The method and created software for automatic construction of numerical models for systems described by Markov chains together with queuing approximation allows:

- to analyse complex Non-Markovian queuing systems applying Markov chains theory;
- queuing systems with the infinite space of states can be approximated by the countable space of states.

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NEMARKOVIŠKŲ APTARNAVIMO SISTEMŲ MODELIAVIMAS

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Santrauka

Eilių teorijos modeliai plačiai taikomi įvairioms sudėtingoms sistemoms analizuoti. Beveik visi modeliai kuriami su prielaida, kad stochastinis procesas, vykstantis sistemoje, yra Markovo procesas. Tačiau dažniausiai ši prielaida nepasitvirtina. Straipsnyje pateikta metodika, kaip nemarkoviškus aptarnavimo sistemų modelius aproksimuoti markoviškais modeliais, naudojant eksponentinių fazių skirstinius. Sistemos funkcionavimas aprašomas įvykių kalboje. Sukurtoji programinė priemonė C++ kalboje pagal aprašymą generuoja sistemos galimų būsenų erdvę, perėjimo intensyvumų tarp jų matricą bei suskaičiuoja stacionariąsias būsenų tikimybes. Pateikti du aptarnavimo sistemų pavyzdžiai, iliustruojantys pateiktąją metodiką.

Reikšminiai žodžiai: eilių modelių aproksimavimas, fazių tipo skirstiniai, skirstinių suderinimas, Markovo grandinės, skaitmeninis modelis.

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