HEURISTIC ANALYSIS OF INVESTMENT STRATEGY

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Abstract. The present article investigates the problem of optimal investment, when, given a limited amount of funds, a decision must be taken to which projects and what amounts of funds are to be invested. Supposing that the expected average profit depends on several possible different market conditions, a matrix "game against nature" has been selected as the initial mathematical model. With a view to develop the optimal investment strategy, a linear programming task is formulated. The sensitivity of solutions to profitability coefficients is analysed by means of formulating a dual task for this task. The present article considers the stability and dynamics of the optimum investment strategy given a varying amount of the funds allocated to investment and the profitability of specific projects.

Keywords: optimal investment, matrix game, parameter programming, stability and dynamics, average profit.

1. Introduction

The finding of the optimal investment plan is usually referred to as a portfolio task. In scientific publications related to the field of financial mathematics, this problem is mentioned fairly frequently. The heuristic methods used for the construction of an optimal portfolio are described in article [1]. Article [2] suggests an algorithm for dynamic multi-period portfolio optimisation. The relationship between fuzzy sets and matrix games theories is offered for multicriteria decision making in paper [3].

The matrix game, linear and stochastic programming are described as models for optimal financing of labour safety means problem in articles [4, 5].

Many financial economic decision problems were approached by operational research techniques [6, 7]. In the paper [8] the Mean Absolute Deviation model is investigated using the bounds of optimal objective function value, an algorithm for this model is developed. The linear programming solvability of Portfolio optimization problem is discussed in paper [9]. Partially integer mathematical programming is employed in article [10].

Various mathematical models are possible to solve the problem of selection the optimal investment policy. The selection of a model depends on the kind of the presumptions made with regard to the known or partially known parameters of a task. The present paper examines the case when the average expected profit is not precisely known, because it depends on the market conditions in the future and the probabilities of possible market conditions are unknown either. In the case of such uncertainty the mathematical model of the problem may be a matrix "game against nature" which is sometimes referred to a statistical game. In order to develop the optimal investment strategy a linear programming task is formulated, which optimal value of the objective function is guaranteed by average profit which does not depend on market conditions.

2. The mathematical models required for the analysis of the problem

Let's suppose that $c_1, c_2, ..., c_m$ are the amounts of the funds required for full financing of projects $P_1, P_2, ..., P_m$. The prospective profit depends on market conditions $R_1, R_2, ..., R_n$ which probabilities are unknown. Let the average profit

which we will obtain provided be fully financed project P_i (we are going to allocate the whole required sum c_i) and the market condition is R_j be marked a_{ij}, i.e., we arrive at the profitability coefficients matrix $\left| a_{ij} \right|_{mxn}$. Where the funds allocated for investment C are less than sum $\sum\limits_{i=1}^{m} c_i$, i.e., the funds for the financing of all projects are insufficient, a problem arises what projects need to be selected. Let's presume that partial financing of each project with a proportionally smaller profit is possible.

Let's mark variables $x_i, i = 1, 2, ..., m$ as the part of the funds required for full financing of i project. Then product $c_i x_i$ is the amount of the funds allocated to project P_i, and the variables satisfy inequalities $0 \leq x_i \leq 1, i = 1, 2, ..., m$. The expected average profit is going to be $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} a_{ij} x_i z_j$, where $z_j, \ j = 1, 2, ..., n$, are probabilities of market conditions.

The present paper examines a parametric programming model which formation is based on looking for a modified mixed maximin strategy that would guarantee a specific amount of the average profit, which does not depend on market conditions [3]. Let W be the profit which will be obtained selecting the investment plan ($x_1, x_2, ..., x_m$). In order to obtain the optimal investment plan, the following linear programming problem must be resolved:

$$\max W$$

$$\sum_{i=1}^{m} c_i x_i \leq C ,$$

$$\sum_{i=1}^{m} a_{ij} x_i - W \geq 0, \quad j = 1, 2, ..., \text{n}, \qquad (1)$$

$$0 \leq x_i \leq 1, \quad i = 1, 2, ..., m.$$

Optimal value W_0 of the objective function is the guaranteed average profit.

Component $\overline{y}$ of solution ($\overline{y}, \overline{z_1}, \overline{z_2}, ..., \overline{z_n}, \overline{v_1}, \overline{v_2}, ..., \overline{v_m}$) of a linear programming problem dual to problem (1):

$$\min (Cy + \sum_{i=1}^{m} v_i)$$

$$c_i y - \sum_{j=1}^{n} a_{ij} z_j + v_i \geq 0, i = 1, 2, ..., m,$$

$$\sum_{j=1}^{n} z_j = 1, \qquad (2)$$

$$y \geq 0, \ z_j \geq 0, j = 1, 2, ..., \text{n},$$

$$v_i \geq 0, i = 1, 2, ..., m,$$

shows the value of money unit of the available sum C in the event of partial financing (with regard to optimal investment into projects $\overline{P_1, P_2, ..., P_m}$). Components $\overline{z_j}, \ j = 1, 2, ..., n$ are the most unfavourable to the investor probabilities of market conditions, $\overline{v_i}, \ i = 1, 2, ..., m$ are the additional values of separate projects conditional upon $\overline{x_i} = 1$.

On the basis of duality theorems we get that the guaranteed profit is $W_0 = \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} a_{ij} \overline{x_i} \overline{z_j}$, where $\overline{x_1}, \overline{x_2}, ..., \overline{x_m}$ are optimal values of the variables of the problem (1).

If the probabilities of market conditions are not the most unfavourable, i.e., irrespective of the way a market condition is realised, the investment plan ($\overline{x_1}, \overline{x_2}, ..., \overline{x_m}$) will yield the average profit not less than W_0.

In order to find out how the optimal investment policy and the amount of guaranteed average profit change, where the amount of the funds allocated for investment increases, we need to solve the following parametric programming problem:

$$\max W$$

$$\sum_{i=1}^{m} c_i x_i \leq C + t ,$$

$$\sum_{i=1}^{m} a_{ij} x_i - W \geq 0, \quad j = 1, 2, ..., n, \qquad (3)$$

$$0 \leq x_i \leq 1, \quad i = 1, 2, ..., m.$$

The result is the breaking down of interval (0; $\sum\limits_{i=1}^{m} c_i$) into partial intervals (0; C_1), (C_1; C_2), ..., (C_{k-1}; C_k), within each of which the optimal investment policy is unchanged (to fully finance certain specific projects, others – to finance in part, third projects – not to finance), but changes moving to another interval. In each of these intervals and subject to increase of parameter t, the optimal values of dual variables $\overline{y}, \overline{z_1}, \overline{z_2}, ..., \overline{z_n}, \overline{v_1}, \overline{v_2}, ..., \overline{v_m}$ remain fixed, however they change moving from one interval to another.

Within each of interval (C_{l-1}; C_l), the following linear dependencies are true:

$$W_0(C) = k_l C + d_l \text{ and } \overline{x_i}(C) = k_{il} C + d_{il}.$$

By dividing restrictions 1, 2,..., m of task (2) by $c_1, c_2, ..., c_m$, we will get:

$$\min (Cy + \sum_{i=1}^{m} v_i)$$

$$y - \sum_{j=1}^{n} \frac{a_{ij}}{c_i} z_j + \frac{v_i}{c_i} \geq 0, i = 1, 2, ..., m,$$

$$\sum_{j=1}^{n} z_j = 1, \qquad (4)$$

$$y \geq 0, \; z_j \geq 0, \; j = 1, 2, \ldots, n,$$

$$v_i \geq 0, \; i = 1, 2, \ldots, m.$$

Let's formulate a task which would be dual to task (4):

$$\max W$$

$$\sum_{i=1}^{m} \frac{a_{ij}}{c_i} X_i - W \geq 0, \quad j = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{m} X_i \leq C, \qquad (5)$$

$$0 \leq X_i \leq c_i, \quad i = 1, 2, \ldots, m.$$

Task (5) is equivalent to task (1), and its variable X_i shows the amount of funds which must be allocated to project P_i.

Let's suppose that in respect of the fixed amount of available funds $C = C_0$, letter J stands for a set of indexes i for which probability $\overline{z}_j(C_0)$ is greater than zero, while letter J_0 stands for a set of indexes i for which probability $\overline{z}_j(C_0) = 0$. Having employed duality theorems for tasks (4) and (5) in respect of each k belonging to set J, we obtain the following linear programming task:

$$\max \sum_{i=1}^{m} \frac{a_{ik}}{c_i} X_i$$

$$\sum_{i=1}^{m} \frac{a_{il}}{c_i} X_i = W_0(C_0), \; l \in J,$$

$$\sum_{i=1}^{m} \frac{a_{is}}{c_i} X_i \geq W_0(C_0), \; s \in J_0,$$

$$\sum_{i=1}^{m} X_i \leq C_0, \qquad (6)$$

$$0 \leq X_i \leq c_i, \quad i = 1, 2, \ldots, m.$$

Having carried out the sensitivity analysis of the solution of task (6) to the coefficients of the objective function, we find that the intervals within which each quotient $\frac{a_{ij}}{c_i}$, $j \in J_0$, may change (while others remain unchanged), so that the optimal investment plan does not change:

$$\underline{g_{ij}} \leq \frac{a_{ij}}{c_i} \leq \overline{g_{ij}}.$$

The sensitivity analysis of the solution of the following task which is equivalent to task (6)

$$\max \sum_{i=1}^{m} a_{ik} x_i$$

$$\sum_{i=1}^{m} a_{il} x_i = W_0(C_0), \; l \in J,$$

$$\sum_{i=1}^{m} a_{is} x_i \geq W_0(C_0), \; s \in J_0,$$

$$\sum_{i=1}^{m} c_i x_i \leq C_0 \qquad (7)$$

$$0 \leq x_i \leq 1, \; i = 1, 2, \ldots, m.$$

to the coefficients of the objective function allows to find the intervals within which each coefficient a_{ij}, $j \in J_0$ may change (while others remain unchanged), so that the optimal investment plan does not change:

$$\underline{a_{ij}} \leq a_{ij} \leq \overline{a_{ij}}, \quad i = 1, 2, \ldots, m.$$

A linear programming task which is equivalent to tasks (1) and (5) is as follows:

$$\max W$$

$$\sum_{i=1}^{m} \frac{a_{ij}}{c_i} C \xi_i - W \geq 0, \quad j = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{m} \xi_i \leq 1, \qquad (8)$$

$$0 \leq \xi_i \leq \frac{c_i}{C}, \quad i = 1, 2, \ldots, m,$$

where variable is $\xi_i = \frac{c_i}{C} x_i = \frac{X_i}{C}$, and the optimal value of the objective function is the same as that of tasks (1) and (5). By fixing $C = C_0$, we will get a task equivalent to tasks (6) and (7):

$$\max \sum_{i=1}^{m} \frac{a_{ik}}{c_i} C_0 \xi_i$$

$$\sum_{i=1}^{m} \frac{a_{il}}{c_i} C_0 \xi_i = W_0(C_0), \; l \in J_0,$$

$$\sum_{i=1}^{m} \frac{a_{is}}{c_i} C_0 \xi_i \geq W_0(C_0), \; s \in J_0,$$

$$\sum_{i=1}^{m} \xi_i \leq 1, \qquad (9)$$

$$0 \leq \xi_i \leq \frac{c_i}{C_0}, \quad i = 1, 2, \ldots, m.$$

The sensitivity analysis of the solution of the task to the coefficients of the objective function allows to find the intervals within which each quotient $\dfrac{a_{ij}}{c_i} C_0$, $j \in I_0$, may change (when others remain unchanged), so that the optimal investment plan does not change:

$$\underline{h_{ij}} \le \frac{a_{ij}}{c_i} C_0 \le \overline{h_{ij}} \ .$$

By parametrising the coefficients of the objective function of task (7)

$$\max \sum_{i=1}^{m} (a_{ik} + ht)x_i$$

$$\sum_{i=1}^{m} a_{il}x_i = W_0(C_0), \quad l \in J_0,$$

$$\sum_{i=1}^{m} a_{is}x_i \ge W_0(C_0), \quad s \in J_0,$$

$$\sum_{i=1}^{m} c_i x_i \le C_0 \qquad (10)$$

$$0 \le x_i \le 1, \quad i = 1, 2, ..., m,$$

we can find out by what amount all coefficients a_{ij}, $j \in J_0$ may increase or decrease or certain coefficients decrease, whereas others increase (by modifying sign h), so that the solution (the optimal investment plan) will remain stable.

3. Conclusions

1. For each $C \in (C_{l-1}; C_l)$ of the following linear dependencies:

$$W_0(C) = k_l C + d_l \quad \text{and} \quad \overline{x_i}(C) = k_{il}C + d_{il},$$

coefficients are $k_l = \overline{y}$, $d_l = \sum_{i=1}^{m} \overline{v_i}$, $k_{il} = \dfrac{\overline{\xi_i}}{c_i}$, $d_{il} = 0$. Here $\overline{\xi_i}$ is i component of the solution of task (8). The rate of increase of the guaranteed average profit decreases with the increase of C.

2. For each i, sum $\gamma_i = \sum_{j=1}^{n} \dfrac{a_{ij}}{c_i}\overline{z_j}$ remains unchanged as long as $C \in (C_{l-1}; C_l)$. Let's refer to γ_i as the coefficient of the profitability of project P_i within interval $(C_{l-1}; C_l)$. It is possible to introduce, in each interval, the arrangement of projects according to their profitability. Let's state that $P_k \succ P_s$, if $\gamma_k > \gamma_s$, and $P_k \sim P_s$, if $\gamma_k = \gamma_s$. The following statements are true:

$$\overline{x_i} = 0, \text{ if } \gamma_i < \overline{y},$$

$$\overline{x_i} = 1, \text{ if } \gamma_i > \overline{y},$$

$$0 < \overline{x_i} < 1, \text{ if } \gamma_i = \overline{y}.$$

3. The optimal investment plan $(\overline{x_1}\,\overline{x_2}...,\overline{x_m})$ is the solution of the linear equations system:

$$\sum_{i=1}^{m} a_{ij}x_i = C\,\overline{y} + \overline{v_i}, \quad j \in I_0.$$

4. The most unfavourable probabilities $\overline{z_j}(C)$ of the market conditions $R_1, R_2, ..., R_n$, which, as we have seen, depend on the amount of available funds, are mixed minimax (second player) strategies in the matrix game

$$\left[\frac{a_{ij}}{c_i} C \right]_{mxn}.$$

The value of game V coincides with the guaranteed average profit $W_0(C)$. Optimal strategies are obtained solving task (8).

If $\overline{\xi_i}(C)$ is the component of the mixed maximin (first player) strategy of this game, the component of the optimal investment plan will be

$$\overline{x_i} = \overline{x_i}(C) = \frac{C}{c_i} \overline{\xi_i}(C).$$

5. If we get that the lengths of the intervals $(\underline{g_{ij}}; \overline{g_{ij}})$, $(\underline{a_{ij}}; \overline{a_{ij}})$, $(\underline{h_{ij}}; \overline{h_{ij}})$ are not large or that the allowed step h, which has been obtained by solving task (10), is small, stochastic models of the proposed tasks must be considered, because in general, profit coefficients are not necessarily determined values, but rather random ones.

6. The columns of the matrix of profit coefficients may not be necessarily treated as possible different market conditions, but also as different forecasts of several experts with regard to the profitability of projects. If a market condition (or the opinion of an expert) exists which is "worse" than others (than forecasts of other experts) in respect to all projects, the mathematical models under analysis use only the column corresponding to this condition (expert), and tasks are simplified. Instead of the matrix under consideration, we are also going to have a single column, if the probabilities of market conditions are known (or where the forecasts of all experts may be replaced by a single – generalised - one).

4. Example

Let' suppose there are four projects P_1, P_2, P_3, P_4 and three possible market conditions R_1, R_2, R_3.

The profit coefficients matrix is:

$$\begin{bmatrix} 2 & 6 & 4 \\ 3 & 4 & 2 \\ 5 & 1 & 4 \\ 4 & 2 & 5 \end{bmatrix},$$

$c_1 = 40$, $c_2 = 10$, $c_3 = 20$, $c_4 = 30$.

Letters c_i have been selected in such a way that their sum would be 100, therefore, each of them may be considered a per cent of the sum total required to fully finance all projects.

Having solved parametric programming problem (4), we get six intervals of the variation of parameter C, within which the optimal investment strategy remains stable (meaning that the sets of fully and partially financed projects do not change). Table 1 exhibits a respective increase of the guaranteed average profit in different intervals at a different rate (on the basis of the first conclusion).

Table 1. Guaranted average profit in different intervals

Interval number	($C_{l-1};C_l$)	$W_0(C) = k_l C + d_l$
1	(0;23,3333)	$W_0 = 0,2C$
2	(23,3333;50)	$W_0 = 0,125C+1,75$
3	(50;72,5)	$W_0 = 0,1222C+1,89$
4	(72,5;85)	$W_0 = 0,1C+3,5$
5	(85;95)	$W_0 = 0,075C+5,625$
6	(95;100)	$W_0 = 0,05C+8$

Fig 1 illustrates in which projects and what amount of funds it is optimal to invest where available funds C increase.

We can see a respective increase of guaranteed average profit in Fig 2.

Profitability coefficients γ_i of projects in different intervals ($C_{l-1};C_l$) may be seen in Fig 3.

The arrangement of the projects according to the profitability coefficients γ_i is presented in Table 2.

The way to estimate $\overline{y}$ of a money unit to be invested and probabilities $\overline{z_1}, \overline{z_2}, \overline{z_3}$ change with the increase of the value of C are illustrated by the diagrams in Fig 4 and 5.
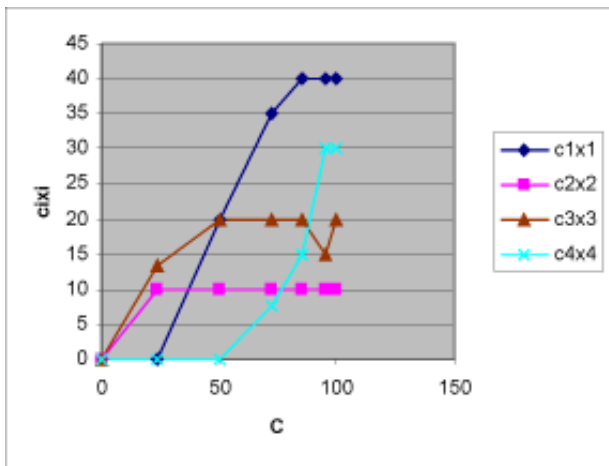

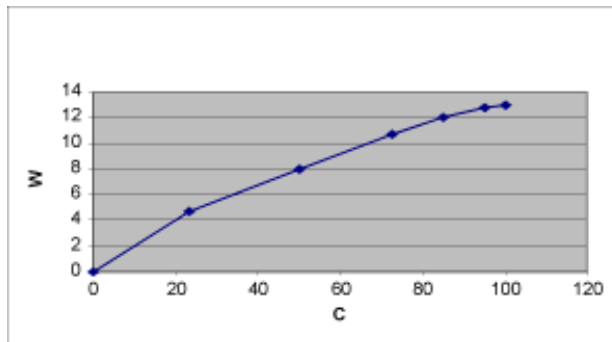**Fig 1.** Dependency of optimal invested amounts on available funds C

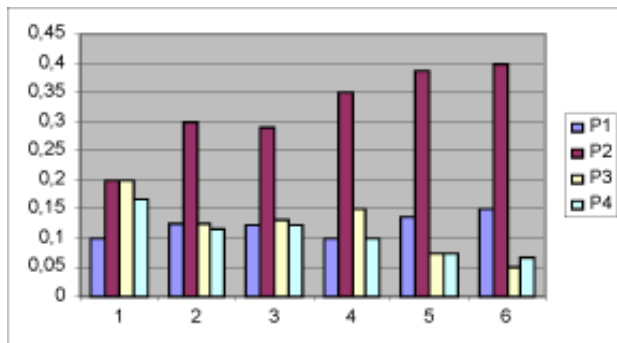**Fig 2.** Dependency of guaranteed average profit on available funds C

Fig 3. Coefficients of projects profitability in different intervals of values of C

Table 2. Arrangement of projects

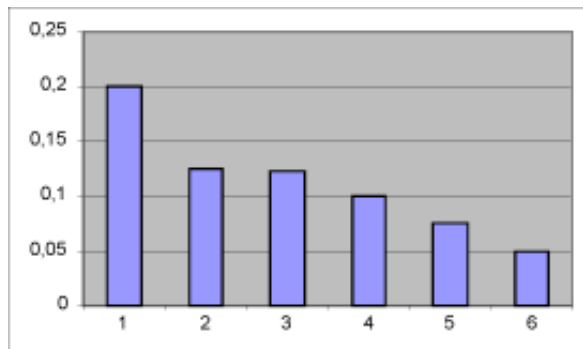| Interval number	($C_{l-1};C_l$)	Preference relations
1	(0;23,33)	$P_2 \sim P_3 \succ P_4 \succ P_1$
2	(23,33;50)	$P_2 \succ P_1 \sim P_3 \succ P_4$
3	(50;72,5)	$P_2 \succ P_3 \succ P_1 \sim P_4$
4	(72,5;85)	$P_2 \succ P_3 \succ P_1 \sim P_4$
5	(85;95)	$P_2 \succ P_1 \succ P_3 \sim P_4$
6	(95;100)	$P_2 \succ P_1 \succ P_4 \sim P_3$

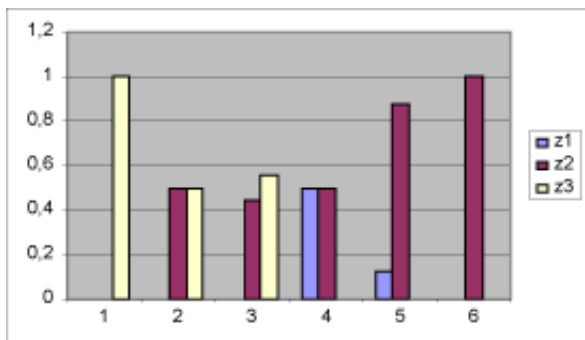**Fig 4.** Estimate of a money unit in different intervals of values of C

Fig 5. Most unfavourable probabilities of market conditions in different intervals of values of C

Let's examine the sensitivity of the solution of task (7) to the coefficients of the objective function, when $C_0 = 40$, which follows that $\overline{z_1} = 0, \overline{z_2} = 0.5, \overline{z_3} = 0.5$ and $W_0 = 6.75$.

The optimal investment plan turns out to be: $x_1 = 0,3125$, $x_2 = 1$, $x_3 = 0,875$, $x_4 = 0$, is not going to change as long as one profit coefficient a_{ij} is going to change within the interval referred to in Table 3 with other coefficients unchanged.

Table 3. Intervals for profit coefficients

$\underline{a_{ij}}$	a_{ij}	$\overline{a_{ij}}$
4	a_{12}	$+\infty$
0,5	a_{22}	$+\infty$
0,5	a_{32}	**6**
$-\infty$	a_{42}	**2,5**
0	a_{13}	**8**
0	a_{23}	$+\infty$
3,6	a_{33}	$+\infty$
$-\infty$	a_{43}	**5,5**

The solution of parameter task (10) shows that the optimal investment plan has to change if a_{13}, a_{23}, a_{33} all decrease by the amount of 0,210536, and a_{43} increases by

this amount or if a_{12}, a_{22}, a_{32} all decrease by the amount of 0,222222, and a_{42} increases by this amount. Therefore, we see that when several coefficients change, a small step (h) is allowed, so it would be important the examine the recommended mathematical models as stochastic ones.

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INVESTAVIMO STRATEGIJOS EURISTINĖ ANALIZĖ

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S a n t r a u k a

Nagrinėjama optimalaus investavimo problema, kai, turint ribotą lėšų kiekį, reikia nuspręsti, į kuriuos projektus ir kokias sumas turėtume investuoti. Teigiant, kad laukiamas vidutinis pelnas priklauso nuo kelių galimų skirtingų rinkos būsenų, pradiniu matematiniu modeliu pasirenkamas matricinis lošimas su gamta. Optimaliai investavimo strategijai gauti formuluojamas tiesinio programavimo uždavinys, kurio optimali tikslo funkcijos reikšmė yra garantuotas, nuo rinkos būsenos nepriklausantis vidutinis pelnas. Naudojant šiam uždaviniui dualųjį uždavinį, tiriamas sprendinio jautrumas pelno koeficientams. Parametrizuojant abiejų, tiesioginio ir dualiojo, tiesinio programavimo uždavinių koeficientus, nagrinėjamas optimalios investavimo strategijos stabilumas ir dinamika, kintant investuotojui skirtų lėšų kiekiui bei atskirų projektų pelningumui.

Pagrindiniai žodžiai: optimalus investavimas, matricinis lošimas, parametrinis programavimas, garantuotas vidutinis pelnas.

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