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MULTI-ATTRIBUTE GROUP DECISION MAKING BASED ON HESITANT FUZZY SETS, TOPSIS METHOD AND FUZZY PREFERENCE RELATIONS

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Abstract. Hesitant fuzzy sets (HFSs) are widely applied in pattern recognition, classification, clustering, and multiple attribute decision making. In order to get more accurate decision results, the order relation of HFSs is particularly important. In this paper, some defects of the existing order relations for HFSs are discussed. In order to solve these problems, by employing a distance measure and the TOPSIS method, we propose a new order relation extraction method based on a new additive consistency fuzzy preference relation for hesitant fuzzy elements (HFEs). Then, the proposed additive consistency fuzzy preference relation is applied to integrate group decision information. In multi-attribute group decision making (MAGDM), it is particularly important to ensure the consensus of the decision-makers (DMs), and the consistency of the decision process is the precondition for DMs to reach consensus. The proposed method can maintain the consistency of the decision process for MAGDM under hesitant fuzzy environments, so as to get the consensus of DMs, besides, it can overcome the limitations of the existing order relations for HFSs. At the end of this paper, a numerical example is used to illustrate the effectiveness and feasibility of the new approach, and some comparative analyses are given. The obtained results confirm the theoretical and numerical analyses and emphasize the advantages, which can ensure the consistency of the whole decision process and avoid the original decision information change and loss of the proposed method, so as to be more in line with the actual situation.

Keyword: multi-attribute group decision making, hesitant fuzzy sets, TOPSIS, distance measure, fuzzy preference relation, additive consistency.

JEL Classification: C02, C44, C63, D71, D81.

Introduction

Many studies on the fuzzy set theory have been conducted (Kacprzyk & Orlovski, 1987; Turksen, 1986) and have achieved great success (Roy & Maji, 2007; Deschrijver & Kerre, 2003; Erceg, 1979), since the fuzzy set theory was advanced by Zadeh (1965). With the

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continuous improvement of the level of human knowledge, the theory of fuzzy sets has been extended by scholars. Therefore, many other forms of fuzzy sets were developed, such as interval fuzzy sets (Chen, Xu, & Xia, 2013; Turksen, 1986), intuitionistic fuzzy sets (Atanassov & Rangasamy, 1986), interval-valued intuitionistic fuzzy sets (IVIFS) (Atanassov, 1989), 2-type fuzzy sets (Yager, 1986), N-type fuzzy sets, fuzzy multisets (Xu & Xia, 2011), etc. In reality, when making decisions, the DMs are usually indecisive and tend to hesitate in the face of several possible choices, so it is difficult to reach a final agreement. For example, two DMs discuss the membership of one element to one set, one is to allocate 0.4, and the other wants 0.7. To deal with such cases, Torra (2010) put forward the definition of hesitant fuzzy sets (HFSs), developed the basic algorithm and discussed the application of HFSs in decision making (Torra & Narukawa, 2009; Turksen, 1986). In multi-attribute group decision making (MAGDM), when several DMs are different in knowledge and experience, it is difficult to reach a consensus. HFSs can allow DMs to have multiple membership degrees for alternatives.

Since then, many scholars have paid attention to HFSs and additional studies have appeared in the literature (Chen, Xu, & Xia, 2013a; Farhadinia, 2013; Kessler et al., 2011; Qian, Wang, & Feng, 2013; Rodriguez, Martinez, & Herrera, 2012; Wu, Jin, & Xu, 2018; Xu & Zhang, 2013; Xu & Xia, 2012; Zhang, Xie, & Wu, 2015a), such as the related aggregation operators, the order relationship of HFSs and distance and similarity measures (Xu & Xia, 2011a) of the HFSs. It has also been diffusely and successfully used to a lot of MAGDM problems. In general, hesitant fuzzy MAGDM problems have two phases: aggregation and exploitation. In the aggregation phase, the information is grouped to reflect a collective value for each alternative or criterion, while in the exploitation phase, the best alternative is selected as a solution to the decision problem by using the collective values obtained in the previous phase (Bedregal, Reiser, Bustince, & Lopez-Molina, 2014).

There are two main ways to integrate hesitant fuzzy decision information. Firstly, extending the existing multi-attribute aggregation operators. For examples, Xia and Xu (2011) presented the hesitant fuzzy weighted averaging operator and the hesitant fuzzy weighted geometric operator and promoted the two operators, so as to obtain the generalized hesitant fuzzy weighted averaging operator, the generalized hesitant fuzzy weighted average geometric operator, the hesitant fuzzy hybrid averaging operator and the hesitant fuzzy hybrid geometric operator. Wei (2012) proposed the hesitant fuzzy prioritized weighted average and hesitant fuzzy prioritized weighted geometric integrated operator in which he considered the priority between the different attributes. Qin, Liu, and Pedrycz (2016) developed hesitant fuzzy aggregation operators based on Frank operations. Zhu, C., Zhu, L., and Zhang (2016) discussed multi-attribute decision making problems with linguistic hesitant fuzzy information and proposed a series of linguistic hesitant fuzzy power aggregation operators. Farhadinia (2016) introduced a novel HFS ranking technique in view of lexicographical ordering. Zhang, Wang, Tian, & Li (2014) developed a series of induced generalized aggregation operators for hesitant fuzzy or interval-valued hesitant fuzzy information. Tang, Fu, Xu, & Yang (2017) analyzed fuzzy Hamacher aggregation functions for uncertain multi-attribute decision making. Yu, Wu, and Zhou (2011) proposed a new hesitant fuzzy aggregation operator based on the Choquet integral which included the importance of the elements, their ordered positions and a fuzzy measure. Zhu, Xu, and Xia (2012) based on the Bonferroni average (BM), which have a great impact on multi-criteria decision making, they proposed the definition of the hesitant fuzzy geometric Bonferroni mean (HFGBM) and the hesitant fuzzy Choquet geometric Bonferroni mean (HFCGBM). In addition, the weighted hesitant fuzzy geometric Bonferroni mean (WHFGBM) and the weighted hesitant fuzzy Choquet geometric Bonferroni mean (WHFCGBM) are also proposed.

Another application involves using the distance or similarity measure to aggregate decision information. Xu and Xia (2011a) presented a distance measurement method of fuzzy sets according to distance axioms, advanced the corresponding similarity measurement method, and studied the relationship and the character of the HFSs distance formula and similar metrics. Li, D., Zeng, and Li, J. (2015) presented some new distance measures between HFSs and a novel generalized hesitant fuzzy synergetic weighted distance measure. Peng, Gao, C. Y., and Gao, Z. F. (2013) presented a generalized hesitant fuzzy synergetic weighted distance measure. Farhadinia (2014) introduced a series of distance measures and similarity measures for the higher order HFSs. Zhang, Li, Chen, Sun, and Attey (2017) proposed a new hesitant distance set in which the distances between different HFSs can be characterized by a series of different values. Xu and Zhang (2013) developed a novel approach based on TOPSIS and the maximizing deviation method for solving MAGDM problems. Yue (2016) proposed a geometric method based on TOPSIS that could be used for sorting interval-valued intuitionistic fuzzy numbers. Jin, Ni, Chen, Li, and Zhou (2016) further constructed several information measure formulas for interval-valued hesitant fuzzy elements (IVHFEs) on the basis of the continuous ordered weighted averaging operator to deal with MAGDM whose attribute values are IVHFs forms. Dong, Chen, and Herrera (2015) proposed a novel distance-based consensus measure for hesitant linguistic group decision making. Zhang and Xu (2015) believed that distance and similarity measures were inaccurate in some cases. Therefore, a new concept of fuzzy set hesitant fuzzy index was proposed to measure the hesitation degree among the possible values in each hesitant fuzzy element of the HFS. According to their indecisive and index considerations, a new method for measuring the distance between HFSs was proposed. There were still a lot of works about distance and similarity measures (Singh, 2015; Su, Xu, Liu, & Liu, 2015; Zeng, Li, & Yin, 2016; Papakostas, Hatzimichailidis, & Kaburlasos, 2013; Hesamian & Shams, 2015; Williams & Steele, 2002) for HFSs.

In addition to the two main methods of dealing with hesitant fuzzy information, fuzzy preference relations are also research hotspots. Zhang, Wang, and Tian (2015b) proposed two group decision making support models with hesitant fuzzy preference relations (HFPRs) based on Tanino's additive consistency concept and the β-normalization based method and then used them to MAGDM. Wu and Xu (2016) studied separate consistency and consensus processes for hesitant fuzzy linguistic preference relations. Zhou, Xu, and Chen (2015) proposed a hesitant-intuitionistic fuzzy number and simultaneously proposed a proposal for the hesitant-intuitionistic fuzzy preference relation and its complementary form. Xu, Chen, Herrera, and Wang (2016) suggested the incomplete HFPR, which is a new type of fuzzy preference structure, and defined the additive consistency of incomplete HFPRs and the concept of consistency of multiple incomplete HFPRs, and they were used to solve the problem of group decision making. The above-mentioned models are based on consistency, but they cannot deal with inconsistent situations. To resolve these limitations, Meng and An (2017) proposed 0-1 mixed programming models to judge the multiplicative consistency of HFPRs. Its main feature is that the concept does not increase the value of the HFEs, nor does it ignore any information from DMs. To obtain more consistent HFPRs, Xu, Cabrerizo, and Herrera-Viedma (2017) gave a consensus model of HFPRs and revised the definition of a HFPR. Then they proposed a revised additive consistency to estimate the unknown values, which are added to the short HFSs, which normalized the HFPRs, and based on this, they developed a model for MAGDM problems.

However, whether the aggregate operators, distance and similarity measures and fuzzy preference relations, when applied to MAGDM problems, there will be the following problems:

(1) When using aggregation operators or score functions to rank HFSs, the results are not consistent with the actual situation (Lan, Jin, Zheng, & Hu, 2017);

(2) When some MAGDM methods of using distance and similarity measures, HFEs need to be extended to the same length (Li et al., 2015; Zeng et al., 2016; Zhang & Xu, 2015), these methods have changed the original decision information, which may lead to inaccurate decision results;

(3) When using fuzzy preference relations for MAGDM problems, in the decision process, it is often difficult to ensure consistency, and thus the algorithm is modified (Khalid & Beg, 2016; Zhang et al., 2015b), which resulting in the changes in decision information, which may lead to inconsistent results, thus may make decision failure.

To overcome the flaws, the rest of this paper is arranged as follows: In Section 1, some basic concepts of HFSs and fuzzy preference relations are reviewed. Examples are used to analyze the rationality of the existing order relations of HFSs in Section 2. Section 3 builds a new fuzzy preference relation for HFSs. Section 4 develops an approach to hesitant fuzzy MAGDM based on FPRs. In Section 5, an example is given to illustrate the rationality and applicability of the new method. Finally, the conclusions are drawn in the end of the paper.

1. Preliminaries

1.1. HFS and its order relations

In this section, the definition of HFS and its order relations are introduced. Torra (2010) first proposed the definition of HFS, which takes the following form:

Definition 1. (Torra, 2010) Set a discourse X, then a HFS on X is in terms of a function that when applied to X returns a subset of [0,1].

To make calculations easier, Xia and Xu (2011) further refined the previous definition by redefining HFS through a mathematical expression, the form is as follows:

$$\mathrm{E}=\left(x,h_E\left(x\right)|x\in X\right). \tag{1}$$

where $h_E\left(x\right)$ is a membership function which returns all the possible memberships of $x\in X$ to the set E, and Xia and Xu (2011) named $h\left(x\right)=h_E\left(x\right)$ a hesitant fuzzy element (HFE).

Example 1. Let $X=\{x_1,x_2,x_3\}$ be the discourse set, and $h_E(x_1)=\{0.3,0.5,0.8\}$, $h_E(x_2)=\{0.4,0.6\}$ and $h_E(x_3)=\{0.2,0.3,0.6\}$ are three HFEs of $x_i(i=1,2,3)$ to a set E. We express E to be a HFS, i.e,

$$\mathrm{E}=\{x_1,\{0.3,0.5,0.8\},\ x_2,\{0.4,0.6\},\ x_3,\{0.2,0.3,0.6\}\}.$$

In MAGDM problems under hesitant fuzzy environment, the order relations of HFSs tend to be involved. Thence, it is especially important to rank HFSs. In the following, we review the existing order relations of HFSs.

Definition 2. (Torra, Xu, & Herrera, 2014) Given two HFSs H_1 and H_2 on X of the same cardinality, we define that $H_1\geq H_2$ if $H_1(x)\geq H_2(x)$ for all x. Note that $H_1(x)$ and $H_2(x)$ are HFEs. Here, $H_1(x)\geq H_2(x)$ for HFEs $H_1(x)$ and $H_2(x)$ if $H_1(x)^{\sigma(j)}\geq H_2(x)^{\sigma(j)}$ for all $\mathrm{j}=\{1,\cdots,|H_1|\}$, where $H(x)^{\sigma(j)}$ is the jth element in $H(x)$ when they are ordered in decreasing order.

Definition 3. (Torra et al., 2014) Let $\varnothing$ be a function on HFSs such that the cardinality of $\varnothing$ is the same for all HFSs. We then say that $\varnothing$ is monotonic when $\varnothing(\mathrm{E})\geq\varnothing(\mathrm{E}')$ for all $\mathrm{E}=\{H_1,\cdots,H_n\}$ and $\mathrm{E}'=\{H_1',\cdots,H_n'\}$ such that $H_i\geq H_i'$ for all $i=\{1,\cdots,n\}$.

Definition 4. (Torra et al., 2014) Let $\mathrm{E}=\{H_1,\cdots,H_n\}$ be a set of n HFSs and Θ a function, $\Theta:[0,1]^n\to[0,1]$; we then export Θ on fuzzy sets to HFSs defining:

$$\Theta_E=\cup_{\gamma\in H_1(x)\times\cdots\times H_n(x)}\{\Theta(\gamma)\}. \tag{2}$$

Property 1. (Torra et al., 2014) Let $\mathrm{E}=\{H_1,\cdots,H_n\}$ and $\mathrm{E}'=\{H_1',\cdots,H_n'\}$ such that $H_i'\geq H_i$ for all $i=\{1,\cdots,n\}$. Then, if Θ is a monotonic function, Θ_E is monotonic.

In practical problems, the number of elements in several HFSs may not all be the same, and Xu and Xia (2011) have given an expansion rule by adding the minimum, maximum or any of them until they have the same length. The choice of this value depends primarily on the risk appetite of the DMs. Optimists expect the desired result to be achieved and may increase the maximum, while the pessimists are expected to produce negative results and possibly increase the minimum. However, when the expansion rule is applied to the calculation, it still changes the information of the original decision matrix, thus leads to inaccuracy in decision making. Therefore, Xia and Xu (2011) defined a score function to describe the order relation of HFSs. The definition is as follows:

Definition 5. (Xia & Xu, 2011) Let h be a HFE. The score function of h is defined by:

$$\mathrm{s}(h)=\frac{1}{l(h)}\sum_{\gamma\in h}\gamma, \tag{3}$$

where $l(h)$ is the number of elements in h.

Let h_1 and h_2 be two HFEs; then,
if $\mathrm{s}(h_1)>\mathrm{s}(h_2)$, then $h_1>h_2$;
if $\mathrm{s}(h_1)=\mathrm{s}(h_2)$, then $h_1=h_2$.

Farhadinia (2013) extended the score function from a HFE to a HFS as follows:

Definition 6. (Farhadinia, 2013) Let E_1 and E_2 be two HFSs on $X=\{x_1,x_2,\cdots,x_n\}$.

$E_1 \succeq E_2$ if and only if $\mathrm{Score}(E_1) \geq \mathrm{Score}(E_2)$, where $\mathrm{Score}(E_1) = \frac{1}{n}\sum_{i=1}^{n} s\left(h_{E_1}(x_i)\right)$ and $\mathrm{Score}(E_2) = \frac{1}{n}\sum_{i=1}^{n} s\left(h_{E_2}(x_i)\right)$.

To get the same purpose, besides the score function, many other aggregation operators (Torra & Narukawa, 2007; Wei, 2012; Xia & Xu, 2011) have been also used to solve this problem.

1.2. Fuzzy preference relation

The fuzzy preference relation is widely used in the decision making problems. Tanino (1984) gave its definition in 1984.

Definition 7. (Tanino, 1984) A fuzzy preference relation R on a finite set of alternatives $\mathrm{A} = \{A_1, A_2, \cdots, A_k\}$ is a fuzzy relation on the product set A×A with membership function $u_R : \mathrm{A} \times \mathrm{A} \to [0,1]$, $u_R(A_i, A_j) = r_{ij}$, and r_{ij} satisfies the following conditions:

$$r_{ij} \geq 0, r_{ij} + r_{ji} = 1, i, j = 1, 2, \cdots, k. \tag{4}$$

Usually, a preference relation is represented by a $k \times k$ matrix $R = (r_{ij})_{k\times k}$, in which r_{ij} represents the preference degree of A_i over A_j. Specifically, $0 \leq r_{ij} < 0.5$ denotes a definite preference of A_j over A_i. In particular, $r_{ij} = 0$ denotes that A_j is totally preferred to A_i, and $r_{ij} = 0.5$ denotes that A_j and A_i are equally important.

Definition 8. (Tanino, 1984) A fuzzy preference relation $R = (r_{ij})_{k\times k}$ is called an additive consistency preference relation if and only if:

$$r_{ij} = r_{it} + r_{tj} - 0.5, i, j, t = 1, 2, \cdots, k. \tag{5}$$

Definition 9. (Huo, Lan, & Wang, 2011) A fuzzy preference relation $R = (r_{ij})_{k\times k}$ is an additive consistency fuzzy preference relation if and only if there exists a positive normalized weight vector of k dimensions $(w_1, w_2, \cdots, w_k)$, such that

$$r_{ij} = \ln w_i - \ln w_j + 0.5, \ i, j = 1, 2, \cdots, k. \tag{6}$$

where $w_i = \dfrac{\exp\left(\frac{1}{k}\sum_{j=1}^{k} r_{ij}\right)}{\sum_{t=1}^{k}\exp\left(\frac{1}{k}\sum_{j=1}^{k} r_{tj}\right)}$, $w_i\,(i = 1, 2, \cdots, k)$ are called the priority weights.

Xu (2015) studied the properties of additive consistency fuzzy preference relation in detail, and gave the following properties and theorems.

Property 2. (Xu, 2015) Let $R = (r_{ij})_{k\times k}$ be an additive consistency fuzzy preference relation. Then, $w_i > w_j \Leftrightarrow r_{ij} > 0.5$, $w_i = w_j \Leftrightarrow r_{ij} = 0.5$.

Definition 10. (Xu, 2015) Let $R = (r_{ij})_{k\times k}$, and b is a nonnegative real number. Then, bR is defined by:

$$bR = (br_{ij})_{k\times k}. \tag{7}$$

Theorem 1. (Xu, 2015) Let $R_1=\left(r_{ij}^{(1)}\right)_{k\times k}$, $R_2=\left(r_{ij}^{(2)}\right)_{k\times k}$, $\cdots$, $R_n=\left(r_{ij}^{(n)}\right)_{k\times k}$ be a set of additive consistency fuzzy preference relations, $w_l\in[0,1], l=1,2,\cdots,n$ and $\sum_{l=1}^{n} w_l=1$. Then, $R=\sum_{l=1}^{n} w_l R_l=\left(r_{ij}\right)_{k\times k}$ is an additive consistency fuzzy preference relation.

2. Analysis on the existing order relations of HFSs

As mentioned in the introduction, many scholars have suggested ranking methods for HFSs. However, there are some defects when using the existing order relations to rank HFSs. In this section, we will analyze the limitations of the order relations of HFSs through examples.

Example 2 (Adapted from Ref. Lan et al. (2017)) Jack and Tom play a game that has three turns. Jack has three cards: a 9 of spades, a 6 of spades and a 3 of spades; and Tom has three cards: an 8 of spades, a 5 of spades and a 2 of spades. Using the following game rules: (1) Each person can only select one of their own cards to play in each turn; (2) The card with the higher points wins in each turn; (3) The person who wins two turns is the final winner. Although Jack's cards have a point' advantage, it is not certain that Jack can win the game, we can only say that his probability of winning is 0.6667.

The above example can be transformed into a comparison of two HFSs as follows. Jack corresponds to the HFE $h_1=\{0.9,0.6,0.3\}$, and Tom corresponds to the HFE $h_2=\{0.8,0.5,0.2\}$. In line with definition 2 or the score function, we can conclude that $h_1\succ h_2$. Obviously, the result is not consistent with reality. We are not sure that Jack can beat Tom, and Tom has a chance to win. Using the proposed order relations or score function to judge the hesitant preference relation between the HFSs always tends to ignore this possibility and ignores some information. Thus, the results obtained are sometimes inconsistent with reality.

In fact, there are real elements $(u,v)\in h_1\times h_2$ satisfying $u<v$, for example, (0.6,0.8). Example 2 is a numerical example to show that by using the score function, there is a defect in ranking HFSs.

Example 3. Assume that $X=\{x_1,x_2\}$. $E_1=\left(x_1,\{0.5,0.4,0.3\},x_2,\{0.9,0.8,0.7,0.1\}\right)$, $E_2=\left(x_1,\{0.5,0.3\},x_2,\{0.6,0.5,0.3\}\right)$ and $E_3=\left(x_1,\{0.8,0.7,0.4,0.3\},x_2,\{0.7,0.4,0.2\}\right)$ are three HFSs on *X*.

In line with Definition 6, we find that $\text{Score}(E_1)=0.5125$, $\text{Score}(E_2)=0.4333$, $\text{Score}(E_3)=0.4917$ and then $E_2\prec E_3\prec E_1$.

However, elements $(u_1,v_1)\in h_{E1}(x_1)\times h_{E3}(x_1)$ exist that satisfy $u_1<v_1$ and $(u_2,v_2)\in h_{E1}(x_2)\times h_{E3}(x_2)$, satisfying $u_2<v_2$. Therefore, from the perspective of decision making, we still cannot be sure that $E_1\succ E_3$, so we can only say that E_1 has an advantage over E_3.

Many other aggregation operators (Xia & Xu, 2011; Wei, 2012; Tang et al., 2017) have been used to integrate hesitant fuzzy information, and the score function is used to deal with the order relations of HFSs. However, by using aggregation operators and the score function, situation similar to the above examples cannot be avoided. This shows that using aggregation operators and the score function to deal with the ordering relations of HFSs is inappropriate.

3. A new fuzzy preference relation for HFEs

When comparing the differences between two objects, many methods have been proposed, of which distance measure is often used. To rank HFSs, in this section we will solve the order relation of the HFEs based on the TOPSIS method.

Xu and Xia (2011) came up with the hesitant normalized Hamming distance, Euclidean distance and generalized hesitant normalized distance (Xu and Xia, 2011a), and their definitions are as follows:

Definition 11. (Xu & Xia, 2011, 2011a) Let M and N be two HFSs on $X=\{x_1,x_2,\cdots,x_n\}$.

Then,

$$d_h(M,N)=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|h_M^{\sigma(j)}(x_i)-h_N^{\sigma(j)}(x_i)\right|\right]; \quad (8)$$

$$d_e(M,N)=\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|h_M^{\sigma(j)}(x_i)-h_N^{\sigma(j)}(x_i)\right|^2\right)\right]^{\frac{1}{2}}; \quad (9)$$

$$d_g(M,N)=\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|h_M^{\sigma(j)}(x_i)-h_N^{\sigma(j)}(x_i)\right|^\lambda\right)\right]^{\frac{1}{\lambda}} \quad (10)$$

are called the hesitant normalized Hamming distance, hesitant normalized Euclidean distance and the generalized hesitant normalized distance, respectively, in which $\lambda\geq 1$, $h_M^{\sigma(j)}(x_i)$ and $h_N^{\sigma(j)}(x_i)$ are the jth values in $h_M(x_i)$ and $h_N(x_i)$, respectively, and $l_{x_i}=\max\{l(h_M(x_i)),l(h_N(x_i))\}$.

Hwang and Yoon (1981) introduced the TOPSIS method, and its main idea is to choose the most ideal alternative. Deviation variables are introduced into the algorithm, that is, the shortest distance to the ideal solution and the farthest distance to the negative ideal solution, avoiding the loss of decision information and reduce the computational complexity caused by the traditional distance measures.

In the traditional TOPSIS method, based on the length of the HFEs, the positive and negative ideal points of the corresponding dimensions are obtained, therefore, when the length of the HFEs are different, it is necessary to extend the HFEs to the same length. This section has made the following change based on the thought of the traditional TOPSIS, the dimension of positive and negative ideal points varies according to the length of HFEs, and their values are still 1 and 0, respectively, which avoids the extension of the HFEs.

Now, we consider the use of the TOPSIS method to solve the HFEs ordering problem.

If $h_j\,(j=1,2,\cdots,n)$ are a set of HFEs, then we have $h_N=\{0\}$, $h_P=\{1\}$, which are the negative ideal solution and positive ideal solution of the HFEs, respectively.

$$d_{h_p}^{-}(\lambda)=d_g(h_p,h_N)=\left(\frac{1}{l(h_p)}\sum_{\gamma\in h_p}\gamma^\lambda\right)^{\frac{1}{\lambda}},\ (p=1,2,\cdots,n) \quad (11)$$

represents the distance between h_p and the negative ideal solution, and

$$d_{h_p}^{+}(\lambda)=d_g\left(h_p,h_P\right)=\left(\frac{1}{l\left(h_p\right)}\sum_{\gamma\in h_p}(1-\gamma)^{\lambda}\right)^{\frac{1}{\lambda}},\ \left(p=1,2,\cdots,n\right) \tag{12}$$

represents the distance between h_p and the positive ideal solution, in which $\lambda\geq 1$ and the number of elements in h_p is represented by $l\left(h_p\right)$.

The relative closeness coefficient of the HFS h_p with respect to the positive ideal solution h_p is defined as the following formula:

$$f_{h_p}(\lambda)=\frac{d_{h_p}^{-}(\lambda)}{d_{h_p}^{-}(\lambda)+d_{h_p}^{+}(\lambda)}, \tag{13}$$

where $0\leq f_{h_p}(\lambda)\leq 1$.

Then we employ the closeness coefficient $f_{h_p}(\lambda)$ to rank the orders of $h_j\left(j=1,2,\cdots,n\right)$. The ranking order rules are as follows:

Definition 12. Let $h_j\left(j=1,2,\cdots,n\right)$ be a collection of HFEs.

(1) If $f_{h_p}(\lambda)>f_{h_q}(\lambda)$, then $h_p\succ h_q$, $p,q=1,2,\cdots,n$.

(2) If $f_{h_p}(\lambda)=f_{h_q}(\lambda)$, then $h_p\simeq h_q$, $p,q=1,2,\cdots,n$.

where $f_{h_p}(\lambda)=\frac{d_{h_p}^{-}(\lambda)}{d_{h_p}^{-}(\lambda)+d_{h_p}^{+}(\lambda)}$, $d_{h_p}^{-}(\lambda)=\left(\frac{1}{l\left(h_p\right)}\sum_{\gamma\in h_p}\gamma^{\lambda}\right)^{\frac{1}{\lambda}}$ and

$$d_{h_p}^{+}(\lambda)=\left(\frac{1}{l\left(h_p\right)}\sum_{\gamma\in h_p}(1-\gamma)^{\lambda}\right)^{\frac{1}{\lambda}},\ \left(p=1,2,\cdots,n\right),$$

when $\lambda=1$, $d_{h_p}^{-}(1)=s\left(h_p\right)$, $d_{h_p}^{+}(1)=1-s\left(h_p\right)$ and $f_{h_p}(1)=s\left(h_p\right)$.

Definition 12 shows that the score function is only a special case of the above ranking order rules.

Example 4. Assume that: $h_1=\{0.5,0.4,0.3\}$, $h_2=\{0.9,0.8,0.7,0.1\}$, $h_3=\{0.8,0.7,0.4,0.3\}$.

Take λ = 1, then we can calculate:

$d_{h_1}^{-}(1)=s\left(h_1\right)=0.4$, $d_{h_2}^{-}(1)=s\left(h_2\right)=0.625$, $d_{h_3}^{-}(1)=s\left(h_3\right)=0.55$;

$d_{h_1}^{+}(1)=1-s\left(h_1\right)=0.6, d_{h_2}^{+}(1)=1-s\left(h_2\right)=0.375$;

$d_{h_3}^{+}(1)=1-s\left(h_3\right)=0.45$;

$f_{h_1}(1)=s\left(h_1\right)=0.4$, $f_{h_2}(1)=s\left(h_2\right)=0.625$, $f_{h_3}(1)=s\left(h_3\right)=0.55$.

Since $f_{h_2}(1)>f_{h_3}(1)>f_{h_1}(1)$, we have $h_2\succ h_3\succ h_1$.

Take λ = 3, then we can calculate:

$d_{h_1}^{-}(3)=0.4160, d_{h_2}^{-}(3)=0.7345, d_{h_3}^{-}(3)=0.6184$;

$d_{h_1}^{+}(3)=0.6109, d_{h_2}^{+}(3)=0.5761, d_{h_3}^{+}(3)=0.5296$;

$f_{h_1}(3)=0.4051$, $f_{h_2}(3)=0.5604$, $f_{h_3}(3)=0.5387$.

Since $f_{h_2}(3)>f_{h_3}(3)>f_{h_1}(3)$, we have $h_2\succ h_3\succ h_1$.

In fact, as described in section 2, h_1, h_2, h_3 are not strictly dominant; therefore, merely ranking HFEs by the proposed closeness coefficient or score function is inappropriate. To overcome the limitation, based on the relative closeness coefficient, a new fuzzy preference relation for HFEs is built:

Definition 13. Suppose that $h_j\left(j=1,2,\cdots,n\right)$ is a collection of HFEs. The mapping $R:\left(h_i,h_j\right)\to\left[0,1\right]$ is defined by

$$R\left(h_i,h_j\right)=\frac{g\left(f_{h_i}\left(\lambda\right)\right)-g\left(f_{h_j}\left(\lambda\right)\right)}{2}+0.5,\left(i,j=1,2,\cdots,n\right). \tag{14}$$

where $g:\left[0,1\right]\to\left[0,1\right]$ is strictly a monotone increasing function, $f_{h_p}\left(\lambda\right)=\frac{d_{h_p}^{-}\left(\lambda\right)}{d_{h_p}^{-}\left(\lambda\right)+d_{h_p}^{+}\left(\lambda\right)}$, $d_{h_p}^{-}\left(\lambda\right)=\left(\frac{1}{l\left(h_p\right)}\sum_{\gamma\in h_p}\gamma^{\lambda}\right)^{\frac{1}{\lambda}}$, $d_{h_p}^{+}\left(\lambda\right)=\left(\frac{1}{l\left(h_p\right)}\sum_{\gamma\in h_p}\left(1-\gamma\right)^{\lambda}\right)^{\frac{1}{\lambda}}$, $\left(p=1,2,\cdots,n\right)$.

Denote $r_{ij}=R\left(h_i,h_j\right)$, and r_{ij} represents the preference degree of h_i over h_j. Obviously, if $r_{ij}\geq 0.5\Leftrightarrow g\left(f_{h_i}\left(\lambda\right)\right)\geq g\left(f_{h_j}\left(\lambda\right)\right)\Leftrightarrow f_{h_i}\left(\lambda\right)\geq f_{h_j}\left(\lambda\right)\Leftrightarrow h_i\succeq h_j$.

Property 3. Suppose that $r_{ij}=\frac{g\left(f_{h_i}\left(\lambda\right)\right)-g\left(f_{h_j}\left(\lambda\right)\right)}{2}+0.5,\ i,j=1,2,\cdots,n$; then r_{ij} has the following properties:

(1) $0\leq r_{ij}\leq 1, i,j=1,2,\cdots,n$,

(2) $r_{ij}+r_{ji}=1, i,j=1,2,\cdots,n$,

(3) $r_{ij}=r_{it}+r_{tj}-0.5, i,j,t=1,2,\cdots,n.$

Proof

(1) Since $-0.5\leq\frac{g\left(f_{h_i}\left(\lambda\right)\right)-g\left(f_{h_j}\left(\lambda\right)\right)}{2}\leq 0.5$, we have $0\leq r_{ij}\leq 1, i,j\in I$.

(2) $r_{ij}+r_{ji}=\frac{g\left(f_{h_i}\left(\lambda\right)\right)-g\left(f_{h_j}\left(\lambda\right)\right)}{2}+0.5+\frac{g\left(f_{h_j}\left(\lambda\right)\right)-g\left(f_{h_i}\left(\lambda\right)\right)}{2}+0.5=1$.

(3) $r_{it}+r_{tj}-0.5=\frac{g\left(f_{h_i}\left(\lambda\right)\right)-g\left(f_{h_t}\left(\lambda\right)\right)}{2}+0.5+\frac{g\left(f_{h_t}\left(\lambda\right)\right)-g\left(f_{h_j}\left(\lambda\right)\right)}{2}+0.5-0.5=$

$$\frac{g\left(f_{h_i}\left(\lambda\right)\right)-g\left(f_{h_j}\left(\lambda\right)\right)}{2}+0.5=r_{ij},$$

which completes the proof.

Through property 3, we know that $R=\left(r_{ij}\right)_{n\times n}$ is an additive consistency fuzzy preference relation. And according to it, we can use the priority weights $w_i=\frac{\exp\left(\frac{1}{k}\sum_{j=1}^{k}r_{ij}\right)}{\sum_{t=1}^{k}\exp\left(\frac{1}{k}\sum_{j=1}^{k}r_{tj}\right)}$, $\left(i=1,2,\cdots,n\right)$ instead of $f_{h_i}\left(\lambda\right)$ to rank the order relation of $h_1,h_2,\cdots,h_n$.

Example 5. In Example 4, we can set up its additive consistency fuzzy preference relation as follows:

Take $\lambda = 3$, $g(x) = \dfrac{\exp(x)}{3}$, and we have:

$$R = \begin{pmatrix} 0.5000 & 0.4580 & 0.4643 \\ 0.5420 & 0.5000 & 0.5063 \\ 0.5357 & 0.4937 & 0.5000 \end{pmatrix}.$$

Based on the relationship between elements and the priority weights, we have $w_1 = 0.3248$, $w_2 = 0.3387$, $w_3 = 0.3366$. So, $h_2 \succeq_{(0.5063)} h_3 \succeq_{(0.5420)} h_1$. In fact, as discussed in this section, the preference of h_2 over h_3 is relative, not absolute. The preference degree of h_2 over h_3 is 0.5063, and conversely, the preference degree of h_3 over h_2 is 0.4937.

Theoretically speaking, the method proposed in this section has the following differences and advantages compared with the use of TOPSIS method, aggregation operators, distance and similarity measures, and fuzzy preference relations to rank HFSs.

(1) According to the introduction of Feng, X., Zuo, Wang, and Feng, L. (2014), traditional TOPSIS method as well as a variety of modified TOPSIS method, on the theoretical flow of the algorithm, in the traditional and modified TOPSIS methods, after the relative closeness is obtained, then the HFSs are ranked directly according to the size of the relative closeness, while the new fuzzy preference relation put forward in this part is based on TOPSIS, using the relative closeness to construct an additive consistency fuzzy preference relation, then the priority weights are extracted and the HFSs are ranked by them, which can ensure the consistency of the algorithm process, so as to ensure the consensus of DMs.

(2) Using the aggregation operators to rank HFSs, they are usually used in conjunction with score function, it may not always ensure the consistency in the decision-making process, which may not always ensure the consensus of DMs, while the approach provided in this section ensures consistency throughout the decision process, and the theoretical proof is given.

(3) Using the distance and similarity measures to rank HFSs, it is necessary to expand the HFEs to the same length, both optimistic and pessimistic approaches have changed the original decision-making information, however, the new method proposed in this section doesn't need to extend the HFEs.

(4) Using the fuzzy preference relations, it is often difficult to get consensus for DMs, and inconsistent may appear in the decision process, so the decision process need to be revised and adjusted, the process of revision has actually changed the decision information of the original DMs, thus may lead to the failure of decision making. While the proposed method in this section can ensure the consistency of the whole decision process, and don't need to be revised.

4. A new approach to MAGDM

In recent years, more and more researchers have studied the MAGDM problems because of its good practicability (Ballesteros-Pérez, Campo-Hitschfeld, Mora-Melià, & Domínguez, 2015; Schubert et al., 2015; Pramanik, S., Pramanik, S., & Giri, 2016). We arrange this section by advancing a new hesitant fuzzy MAGDM method due to fuzzy preference relations.

We use $A=\{A_1,A_2,\cdots,A_m\}$ to represent dispersed alternatives, $C=\{C_1,C_2,\cdots,C_n\}$ to represent the attribute sets, and finally $E=\{e_1,e_2,\cdots,e_k\}$ to represent the expert sets. For each alternative $A_i \in A$, expert e_t gives a preference value a_{ij}^t with respect to attribute $C_j \in C$. Assume that $w=(w_1,w_2,\cdots,w_k)^T$ is the weight vector of experts, where $w_t \in [0,1]$, $(t=1,2,\cdots,k)$ and $\sum_{t=1}^{k} w_t = 1$. Note that a_{ij}^t represents the assessment information given by expert e_t about attribute C_j of alternative A_i, where a_{ij}^t is a HFE; all of the preference values of the alternatives make up a series of the decision making matrix $D^t=\left(a_{ij}^t\right)_{m\times n}$, $(t=1,2,\cdots,k)$ (Lan et al., 2017).

4.1. The method to determine attribute weights

Generally, in MAGDM problems, if the preference value difference of all alternatives under attribute $\overline{C}$ is smaller, then attribute $\overline{C}$ has less influences on the decision so as to the rank of the alternatives. Conversely, if the preference value difference of all alternatives under attribute $\overline{C}$ is larger, then attribute $\overline{C}$ has more of an effect on the decision and ranking of the alternatives (Wang, 2012). Therefore, if the preference value difference of all alternatives under attribute $\overline{C}$ is smaller, then the weight value of attribute $\overline{C}$ is small. If the preference value difference of all alternatives under attribute $\overline{C}$ is larger, then the weight value of attribute $\overline{C}$ is larger.

Let $D^t=\left(a_{ij}^t\right)_{m\times n}$ be a hesitant fuzzy decision making matrix, where a_{ij}^t represents the assessment information given by expert e_t about attribute C_j of alternative A_i. Attribute C_j corresponds to HFEs a_{ij}^t, $(i=1,2,\cdots,m)$. Suppose that $a_{ij}^t=\left\{\gamma_{iq}^{t(j)} \mid q=1,2,\cdots,l\left(a_{ij}^t\right)\right\}$, $\alpha_{ip}^{t(j)}=\frac{1}{l\left(a_{ij}^t\right)l\left(a_{pj}^t\right)}\sum_{q=1}^{l\left(a_{ij}^t\right)}\sum_{u=1}^{l\left(a_{pj}^t\right)}\left|\gamma_{iq}^{t(j)}-\gamma_{pu}^{t(j)}\right|$, where $l\left(a_{ij}^t\right)$ is the number of elements in a_{ij}^t. $\alpha_{ip}^{t(j)}$ represents the average deviation between a_{ij}^t and a_{pj}^t, and the total average deviation about attribute C_j is $\beta_j^t=\frac{2}{m(m-1)}\sum_{i<p}\alpha_{ip}^{t(j)}$.

For expert e_t, $(t=1,2,\cdots,k)$, calculate the weight of attribute C_j, and the formula is: $\overline{w}_j^t=\frac{\beta_j^t}{\sum_{p=1}^{n}\beta_p^t}$, $(j=1,2,\cdots,n)$.

4.2. A new method for MAGDM

In order to solve the unknown attribute weight of MAGDM problems, we propose a new method for MAGDM problems, in which the attribute values are given in the form of HFS, and the attribute weight information is unknown.

Step 1: Let $D^t = \left(a_{ij}^t\right)_{m\times n}$, $\left(t = 1,2,\cdots,k\right)$ be a series of hesitant fuzzy decision making matrixes, in which a_{ij}^t represents the assessment information provided by expert e_t about attribute C_j of alternative A_i.

Step 2: For a given $\lambda \ge 1$ and attribute C_j, $\left(j = 1,2,\cdots,n\right)$, calculate $d_{ij}^{t-}\left(\lambda\right) = d_g\left(a_{ij}^t, 0\right)$, $d_{ij}^{t+}\left(\lambda\right) = d_g\left(a_{ij}^t, 1\right)$ and the relative closeness coefficient $f_{ij}^t\left(\lambda\right) = \dfrac{d_{ij}^{t-}\left(\lambda\right)}{d_{ij}^{t-}\left(\lambda\right) + d_{ij}^{t+}\left(\lambda\right)}$, $\left(i = 1,2,\cdots,m; j = 1,2,\cdots,n; t = 1,2,\cdots,k\right)$. Let $F^t = \left(f_{ij}^t\right)_{m\times n}$, $\left(t = 1,2,\cdots,k\right)$ be a series of the coefficient matrixes for each expert.

Step 3: For a given strictly monotone increasing function $g:\left[0,1\right] \to \left[0,1\right]$, construct the fuzzy preference relation $R_j^t = \left(r_{uv}^{j(t)}\right)_{m\times m}$ of attribute C_j, where: $r_{uv}^{j(t)} = \dfrac{g\left(f_{uj}^t\left(\lambda\right)\right) - g\left(f_{vj}^t\left(\lambda\right)\right)}{2} + 0.5$, $\left(u, v = 1,2,\cdots,m\right)$.

Step 4: According to section 4.1, calculate the weight of attribute C_j, $\overline{w}_j^t = \dfrac{\beta_j^t}{\sum_{p=1}^{n}\beta_p^t}$, $\left(j = 1,2,\cdots,n\right)$.

Step 5: Calculate $R^t = \sum_{j=1}^{n}\overline{w}_j^t R_j^t$, $\left(t = 1,2,\cdots,k\right)$, and then calculate $R = \sum_{t=1}^{k} w_t R^t = \left(r_{ij}\right)_{m\times m}$.

Step 6: According to R, calculate the priority weight of each alternative

$$\omega_i = \frac{\exp\left(\frac{1}{m}\sum_{j=1}^{m} r_{ij}\right)}{\sum_{t=1}^{k}\exp\left(\frac{1}{m}\sum_{j=1}^{m} r_{tj}\right)}.$$

Step 7: Rank alternatives according to ω_i.

Step 8: End.

Remark 1. According to Theorem 1, for expert e_t about attribute C_j, of alternative A_i, $R_j^t\left(j = 1,2,\cdots,n; t = 1,2,\cdots,k\right)$ are additive consistency FPRs; thus, $R^t = \sum_{j=1}^{n}\overline{w}_j^t R_j^t$, $\left(t = 1,2,\cdots,k\right)$ satisfies additive consistency, so $R = \sum_{t=1}^{k} w_t R^t = \left(r_{ij}\right)_{m\times m}$.

5. Illustrative example and comparison analysis

5.1. Illustrative example

To show the feasibility and validity of the above MAGDM method, in this part, we give a practical example as follows:

Example 6. Let us consider the group decision making problem of evaluating several fund management companies. Suppose that there are four companies (alternatives) $A_i\left(i = 1,2,3,4\right)$ to be assessed from which the best fund management company to invest in financial management is to be chosen. There are four attributes to be considered, C_1: revenue ability, C_2: asset size, C_3: stabilization, and C_4: quality of service. Four economics experts are invited to evaluate the choices. Suppose that $\mathrm{E} = \left\{e_1, e_2, e_3, e_4\right\}$ is the expert set, and suppose that the weight vector of e_1, e_2, e_3, e_4 is $w = \left(0.15, 0.3, 0.2, 0.35\right)$. The assessment values provided by the experts are contained in hesitant fuzzy decision matrices, shown in Tables 1, 2, 3, and 4.

Table 1. Hesitant fuzzy decision matrix of expert e_1

	C_1	C_2	C_3	C_4
A_1	$\{0.5,0.4,0.3\}$	$\{0.9,0.8,0.7,0.1\}$	$\{0.5,0.4,0.2\}$	$\{0.9,0.6,0.5,0.3\}$
A_2	$\{0.5,0.3\}$	$\{0.9,0.7,0.6,0.5,0.2\}$	$\{0.8,0.6,0.5,0.1\}$	$\{0.7,0.4,0.3\}$
A_3	$\{0.8,0.7,0.4,0.3\}$	$\{0.7,0.4,0.2\}$	$\{0.8,0.1\}$	$\{0.9,0.8,0.6\}$
A_4	$\{0.9,0.7,0.6,0.3,0.1\}$	$\{0.8,0.7,0.6,0.4\}$	$\{0.9,0.8,0.7\}$	$\{0.9,0.7,0.6,0.3\}$

Table 2. Hesitant fuzzy decision matrix of expert e_2

	C_1	C_2	C_3	C_4
A_1	$\{0.8,0.6,0.4\}$	$\{0.7,0.5,0.4\}$	$\{0.6,0.4,0.3\}$	$\{0.7,0.6,0.5,0.4\}$
A_2	$\{0.7,0.5,0.4\}$	$\{0.7,0.6\}$	$\{0.6,0.5,0.3\}$	$\{0.7,0.5,0.3\}$
A_3	$\{0.7,0.4,0.3\}$	$\{0.6,0.4,0.3\}$	$\{0.8,0.6,0.1\}$	$\{0.8,0.6\}$
A_4	$\{0.9,0.8\}$	$\{0.5,0.4\}$	$\{0.7,0.6,0.4\}$	$\{0.8,0.7,0.6\}$

Table 3. Hesitant fuzzy decision matrix of expert e_3

	C_1	C_2	C_3	C_4
A_1	$\{0.9,0.6,0.4\}$	$\{0.6,0.5,0.3\}$	$\{0.7,0.4,0.3\}$	$\{0.7,0.6,0.5\}$
A_2	$\{0.8,0.6,0.4\}$	$\{0.6,0.5,0.2\}$	$\{0.8,0.7,0.5,0.4\}$	$\{0.8,0.5,0.4\}$
A_3	$\{0.8,0.7\}$	$\{0.7,0.6,0.4\}$	$\{0.8,0.7,0.5\}$	$\{0.4,0.3,0.1\}$
A_4	$\{0.8,0.7,0.6\}$	$\{0.7,0.5,0.4,0.1\}$	$\{0.7,0.5,0.4\}$	$\{0.8,0.6,0.5,0.3\}$

Table 4. Hesitant fuzzy decision matrix of expert e_4

	C_1	C_2	C_3	C_4
A_1	$\{0.9,0.7,0.6\}$	$\{0.4,0.3,0.1\}$	$\{0.7,0.5,0.3\}$	$\{0.7,0.6,0.4,0.3\}$
A_2	$\{0.8,0.5,0.3\}$	$\{0.6,0.5,0.2\}$	$\{0.6,0.5,0.1\}$	$\{0.6,0.4,0.3\}$
A_3	$\{0.9,0.7,0.5\}$	$\{0.7,0.5,0.2\}$	$\{0.6,0.4\}$	$\{0.5,0.4,0.1\}$
A_4	$\{0.8,0.7,0.6\}$	$\{0.9,0.7,0.6,0.4\}$	$\{0.3,0.2,0.1\}$	$\{0.9,0.7,0.6\}$

Using the above proposed MAGDM method to deal with the problem, the calculation processes are as follows:

Step 1: We obtain a series of hesitant fuzzy decision making matrices as the above Tables 1–4.

Step 2: For expert e_1, according to attribute C_1, take $\lambda = 3$ and calculate:

$$d_{11}^{1-}(3) = 0.4160, d_{11}^{1+}(3) = 0.6109, f_{11}^{1}(3) = 0.4051;$$
$$d_{21}^{1-}(3) = 0.9598, d_{21}^{1+}(3) = 1.0231, f_{21}^{1}(3) = 0.4840;$$
$$d_{31}^{1-}(3) = 1.2081, d_{31}^{1+}(3) = 1.2146, f_{31}^{1}(3) = 0.4986;$$
$$d_{41}^{1-}(3) = 1.4271, d_{41}^{1+}(3) = 1.3872, f_{41}^{1}(3) = 0.5071.$$

For each attribute, the corresponding coefficient is calculated, and let $F^t = \left(f_{ij}^t\right)_{m\times n}$, $(t = 1,2,\cdots,k)$ be a series of the coefficient matrices; then, for experts e_1, e_2, e_3, e_4, the coefficient matrices are obtained, and the results are shown in the Appendix.

Step 3: Take expert e_1 as an example. Take $g(x) = \frac{\exp(x)}{3}$, and then build fuzzy preference relations about attributes C_1, C_2, C_3, C_4 of each alternative $A_i\,(i = 1,2,3,4)$, respectively. The results are shown in the Appendix.

Step 4: For each expert, calculate the weight of each attribute.

$$\overline{w}_1^1 = 0.0419, \overline{w}_2^1 = 0.1138, \overline{w}_3^1 = 0.3252, \overline{w}_4^1 = 0.5191;$$
$$\overline{w}_1^2 = 0.0672, \overline{w}_2^2 = 0.1752, \overline{w}_3^2 = 0.2867, \overline{w}_4^2 = 0.4710;$$
$$\overline{w}_1^3 = 0.0735, \overline{w}_2^3 = 0.1459, \overline{w}_3^3 = 0.2884, \overline{w}_4^3 = 0.4922;$$
$$\overline{w}_1^4 = 0.0538, \overline{w}_2^4 = 0.1292, \overline{w}_3^4 = 0.3167, \overline{w}_4^4 = 0.5003.$$

Step 5: Calculate $R^t = \sum_{j=1}^{n} \overline{w}_j^t R_j^t$, $(t = 1,2,\cdots,k)$. The results are shown in the Appendix. Then, calculate the additive consistency fuzzy preference relation $R = 0.15R_1 + 0.3R_2 + 0.2R_3 + 0.35R_4$:

$$R = \begin{pmatrix} 0.5000 & 0.5026 & 0.5021 & 0.4989 \\ 0.4974 & 0.5000 & 0.4995 & 0.4963 \\ 0.4979 & 0.5005 & 0.5000 & 0.4968 \\ 0.5011 & 0.5037 & 0.5032 & 0.5000 \end{pmatrix}.$$

Step 6: Compute the priority weight of each alternative:

$$\omega_1 = 0.2502, \omega_2 = 0.2496, \omega_3 = 0.2497, \omega_4 = 0.2505.$$

Step 7: The ranking of the alternatives is:

$$A_4 \succeq_{(0.5011)} A_1 \succeq_{(0.5021)} A_3 \succeq_{(0.5005)} A_2.$$

Step 8: End.

We can see that the ranking results obtained from different λ are shown in Table 5.

This example shows that the ranking of the alternatives may change when the parameter λ changes. Additionally, we can find that the priority weight of alternatives A_1 and A_2 becomes larger as the parameter λ increases, while the priority weight of alternatives A_3 and A_4 becomes smaller as the parameter λ increases. In fact, for alternatives A_1, A_2, A_3, A_4, each attribute corresponding to the HFEs is not absolutely dominant; therefore, the results obtained by the algorithm should be relative, rather than absolute. Moreover, the fuzzy preference relation calculated always satisfies the additive consistency. Finally, through the results, the new MAGDM method is more reasonable and conforms to the actual situation.

Table 5. Results obtained by different λ

	A_1	A_2	A_3	A_4	Rankings
	ω_1	ω_2	ω_3	ω_4	
λ = 1	0.2501	0.2494	0.2496	0.2508	$A_4 \succeq_{(0.5026)} A_1 \succeq_{(0.5020)} A_3 \succeq_{(0.5010)} A_2$
λ = 2	0.2501	0.2495	0.2497	0.2507	$A_4 \succeq_{(0.5020)} A_1 \succeq_{(0.5019)} A_3 \succeq_{(0.5007)} A_2$
λ = 5	0.2505	0.2497	0.2496	0.2502	$A_1 \succeq_{(0.5011)} A_4 \succeq_{(0.5019)} A_2 \succeq_{(0.5005)} A_3$
λ = 10	0.2509	0.2499	0.2494	0.2497	$A_1 \succeq_{(0.5039)} A_2 \succeq_{(0.5007)} A_4 \succeq_{(0.5013)} A_3$
λ = 20	0.2512	0.2506	0.2493	0.2494	$A_1 \succeq_{(0.5043)} A_2 \succeq_{(0.5030)} A_4 \succeq_{(0.5005)} A_3$
λ = 40	0.2513	0.2503	0.2492	0.2493	$A_1 \succeq_{(0.5041)} A_2 \succeq_{(0.5040)} A_4 \succeq_{(0.5002)} A_3$
λ = 100	0.2513	0.2503	0.2492	0.2492	$A_1 \succeq_{(0.5039)} A_2 \succeq_{(0.5043)} A_4 \succeq_{(0.5003)} A_3$

5.2. Comparison analysis

As we have analyzed in section 3, the new method we proposed has an advantage over the use of TOPSIS method, aggregation operators, distance and similarity measure and fuzzy preference relation at the end of section 3, the theoretical analysis has been given**,** next we give an numerical comparison analysis:

Xia, Xu, & Chen (2013) put forward some hesitant fuzzy aggregation operators and applied them to group decision making, now we use their algorithm to calculate the example in the section 5.1, and the main results are as follows:

In the algorithm flow, the hesitant normalized Hamming distance is used, the premise is to extend the two HFEs as comparison calculations to the same length,

Step 1. Calculate the weights $v_{ij}^{(k)}\left(k=1,2,3,4\right)$:

$$V^{(1)}=\left(v_{ij}^{(1)}\right)_{4\times 4}=\begin{pmatrix} 0.2321 & 0.2330 & 0.2133 & 0.2467 \\ 0.2320 & 0.2459 & 0.2524 & 0.2560 \\ 0.2571 & 0.2580 & 0.2535 & 0.2405 \\ 0.2361 & 0.2563 & 0.2351 & 0.2520 \end{pmatrix};$$

$$V^{(2)}=\left(v_{ij}^{(2)}\right)_{4\times 4}=\begin{pmatrix} 0.2608 & 0.2612 & 0.2638 & 0.2550 \\ 0.2576 & 0.2623 & 0.2547 & 0.2516 \\ 0.2411 & 0.2352 & 0.2459 & 0.2568 \\ 0.2427 & 0.2455 & 0.2730 & 0.2514 \end{pmatrix};$$

$$V^{(3)}=\left(v_{ij}^{(3)}\right)_{4\times 4}=\begin{pmatrix} 0.2608 & 0.2644 & 0.2638 & 0.2467 \\ 0.2529 & 0.2313 & 0.2453 & 0.2451 \\ 0.2470 & 0.2489 & 0.2497 & 0.2459 \\ 0.2606 & 0.2455 & 0.2730 & 0.2452 \end{pmatrix};$$

$$V^{(4)}=\left(v_{ij}^{(4)}\right)_{4\times 4}=\begin{pmatrix} 0.2464 & 0.2413 & 0.2592 & 0.2517 \\ 0.2575 & 0.2606 & 0.2476 & 0.2473 \\ 0.2548 & 0.2580 & 0.2509 & 0.2568 \\ 0.2606 & 0.2527 & 0.2189 & 0.2514 \end{pmatrix}.$$

Step 2. Assume that the weights of the attributes have correlations with each other and

$$m(\Phi)=0,\ m(A_1)=0.3,\ m(A_2)=0.3,\ m(A_3)=0.2,\ m(A_4)=0.4$$

then we can calculate:

$$m(\{A_1,A_2\})=0.5414,\ m(\{A_1,A_3\})=0.4609,\ m(\{A_1,A_4\})=0.6219;$$
$$m(\{A_2,A_3\})=0.4609,\ m(\{A_2,A_4\})=0.6219,\ m(\{A_3,A_4\})=0.5479;$$
$$m(\{A_1,A_2,A_3\})=0.7883,\ m(\{A_1,A_2,A_4\})=0.9766,\ m(\{A_1,A_3,A_4\})=0.8844;$$
$$m(\{A_2,A_3,A_4\})=0.8844,\ m(\{A_1,A_2,A_3,A_4\})=1.$$

Step 3. Aggregate all the individual hesitant fuzzy decision matrices into the collective hesitant fuzzy decision matrix.

Table 6. Collective hesitant fuzzy decision matrix

	C_1	C_2	C_3	C_4
A_1	$\{0.7811,0.5782,0.4261\}$	$\{0.6478,0.5216,0.3711,0.0233\}$	$\{0.631,0.4259,0.2787\}$	$\{0.7493,0.6,0.4748,0.2515\}$
A_2	$\{0.7046,0.4789,0.2814\}$	$\{0.7,0.5754,0.2459,0.1230,0.0492\}$	$\{0.6995,0.5743,0.3500,0.1233\}$	$\{0.6998,0.4497,0.3245\}$
A_3	$\{0.8014,0.6277,0.354,0.0771,0.0257\}$	$\{0.6765,0.4756,0.2733\}$	$\{0.7498,0.4488,0.1494\}$	$\{0.6487,0.523,0.1946\}$
A_4	$\{0.8479,0.7243,0.4544,0.0708,0.0236\}$	$\{0.7271,0.5772,0.4036,0.2281\}$	$\{0.6595,0.5322,0.4049\}$	$\{0.8503,0.6755,0.5755,0.1492\}$

Then, through Step 4 and Step 5, we can get the final ranking results:

$$A_4 \succ A_1 \succ A_3 \succ A_2\ .$$

Through the calculation process, we can see the following problems:

(1) First, in the first step, the original HFEs need to be extended to the same length. In this process, the original decision information has been changed.

(2) In the second step, the correlations between different attributes are given, which has a strong subjectivity.

(3) According to the data analysis, the result $A_4 \succ A_1 \succ A_3 \succ A_2$ is too absolute, for example, for expert e_4, under attribute C_1, the HFEs of A_4 are completely less than the HFSs of A_1, thus there is also a possibility that A_4 is less than A_1, while the result presented in this paper is that the possibility of A_4 greater than A_1 is 0.5011, that is $A_4 \succ_{(0.5011)} A_1$, which is more in line with the actual situation.

(4) It increases the flexibility of decision making and avoids the loss of decision information.

Conclusions

Generally speaking, the mathematical model is constructed based on the reality of environment, and most of the reality MAGDM problems, are often set up in a complex environment, therefore, the decision data that DMs can provide is highly uncertain and complex, while HFSs can better deal with DMs' choice of alternative fuzziness. In this paper, we have studied MAGDM problems in which the preference information offered by experts are HFSs. In order to get more reasonable decision results, the ranking of HFSs is particularly important. However, there are some defects in the existing rank relations for HFSs, such as score function, aggregation operator, etc., so that the decision results are inconsistent with the actual situation. Hence, we employ a distance measure and the TOPSIS method through which a new additive consistency fuzzy preference relation for HFSs is built. The TOPSIS method is based on the relative closeness of each alternative to determine the order of all alternatives, which can avoid excessive information loss in the process of information aggregation. Besides that, TOPSIS can effectively avoid high complexity of aggregating hesitant fuzzy information in traditional methods. The advanced new additive consistency fuzzy preference relation can overcome the defect of the existing order relations when ranking HFSs. In addition, this method can maintain the additive consistency of the fuzzy preference relation. Moreover, due to the advanced new additive consistency fuzzy preference relation, a new MAGDM method has been addressed. In MAGDM, whether experts can get the right decision results depends on whether experts can achieve consistent consensus decision, the existing decision making methods can not guarantee the consistency of the decision process, although some can achieve acceptable consistency or revised to achieve consistency, they still change the original decision information, while the method proposed in this paper guarantees the consistency of the whole process of decision making in theory and it is loyal to the original decision information. The new MAGDM method we have developed is not only simple and practical but is also more consistent with reality in integrating group decision information. At the end of this paper, an empirical analysis of a fund management company is carried out to illustrate the practicability and rationality of the model proposed in this paper and the corresponding comparative analysis is given. The proposed MAGDM method based on HFSs can be widely applied to a series of MAGDM problems such as military, transportation and economic management. In the later period, the proposed method can be extended for the decision making on interval valued hesitant fuzzy sets, intuitionistic fuzzy sets and interval valued hesitant fuzzy sets.

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APPENDIX

The partial calculation results of the numerical example used in Section 5 are as follows.

Step 2: For each attribute, the corresponding coefficient is calculated, and let $F^t = \left(f_{ij}^t\right)_{m\times n}$, $\left(t = 1,2,\cdots,k\right)$ be a series of the coefficient matrices. Then, for experts e_1, e_2, e_3, e_4, the coefficient matrices are obtained, and the results are as follows:

$$F^1 = \begin{pmatrix} 0.4051 & 0.5092 & 0.4770 & 0.4971 \\ 0.4840 & 0.4966 & 0.4961 & 0.4938 \\ 0.4986 & 0.4940 & 0.4907 & 0.5068 \\ 0.5071 & 0.5124 & 0.5248 & 0.5286 \end{pmatrix};$$

$$F^2 = \begin{pmatrix} 0.5835 & 0.5583 & 0.5202 & 0.5261 \\ 0.5269 & 0.5436 & 0.5331 & 0.5287 \\ 0.5222 & 0.5145 & 0.5114 & 0.5238 \\ 0.5431 & 0.5381 & 0.5393 & 0.5465 \end{pmatrix};$$

$$F^3 = \begin{pmatrix} 0.6049 & 0.5438 & 0.5215 & 0.5362 \\ 0.5454 & 0.5278 & 0.5354 & 0.5382 \\ 0.5553 & 0.5556 & 0.5628 & 0.5394 \\ 0.5481 & 0.5393 & 0.5388 & 0.5389 \end{pmatrix};$$

$$F^4 = \begin{pmatrix} 0.7068 & 0.5000 & 0.5000 & 0.5000 \\ 0.5053 & 0.4960 & 0.4849 & 0.4806 \\ 0.4998 & 0.4971 & 0.4973 & 0.4868 \\ 0.4988 & 0.5070 & 0.4918 & 0.5028 \end{pmatrix}.$$

Step 3: Take expert e_1 as an example, and then build FPRs about attributes C_1, C_2, C_3, C_4 of alternatives $A_i\left(i=1,2,3,4\right)$ as follows:

$$R_1^1 = \begin{pmatrix} 0.5000 & 0.4795 & 0.4755 & 0.4732 \\ 0.5205 & 0.5000 & 0.4960 & 0.4937 \\ 0.5245 & 0.5040 & 0.5000 & 0.4977 \\ 0.5268 & 0.5063 & 0.5023 & 0.5000 \end{pmatrix};$$

$$R_2^1 = \begin{pmatrix} 0.5000 & 0.5035 & 0.5042 & 0.4991 \\ 0.4965 & 0.5000 & 0.5007 & 0.4956 \\ 0.4958 & 0.4993 & 0.5000 & 0.4949 \\ 0.5009 & 0.5044 & 0.5051 & 0.5000 \end{pmatrix};$$

$$R_3^1 = \begin{pmatrix} 0.5000 & 0.4948 & 0.4963 & 0.4869 \\ 0.5052 & 0.5000 & 0.5015 & 0.4920 \\ 0.5037 & 0.4985 & 0.5000 & 0.4906 \\ 0.5131 & 0.5080 & 0.5094 & 0.5000 \end{pmatrix};$$

$$R_4^1 = \begin{pmatrix} 0.5000 & 0.5009 & 0.4973 & 0.4912 \\ 0.4991 & 0.5000 & 0.4964 & 0.4903 \\ 0.5027 & 0.5036 & 0.5000 & 0.4939 \\ 0.5088 & 0.5097 & 0.5061 & 0.5000 \end{pmatrix}.$$

Step 5: Calculate $R^t = \sum_{j=1}^{n} \overline{w}_j^t R_j^t$, $\left(t=1,2,\cdots,k\right)$

$$R^1 = \begin{pmatrix} 0.5000 & 0.4983 & 0.4969 & 0.4899 \\ 0.5017 & 0.5000 & 0.4985 & 0.4916 \\ 0.5031 & 0.5015 & 0.5000 & 0.4931 \\ 0.5101 & 0.5084 & 0.5069 & 0.5000 \end{pmatrix}.$$

In the same way, we have:

$$R^2 = \begin{pmatrix} 0.5000 & 0.5004 & 0.5044 & 0.4975 \\ 0.4996 & 0.5000 & 0.5039 & 0.4971 \\ 0.4956 & 0.4961 & 0.5000 & 0.4931 \\ 0.5025 & 0.5029 & 0.5069 & 0.5000 \end{pmatrix};$$

$$R^3 = \begin{pmatrix} 0.5000 & 0.5005 & 0.4967 & 0.4996 \\ 0.4995 & 0.5000 & 0.4962 & 0.4991 \\ 0.5033 & 0.5038 & 0.5000 & 0.5029 \\ 0.5004 & 0.5009 & 0.4971 & 0.5000 \end{pmatrix};$$

$$R^4 = \begin{pmatrix} 0.5000 & 0.5074 & 0.5055 & 0.5035 \\ 0.4926 & 0.5000 & 0.4981 & 0.4961 \\ 0.4945 & 0.5019 & 0.5000 & 0.4980 \\ 0.4965 & 0.5039 & 0.5020 & 0.5000 \end{pmatrix}.$$