



HESITANT FUZZY MULTI-CRITERIA DECISION MAKING METHODS BASED ON HERONIAN MEAN

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Abstract. Among several extensions of fuzzy set theory, the concept introduced by Torra and Narukawa (2009) in defining hesitant fuzzy set is interesting and practical. In this paper we introduce and study new methods for dealing with MCDM (multi-criteria decision making) problems under the hesitant fuzzy environment. First, we propose and discuss the notion of hesitant fuzzy Heronian mean operators. By using these operators, we can portray the relationship of the criteria effectively. Then, the numerical examples are provided and comparative analyses with other aggregation operators are not neglected. Furthermore, the weighted forms of the hesitant fuzzy Heronian mean operators are defined for MCDM problem, based on which, new MCDM methods are proposed. The MCDM methods presented in this paper can provide an effective manner to assist the decision maker in making his/her decision. An example about dormitory construction projection selection is given to show the effectiveness of the proposed method.

Keywords: hesitant fuzzy set, multi-criteria decision-making, Heronian mean.

JEL Classification: C43, D81.

Introduction

Torra and Narukawa (2009) developed the concept of hesitant fuzzy set (HFS), which is an extension of fuzzy set, characterized by the membership function expressed by a set of some possible values between $[0,1]$. HFS is an effective tool in expressing the fuzziness in the daily life of the human beings (Zhang, Xu 2012). The starting point for proposing the theory of HFS is that it is sometimes not easy to decide the degree of the membership of an element belonging to a set, however, this dilemma can be covered by a set of possible values between $[0,1]$ (Zhu, Xu 2013a; Liao *et al.* 2014). The relationship between HFS and some other extended fuzzy set, such as intuitionistic fuzzy set (Atanassov 1986), Type-2 fuzzy set (Dubois, Prade 1980; Miyamoto 2005) and fuzzy multiset (Yager 1986; Miyamoto 2000)

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are also discussed in detail by Torra and Narukawa (2009). Furthermore, Torra (2010) have pointed out that the envelope of HFS is an intuitionistic fuzzy set.

In the past few years, HFS theory has been receiving extensively attention from various aspects, including interval-valued HFS (IVHFS) (Chen *et al.* 2013; Farhadinia 2013; Quirós *et al.* 2015), entropy measures and cross-entropy (Xu, Xia 2012), distance and similarity measures (Zhu, Xu 2014; Xia, Xu 2011a; Xu, Xia 2011). Since its powerfulness in dealing with vagueness, HFS theory has been received many application achievements, such as personnel evaluation, medical diagnosis, clustering analysis, pattern recognition, and MCDM. In order to obtain the most suitable candidate in MCDM problem, aggregating the experts' evaluations on each criterion for different alternatives is a very important step (Xu, Zhang 2013; Parameshwaran *et al.* 2015). In other words, aggregation methods are indispensable to MCDM problems (Karsak, Dursun 2015; Liu *et al.* 2015).

Learned from the idea of intuitionistic fuzzy aggregation operators, Xia and Xu (2011b) first proposed many operators for hesitant fuzzy elements (HFEs), such as hesitant fuzzy weighted averaging (HFWA) operator, HFWG, GHFWA and GHFWG operators. Based on the research results of Xia and Xu (2011b), Wei *et al.* (2012) introduced some subjective correlated aggregation operators called Choquet aggregation operators. With the aid of quasi-arithmetic means, Xia *et al.* (2013) introduced the generalized form of hesitant fuzzy aggregation operators. Zhu *et al.* (2012) introduced some objective correlated aggregation operators called geometric Bonferroni means operators and hesitant fuzzy Bonferroni means operators respectively.

Compared with Bonferroni mean (Bonferroni 1950), Heronian mean (HM) (Beliakov *et al.* 2007) is another aggregation technique which can also depict the relationship between aggregated arguments objectively. From the definitions of Bonferroni mean and HM, we find that the BM mainly indicated the correlations between any pairs of c_i and c_j ($i \neq j$). However, it should be pointed out that the correlation between c_i and c_j ($i \neq j$) is exactly the same with the correlation between c_j and c_i ($i \neq j$). In other words, the BM operator takes into account it twice and it generates redundant subsequently. HM can deal with this situation effectively although it has a similar structure with BM. At present, the HM has been applied to intuitionistic fuzzy environment (Yu 2013) rather than any other kinds of arguments, such as hesitant fuzzy environment and this is the focus of this paper.

To do this, the rest of this study is organized as follows: in Section 1, we review some concepts such as HFS theory associated with our research. In Section 2, we propose the hesitant fuzzy Heronian mean operator (HFHM) and the hesitant fuzzy geometric Heronian mean operator (HFGHM) the weighted form of HFHM and HFGHM operators are also proposed in this section. A MCDM method based on the proposed operator is forwarded in the context of hesitant fuzzy environment in Section 3. A comparative example is presented for finding correlations with hesitant fuzzy BM operators (Zhu, Xu 2013b; Zhu *et al.* 2012) and shows the applicability of the proposed MCDM approach. The last Section gives some conclusion remarks.

1. Some basic concepts

Decision makers always face the actual conditions which is vagueness and uncertainty. There exist various techniques to cope with the vagueness and uncertainty phenomenon; however, different techniques are designed for different situations.

Definition 1 (Torra, Narukawa 2009; Xia *et al.* 2013). Let X be an any given set, a HFS is a mathematical function that when refer to X it returns a subset belongs to $[0,1]$, and it can be showed as follows:

$$E = \{ \langle x, h_E(x) \rangle | x \in X \}, \tag{1}$$

where $h_E(x)$ is a group of some possible numbers in $[0,1]$.

Operations on the HFEs are very basic for HFS theory. Xia and Xu (2011b) have studied on this and achieved some research results as follows:

Definition 2. Let h, h_1 and h_2 be three HFEs, then

- 1) $h^\lambda = \bigcup_{\xi \in h} \{ \xi^\lambda \}, \lambda > 0;$
- 2) $\lambda h = \bigcup_{\xi \in h} \{ 1 - (1 - \xi)^\lambda \}, \lambda > 0;$
- 3) $h_1 \oplus h_2 = \bigcup_{\xi_1 \in h_1, \xi_2 \in h_2} \{ \xi_1 + \xi_2 - \xi_1 \xi_2 \};$
- 4) $h_1 \otimes h_2 = \bigcup_{\xi_1 \in h_1, \xi_2 \in h_2} \{ \xi_1 \xi_2 \}.$

In the above operations, λ is a real number greater than zero. Xia and Xu (2011b) introduced the following comparative method.

Definition 3. Suppose h is a given HFE, the score of h is defined as $S(h) = \frac{1}{\#h} \sum_{\xi \in h} \xi$, and $\#h$ is used to represent the number of the elements in HFE h . For any given two HFEs h_1 and h_2 , if $S(h_1) > S(h_2)$, then h_1 is better than h_2 and expressed as $h_1 \succ h_2$; if $S(h_1) = S(h_2)$, then h_1 is equal to h_2 and expressed as $h_1 \sim h_2$.

2. HFHMs

In this section we study the Heronian mean aggregation method and introduce the HF-HMs, the properties of the proposed operators are also discussed in this Section.

Beliakov *et al.* (2007) introduced the basic Heronian mean operator as follows:

Definition 3. Let $a_i (i = 1, 2, \dots, n)$ be a set of real numbers which are greater than zero. If

$$BHM(a_1, a_2, \dots, a_n) = \frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=i}^n \sqrt{a_i a_j}, \tag{2}$$

then BHM is named as Basic Heronian mean.

Based on Definition 3, Yu and Wu (2012) extended the BHM to a more generalized form by introducing two parameters p and q .

Definition 4. Suppose there are two parameters p and $q (p > 0, q > 0)$, and $a_i (i = 1, 2, \dots, n)$ be a set of nonnegative real numbers. If

$$HM^{p,q}(a_1, a_2, \dots, a_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=i}^n a_i^p a_j^q \right)^{1/p+q}, \tag{3}$$

then HM is called the Heronian mean.

It should be noted that, when $p = q = \frac{1}{2}$, the HM reduces to the BHM (Beliakov *et al.* 2007).

Definition 5. Suppose there is a set of HFEs $h_j (j = 1, 2, \dots, n)$, the HFHM is defined as follows:

$$HFHM(h_1, h_2, \dots, h_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=i}^n h_i^p \otimes h_j^q \right)^{1/p+q}. \tag{4}$$

Especially, if $p = q = \frac{1}{2}$, then the HFHM operator reduces to the basic HFHM (BH-FHM):

$$BH-FHM(h_1, h_2, \dots, h_n) = \frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=i}^n \sqrt{h_i \otimes h_j}. \tag{5}$$

Theorem 1. Suppose there are two parameters p and $q (p > 0, q > 0)$, and $h_j (j = 1, 2, \dots, n)$ be a set of HFEs, then the aggregated value by using the HFHM is as follows:

$$HFHM(h_1, h_2, \dots, h_n) = \bigcup_{\xi_i \in h_i, \xi_j \in h_j} \left\{ \left(1 - \prod_{i=1, j=i}^n \left(1 - \xi_i^p \xi_j^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right\}. \tag{6}$$

Proof. According to the operations defined in Definition 2, we know:

$$h_i^p = \bigcup_{\xi_i \in h_i} \{ \xi_i^p \}, \quad h_j^q = \bigcup_{\xi_j \in h_j} \{ \xi_j^q \} \tag{7}$$

and

$$h_i^p \otimes h_j^q = \bigcup_{\xi_i \in h_i, \xi_j \in h_j} \{ \xi_i^p \xi_j^q \}, \tag{8}$$

then

$$\sum_{i=1}^n \sum_{j=i}^n h_i^p \otimes h_j^q = \bigcup_{\xi_i \in h_i, \xi_j \in h_j} \left\{ 1 - \prod_{i=1, j=i}^n \left(1 - \xi_i^p \xi_j^q \right) \right\} \tag{9}$$

and

$$\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=i}^n h_i^p \otimes h_j^q = \bigcup_{\xi_i \in h_i, \xi_j \in h_j} \left\{ 1 - \prod_{i=1, j=i}^n \left(1 - \xi_i^p \xi_j^q \right)^{\frac{2}{n(n+1)}} \right\}. \tag{10}$$

Therefore, we have:

$$HFHM(h_1, h_2, \dots, h_n) = \bigcup_{\xi_i \in h_i, \xi_j \in h_j} \left\{ \left(1 - \prod_{i=1, j=i}^n \left(1 - \xi_i^p \xi_j^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right\}, \tag{11}$$

end.

When the parameters p and q take different values, some exceptions of the HFHGM can be calculated as follows:

Case 1. If $p = q = \frac{1}{2}$, then the HFHGM changes to:

$$HFHGM(h_1, h_2, \dots, h_n) = \bigcup_{\xi_i \in h_i, \xi_j \in h_j} \left\{ 1 - \prod_{i=1, j=i}^n \left(1 - \sqrt{\xi_i \xi_j} \right)^{\frac{2}{n(n+1)}} \right\}, \tag{12}$$

which is the BHFHGM.

Case 2. If $q \rightarrow 0$ then the HFHGM operator changes to:

$$\lim_{q \rightarrow 0} HFHGM(h_1, h_2, \dots, h_n) = \left(\frac{1}{n} \sum_{i=1}^n h_i^p \right)^{\frac{1}{p}} = \bigcup_{\xi_i \in h_i} \left\{ \left(1 - \prod_{i=1}^n \left(1 - \xi_i^p \right)^{\frac{1}{n}} \right)^{\frac{1}{p}} \right\}, \tag{13}$$

which is the generalized hesitant fuzzy weighted operator which has been studied by Xia and Xu (2011b).

Case 3. If $q \rightarrow 0$ and $p = 1$, then the HFHGM changes to:

$$HFHGM(h_1, h_2, \dots, h_n) = \frac{1}{n} \left(\bigoplus_{i=1}^n h_i \right) = \bigcup_{\xi_i \in h_i} \left\{ 1 - \prod_{i=1}^n \left(1 - \xi_i \right)^{\frac{1}{n}} \right\}, \tag{14}$$

which is the hesitant fuzzy average mean operator studied by Xia and Xu (2011b).

Example 1. Let $h_1 = \{0.1, 0.2, 0.3\}$, $h_2 = \{0.8\}$, $h_3 = \{0.4, 0.5\}$ be three HFEs, then by Definitions 5 and Theorem 1, when the parameters take different values, the aggregated HFEs be calculated, for example,

When $p = 1, q = 1$, $HFHGM(h_1, h_2, h_3) = \{0.4987, 0.5285, 0.5187, 0.5478, 0.5411, 0.5694\}$, and the score of the HFE is 0.5340.

When $p = 3, q = 7$, $HFHGM(h_1, h_2, h_3) = \{0.6726, 0.6749, 0.6734, 0.6757, 0.6757, 0.6780\}$, and the score of the HFE is 0.6751.

If we let one of the values of one parameters fixed, take $p = 2$ or $q = 2$ for example, the change trends of scores of the aggregated HFEs obtained by HFHGM are shown in Figure 1.

On the other hand, Figure 2 described the change trends of scores obtained by HFHGM operator when one of the two parameters getting bigger slowly.

Different scores of the HFEs aggregated by HFHGM can be calculated as the parameters p and q changed. Figure 3 illustrates the scores obtained by the HFHGM operator in detail.

Geometric mean is an important aggregation method and has attracted many attentions from scholars (Liu, Jin 2012; Tan 2011; Wang, Liu 2011; Wei 2010; Xu 2000; Yu 2012). Next, we focus on the geometric form of HFHGM operator and introduce the hesitant fuzzy geometric Heronian mean operator (HFGHM).

Definition 6. Suppose there are a group of HFEs $h_j (j = 1, 2, \dots, n)$. If

$$HFGHM(h_1, h_2, \dots, h_n) = \frac{1}{p+q} \left(\bigotimes_{i=1, j=i}^n (ph_i \oplus qh_j)^{\frac{2}{n(n+1)}} \right), \tag{15}$$

then HFGHM is called the HFGHM.

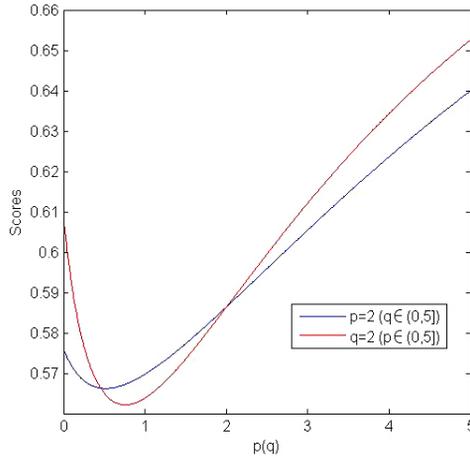


Fig. 1. Change trends of the Scores $p = 2$ and $q = 2$

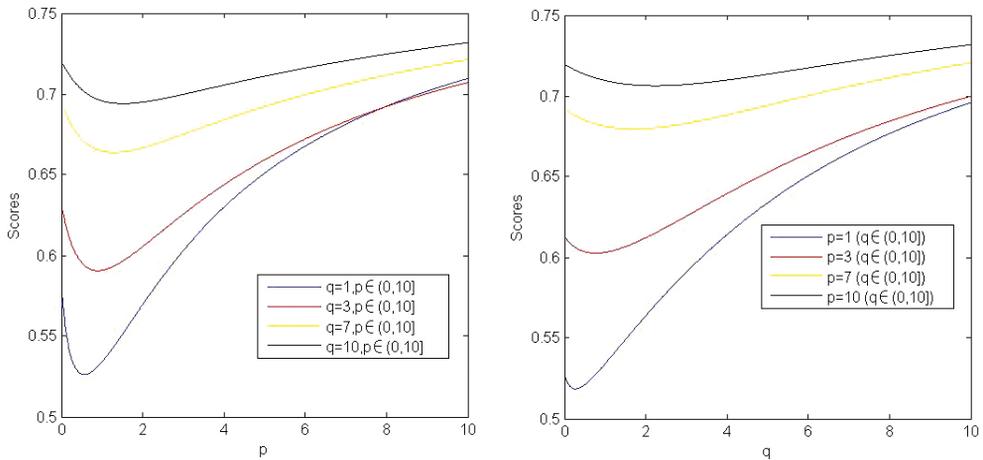


Fig. 2. Change trends of the Scores $p(q) = 1, 3, 7, 10$

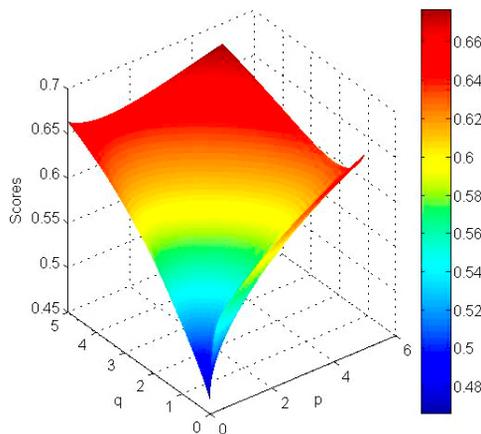


Fig. 3. Scores for HFEs obtained by the HFHM operator ($p \in [0,6], q \in [0,6]$)

Especially, if $p = q = \frac{1}{2}$, then the HFGHM operator reduces to the Basic hesitant fuzzy geometric Heronian mean operator (BHFGHM):

$$\text{BHFGHM}(h_1, h_2, \dots, h_n) = \bigotimes_{i=1, j=i}^n \left(\frac{1}{2} (h_i \oplus h_j) \right)^{\frac{2}{n(n+1)}}. \tag{16}$$

Theorem 2. Suppose there are two parameters p and $q (p > 0, q > 0)$, and $h_j (j = 1, 2, \dots, n)$ be a set of HFEs, then the aggregated value by using the HFGHM is as follows:

$$\text{HFGHM}(h_1, h_2, \dots, h_n) = \bigcup_{\xi_i \in h_i, \xi_j \in h_j} \left\{ 1 - \left(1 - \prod_{i=1, j=i}^n \left(1 - (1 - \xi_i)^p (1 - \xi_j)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right\}. \tag{17}$$

Proof. According to the operations defined in Definition 2, we know,

$$\lambda_f = \bigcup_{\xi \in f} \{ 1 - (1 - \xi)^\lambda \}$$

$$ph_i = \bigcup_{\xi_i \in h_i} \{ 1 - (1 - \xi_i)^p \}, \quad ph_j = \bigcup_{\xi_j \in h_j} \{ 1 - (1 - \xi_j)^p \} \tag{18}$$

and

$$h_i^p \oplus h_j^q = \bigcup_{\xi_i \in h_i, \xi_j \in h_j} \{ 1 - (1 - \xi_i)^p (1 - \xi_j)^q \}. \tag{19}$$

Then:

$$\left(h_i^p \oplus h_j^q \right)^{\frac{2}{n(n+1)}} = \bigcup_{\xi_i \in h_i, \xi_j \in h_j} \left\{ 1 - (1 - \xi_i)^p (1 - \xi_j)^q \right)^{\frac{2}{n(n+1)}} \right\}. \tag{20}$$

Therefore:

$$\frac{1}{p+q} \left(\bigotimes_{i=1, j=i}^n (ph_i \oplus ph_j)^{\frac{2}{n(n+1)}} \right) = \bigcup_{\xi_i \in h_i, \xi_j \in h_j} \left\{ 1 - \left(1 - \prod_{i=1, j=i}^n \left(1 - (1 - \xi_i)^p (1 - \xi_j)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right\}, \tag{21}$$

which completes the proof of Theorem 2.

When the parameters p and q take different values, some special cases of the HFGHM can be calculated as follows:

Case 1. If $p = q = \frac{1}{2}$, then the HFGHM reduces to:

$$\text{HFGHM}(h_1, h_2, \dots, h_n) = \bigcup_{\xi_i \in h_i, \xi_j \in h_j} \left\{ \prod_{i=1, j=i}^n \left(1 - \sqrt{(1 - \xi_i)(1 - \xi_j)} \right)^{\frac{2}{n(n+1)}} \right\}, \tag{22}$$

which we call the Basic hesitant fuzzy geometric Heronian mean operator (BHFGHM).

Case 2. If $q \rightarrow 0$ then the HFGHM operator reduces to:

$$\lim_{q \rightarrow 0} \text{HFGHM}(h_1, h_2, \dots, h_n) = \frac{1}{p} \left(\sum_{i=1}^n (ph_i \frac{1}{n}) \right) = \bigcup_{\xi_i \in h_i} \left\{ 1 - \left(1 - \prod_{i=1}^n \left(1 - (1 - \xi_i)^p \right)^{\frac{1}{n}} \right)^{\frac{1}{p}} \right\}, \tag{23}$$

which is the GHFWG studied by Xia and Xu (2011b).

Case 3. If $p = 1, q \rightarrow 0$, then the HFGHM reduces to:

$$\text{HFGHM}(h_1, h_2, \dots, h_n) = \left(\sum_{i=1}^n (h_i)^{\frac{1}{n}} \right) = \bigcup_{\xi_i \in h_i} \left\{ \prod_{i=1}^n (\xi_i)^{\frac{1}{n}} \right\}, \tag{24}$$

which is the HFGM studied by Xia and Xu (2011b).

Example 2. Let $h_1 = \{0.1, 0.2, 0.3\}, h_2 = \{0.8\}, h_3 = \{0.4, 0.5\}$ be three HFEs, then by Definition 6 and Theorem 2, when the parameters take different values, the aggregated HFEs can be calculated, for example:

When $p = 1, q = 1$, $\text{HFGHM}(h_1, h_2, h_3) = \{0.3533, 0.3817, 0.4196, 0.4522, 0.4694, 0.5047\}$, and the score of the HFE is 0.4301.

When $p = 3, q = 7$, $\text{HFGHM}(h_1, h_2, h_3) = \{0.2301, 0.2335, 0.3169, 0.3252, 0.3877, 0.4069\}$, and the score of the HFE is 0.3167.

When $p = 10, q = 10$, $\text{HFGHM}(h_1, h_2, h_3) = \{0.1743, 0.1748, 0.2663, 0.2680, 0.3525, 0.3588\}$, and the score of the HFE is 0.2658.

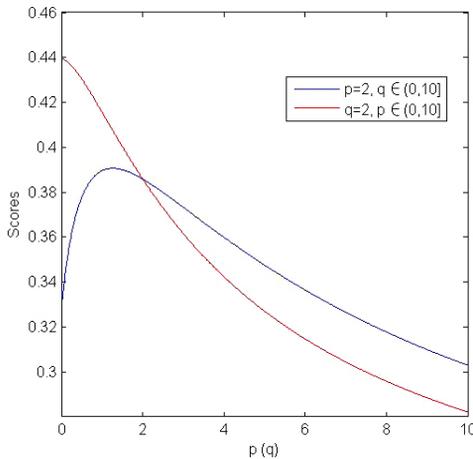


Fig. 4. Change trends of the Scores $p = 2$ and $q = 2$

If we let the value of one parameters fixed, take $p = 2$ and $q = 2$ for example, the change trends of scores of the aggregated HFEs obtained by HFGHM are shown in Figure 4.

On the other hand, Figure 5 described the change trends of scores obtained by HFGHM operator when one of the two parameters getting bigger slowly.

Different scores of the HFEs aggregated by HFGHM can be calculated as the parameters p and q changed. Figure 6 illustrates the scores obtained by the HFGHM operator in detail.

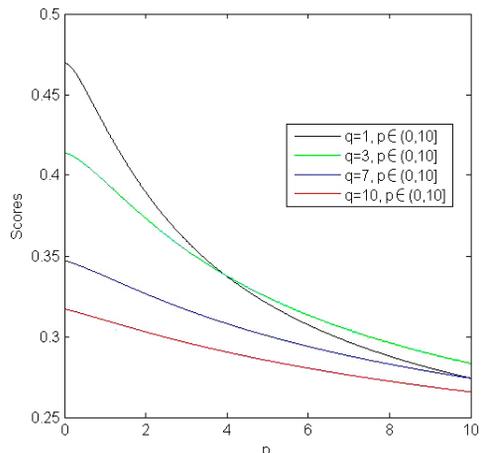
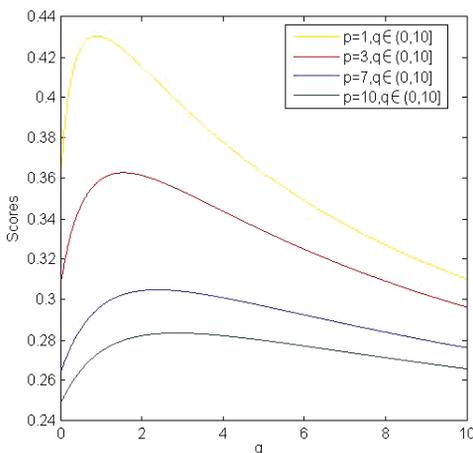


Fig. 5. Change trends of the Scores $p(q) = 1, 3, 7, 10$

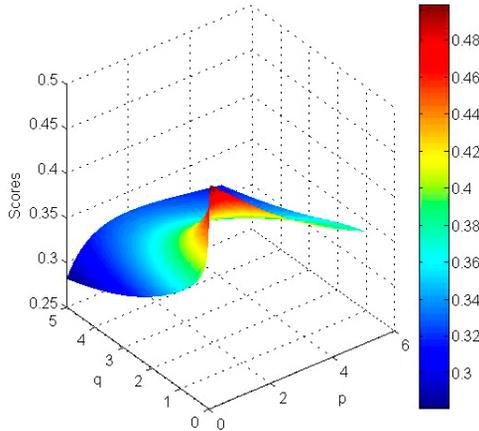


Fig. 6. Scores for HFES obtained by the HFGHM operator ($p \in [0,6], q \in [0,6]$)

3. Hesitant fuzzy MCDM based on weighted Heronian mean operators

HFHM and HFGHM operators can cope with the correlated hesitant fuzzy information. However, they neglected different weights of the aggregated arguments which are not feasible. In this section, the weighted forms of HFHM and HFGHM operators are studied and the hesitant fuzzy weighted Heronian mean operator (HFWHM) and hesitant fuzzy weighted geometric Heronian mean operator (HFWGHM) are proposed. Furthermore, a MCDM method based on the HFWHM and HFWGHM operators are introduced.

Definition 7. Let $h_j (j = 1, 2, \dots, n)$ be a collection of HFES whose weight vector is $w = (w_1, w_2, \dots, w_n)^T$, which satisfies $w_i > 0, i = 1, 2, \dots, n$ and $\sum_{i=1}^n w_i = 1$. If

$$\text{HFWHM}(h_1, h_2, \dots, h_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{j=i}^n (w_i h_i)^p \otimes (w_j h_j)^p \right)^{1/p+q}; \tag{25}$$

$$\text{HFWGHM}(h_1, h_2, \dots, h_n) = \frac{1}{p+q} \left(\bigotimes_{i=1, j=i}^n \left((p h_i)^{w_i} \oplus (q h_j)^{w_j} \right)^{2/n(n+1)} \right). \tag{26}$$

Then HFWHM is called the weighted HFHM (HFWHM) and HFWGHM is called the weighted HFGHM (HFWGHM).

Learned from the Theorems 1 and 2, the following research results can be got easily. Therefore, we omit the detailed proof.

Theorem 3. Let $p, q > 0$, and $h_j (j = 1, 2, \dots, n)$ be a collection of HFES whose weight vector is $w = (w_1, w_2, \dots, w_n)^T$, which satisfies $w_i > 0, i = 1, 2, \dots, n$ and $\sum_{i=1}^n w_i = 1$. Then the aggregated HFES by using the HFWHM or HFWGHM is as follows:

$$\text{HFWHM}(h_1, h_2, \dots, h_n) =$$

$$\cup_{\xi_i \in h_i, \xi_j \in h_j} \left\{ \left(1 - \prod_{i=1, j=i}^n \left(1 - (1 - (1 - \xi_i)^{w_i})^p (1 - (1 - \xi_j)^{w_j})^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right\}; \tag{27}$$

$$\text{HFWGHM}(h_1, h_2, \dots, h_n) = \cup_{\xi_i \in h_i, \xi_j \in h_j} \left\{ 1 - \left(1 - \prod_{i=1, j=i}^n \left(1 - (1 - \xi_i^{w_i})^p (1 - \xi_j^{w_j})^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right\}. \tag{28}$$

For a MCDM problem (Liu 2014; Wang, Liu 2012; Li, Wan 2013), suppose there are m alternatives to be evaluated and $C = \{c_1, c_2, \dots, c_n\}$ is a set of criteria with the weight vector $w = (w_1, w_2, \dots, w_n)^T$. The MCDM matrix is illustrated as follows (Table 1).

Table 1. The decision matrix

	C_1	C_2	...	C_n
X_1	h_{11}	h_{12}	...	h_{1n}
X_2	h_{21}	h_{22}	...	h_{2n}
...
X_m	h_{m1}	h_{m2}	...	h_{mn}

In the decision making matrix, h_{ij} is the performance of criteria C_j regarding alternative x_i expressed by experts. The main decision processes are shown as follows.

Step 1. Fuse all the HFEs $h_{ij}(i = 1, 2, \dots, n)$ of the i th line, and gain the overall HFEs h_i regard to the alternative A_i by the HFWHM or HFWGHM operators.

Step 2. Rank the hesitant fuzzy values $h_i(i = 1, 2, \dots, m)$ and select the most appropriate one.

In the following, we show a practical application of the proposed methodology by considering a dormitory construction projection selection problem in a real company concerning MCDM.

In order to support company’s future development, ABC Company plan to have a new plant C in D city. According to the Headcount Plan, Plant C needs to recruit 4000 employees in the following two years, it’s a really hard work. Below are the existing transportation conditions for plant C:

- 1) Plant C located in the outskirts of the city D, the distance between the company and city downtown is about 2-hour drive.
- 2) The distance between the company and the airport is about 1-hour drive.
- 3) There is only one bus – bus runs from the nearest living zone to the company, which runs 45 minutes between each run. The earliest bus is 7am, and the latest is 6pm.
- 4) Because of the remote location, taxis are barely available, and thus, if someone needs to go out in emergency, he can only drive by himself.

Above poor transportation conditions increased the recruiting difficulties of plant C, many candidates rejected the offer due to the inconvenient. Thus except to enlarge recruiting channel, Building employees' dormitory is another effectively way to attract candidates and also will acting an important role to the employee retention project in the future. Actually, whether dormitory will be provided has become one of the key factors in the hiring process since living cost in city D is very high, especially when hiring blue – collar workers.

Based on above reasons, Plant C raised dormitory construction proposal after fully comparison and analysis and it is now at the bidding stage. There are 4 suppliers to bid for this construction project. According to related requirements, Plant C needs to review them from 3 points (c_1, c_2, c_3) which are listed as below:

- 1) Qualification: supplier should have required qualification, including registered capital and its finical background. This dormitory project need the vendor to bear all the finical risk, thus the vendor should have strong financial resources (c_1).
- 2) Cost: supplier' bidding price is the most important point since cost always acting a very important role in a manufacturing plant (c_2).
- 3) Similar experiences: it's better for supplier to have experiences to complete a similar project (c_3). Since this project is a big one, previous experiences can help this dormitory project smoothly completed.

Assume the weight of the three criteria is $(0.3, 0.4, 0.3)^T$. Three experts are invited to evaluate the four vendors and the hesitant fuzzy decision making matrix are shown in Table 2.

Table 2. The hesitant fuzzy decision matrix

	C_1	C_2	C_3
X_1	{0.7, 0.8}	{0.6, 0.7, 0.8}	{0.8}
X_2	{0.4, 0.6}	{0.5}	{0.3, 0.4}
X_3	{0.6}	{0.5, 0.7}	{0.6, 0.7, 0.8}
X_4	{0.3, 0.4, 0.5}	{0.4, 0.6, 0.7}	{0.5}

Take $h_{31} = \{0.6\}$ for example, it means the three experts expressed the same views on the performance of alternative X_3 on criterion C_1 and the value is 0.6. Therefore, the evaluation of alternative X_3 on criterion C_1 is expressed by HFE $\{0.6\}$. Take $h_{11} = \{0.7, 0.8\}$ for another example, it means two of the three experts expressed the same views on the performance of alternative X_1 on criterion C_1 and the value is 0.7 or 0.8. The third expert expressed 0.8 or 0.7 simultaneously. Therefore, the evaluation of alternative X_1 on criterion C_1 is expressed by HFE $\{0.7, 0.8\}$.

For illustrating the influence of the parameter p and q in the above evaluation problem, the different value of p and q are taken to aggregate the HFE and the ultimate results are shown in Table 3.

Furthermore, let p or q fixed, the change trends of the scores of four alternatives are shown in Figures 7 and 8.

Table 3. Score obtained by the HFWHM operator and the rankings results

	Operators	A_1	A_2	A_3	A_4	Ranking
$p = 0, q = 1$	HFWHM	0.3792	0.1756	0.3004	0.2182	$A_1 \succ A_3 \succ A_4 \succ A_2$
$p = q = 1$	HFWHM	0.3724	0.1875	0.2890	0.2117	$A_1 \succ A_3 \succ A_4 \succ A_2$
$p = q = 5$	HFWHM	0.3829	0.2152	0.3096	0.2533	$A_1 \succ A_3 \succ A_4 \succ A_2$
$p = 0, q = 10$	HFWHM	0.3928	0.2213	0.3251	0.2674	$A_1 \succ A_3 \succ A_4 \succ A_2$
$p = 10, q = 10$	HFWHM	0.3925	0.2273	0.3237	0.2710	$A_1 \succ A_3 \succ A_4 \succ A_2$

As we can see from Table 3 and Figures 7 and 8, the ranking results are the same though different parameter values of p and q are taken. In other words, the MCDM method based on HFWHM operator offers good stability. Different scores of the HFEs aggregated by HFWHM can be calculated as the parameters p and q changed. Figure 9 illustrates the scores obtained by the HFWHM operator in detail.

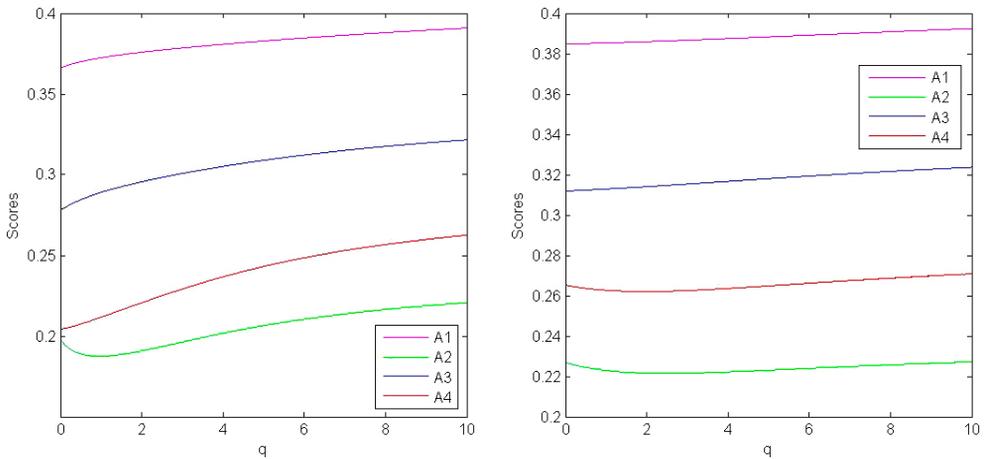


Fig. 7. Scores of four alternatives ($p \in (0,10]$; $p = 1, p = 10$)

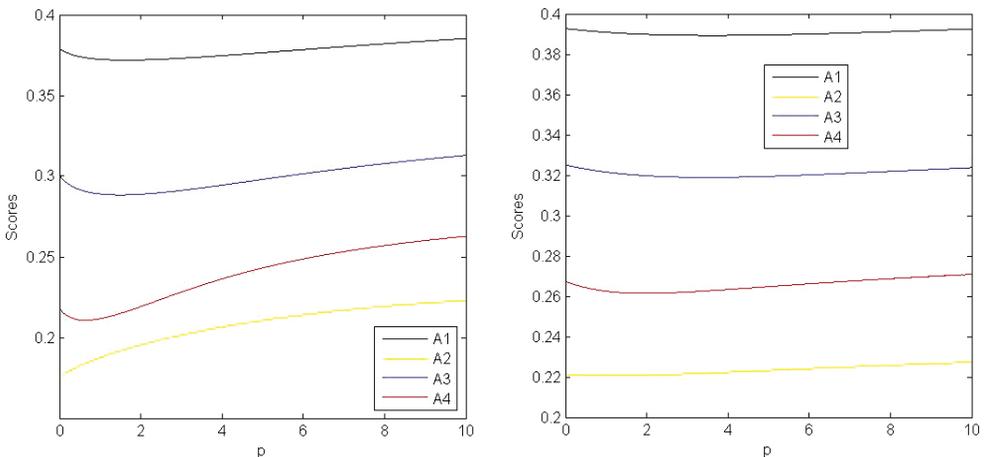


Fig. 8. Scores of four alternatives ($p \in (0,10]$; $q = 1, q = 10$)

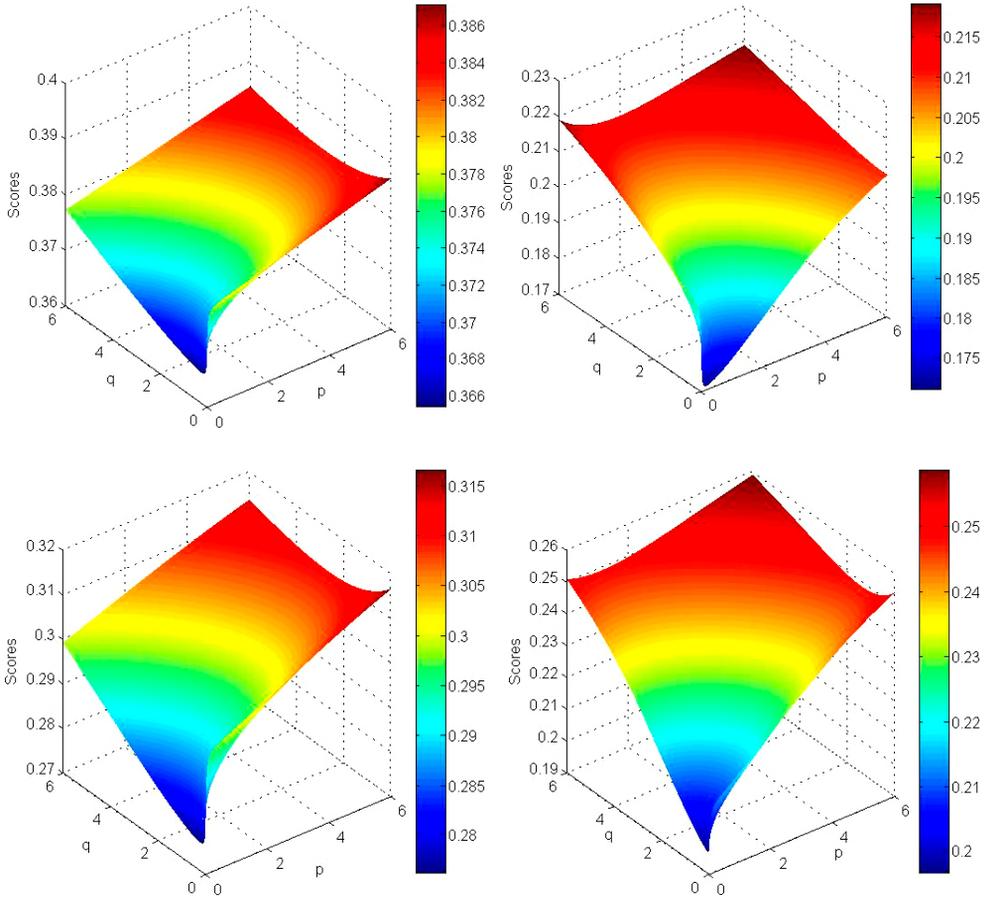


Fig. 9. Scores obtained by the HFWHM operator ($p \in [0,6], q \in [0,6]$)

About the hesitant fuzzy MCDM method, we can compare with the method based on HFWGHM operator, HFWBM operator (Zhu, Xu 2013b) and HFWGBM operator (Zhu *et al.* 2012). For illustrating the influence of the parameters p and q in the HFWGHM, HFWBM, and HFWGBM operators based methods, the different value of p and q are taken to aggregate the HFE and the ultimate results are shown in Table 4.

Let q fixed, the variation tendency of the scores of the four alternatives based on the HFWGHM operator, HFWBM operator and HFWGBM operator are shown as follows.

Obviously, the ranking results of the hesitant fuzzy MCDM methods based on four different aggregation operators have hardly changed. From the Figures 7–8 and 10–12, we can clearly see that the ranking results of the four alternatives and the gaps between each other. Different scores of the HFEs aggregated by HFWGHM, HFWBM, and HFWGBM can be calculated as the parameters p and q changed. Figures 13–15 illustrate the details.

Table 4. Score obtained by the HFWGHM, HFWBM, HFWGBM operators and ranking results

	Operators	A_1	A_2	A_3	A_4	Ranking
$p = 0, q = 1$	HFWBM	0.1767	0.0790	0.1381	0.0876	$A_1 \succ A_3 \succ A_4 \succ A_2$
	HFWGHM	0.9082	0.7506	0.8610	0.7949	$A_1 \succ A_3 \succ A_4 \succ A_2$
	HFWGBM	0.9307	0.8326	0.8971	0.8423	$A_1 \succ A_3 \succ A_4 \succ A_2$
$p = q = 1$	HFWBM	0.2388	0.1216	0.1809	0.1226	$A_1 \succ A_3 \succ A_4 \succ A_2$
	HFWGHM	0.9043	0.7637	0.8539	0.7852	$A_1 \succ A_3 \succ A_4 \succ A_2$
	HFWGBM	0.9225	0.8168	0.8836	0.8206	$A_1 \succ A_3 \succ A_4 \succ A_2$
$p = q = 5$	HFWBM	0.3302	0.1816	0.2555	0.1832	$A_1 \succ A_3 \succ A_4 \succ A_2$
	HFWGHM	0.8831	0.7497	0.8313	0.7651	$A_1 \succ A_3 \succ A_4 \succ A_2$
	HFWGBM	0.8952	0.7752	0.8535	0.7685	$A_1 \succ A_3 \succ A_4 \succ A_2$
$p = 0, q = 10$	HFWBM	0.3378	0.2025	0.2830	0.1718	$A_1 \succ A_3 \succ A_4 \succ A_2$
	HFWGHM	0.8775	0.7380	0.8256	0.7637	$A_1 \succ A_3 \succ A_4 \succ A_2$
	HFWGBM	0.8455	0.7475	0.8311	0.7375	$A_1 \succ A_3 \succ A_4 \succ A_2$
$p = 10, q = 10$	HFWBM	0.3492	0.1967	0.2717	0.1661	$A_1 \succ A_3 \succ A_4 \succ A_2$
	HFWGHM	0.9439	0.7418	0.8893	0.7528	$A_1 \succ A_3 \succ A_4 \succ A_2$
	HFWGBM	1.0000	0.7643	0.9126	0.7509	$A_1 \succ A_3 \succ A_4 \succ A_2$

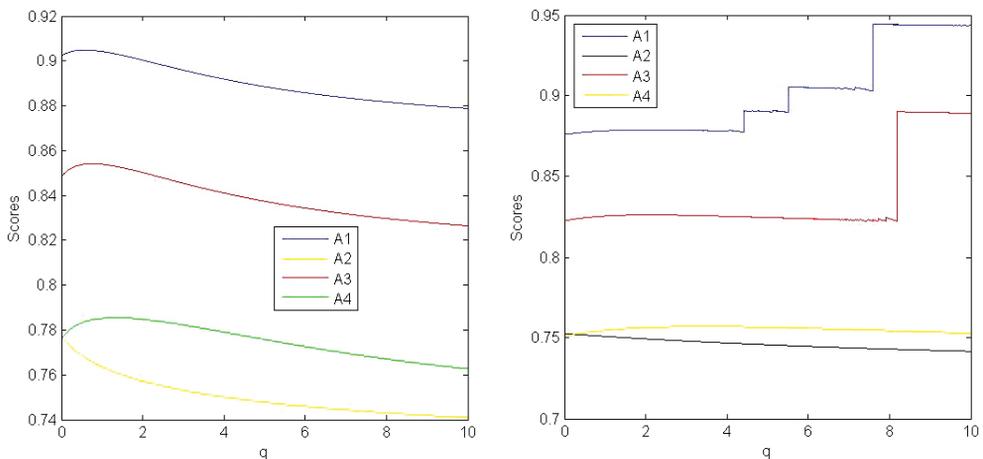


Fig. 10. Results based on HFWGHM operator ($q \in (0,10]$; $p = 1, p = 10$)

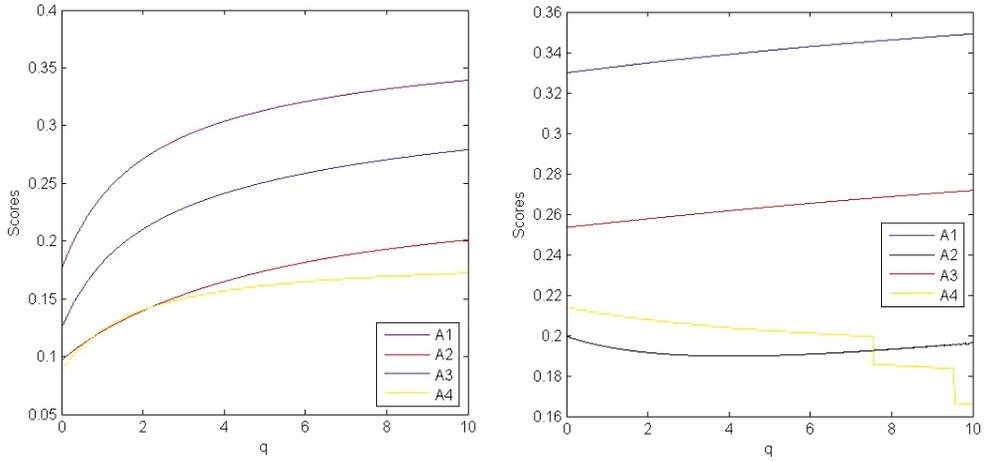


Fig. 11. Results based on HFWM operator ($q \in (0,10]$; $p = 1, p = 10$)

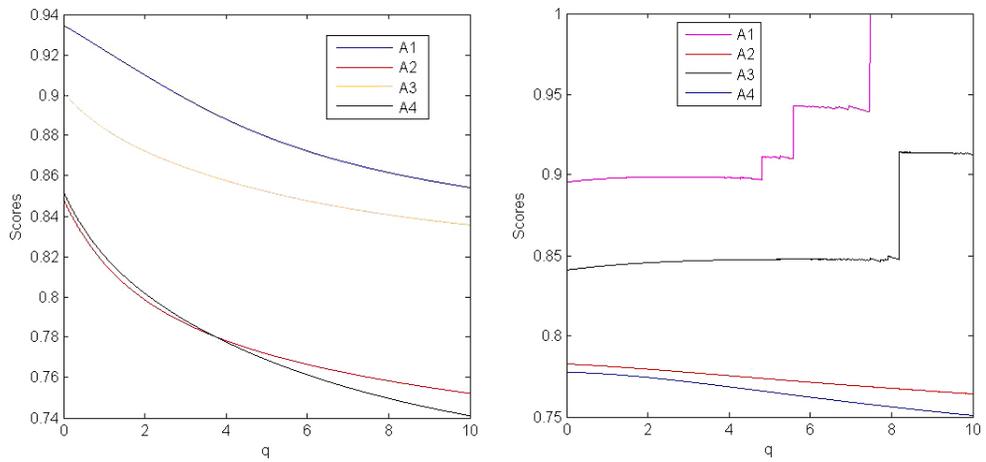


Fig. 12. Results based on HFWGBM operator ($p \in (0,10]$; $p = 1, p = 10$)

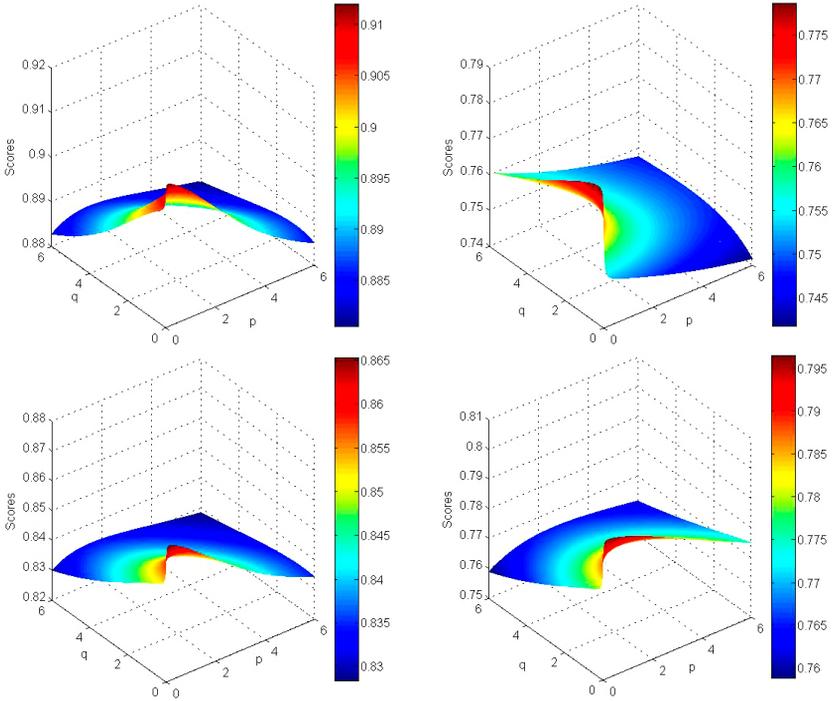


Fig. 13. Scores obtained by the HFWGHM operator ($p \in [0,6], q \in [0,6]$)

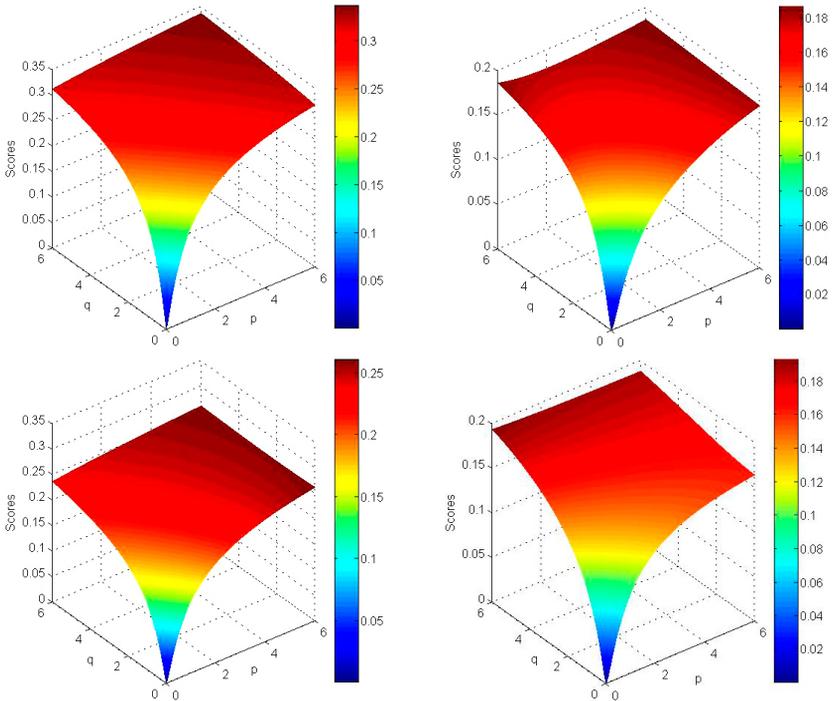


Fig. 14. Scores obtained by the HFWBM operator ($p \in [0,6], q \in [0,6]$)

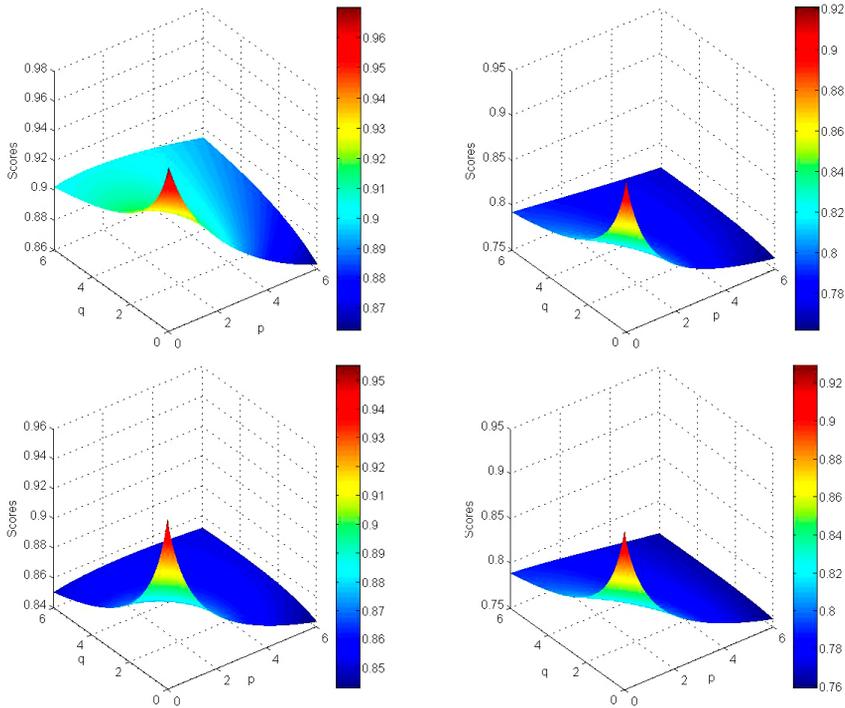


Fig. 15. Scores obtained by the HFWGBM operator ($p \in [0,6], q \in [0,6]$)

Conclusions

In this paper, we have researched new hesitant fuzzy MCDM methods, and we have applied them in dormitory construction projection selection problem. We have put forwarded some operators for HFEs, such as HFHM, HFGHM, HFWHM and HFWGHM operators. We have proved that some common operators, such as HFWA, HFWG, GHFWA and GHFWG operators are some exceptions of the operators put forwarded in this paper. In other words, the Heronian mean operators are more flexible and general. Further, the comparison with other MCDM methods based on HFWBM, HFWGBM operators are also provided. We have forwarded the proposed methods to the selection of dormitory construction projection, but it should be noted that they can also be applied to other evaluation problems, such as the personnel evaluation, investment plans evaluations, and so on. In future research, we will focus on how these methods proposed in this paper can be applied to other evaluation fields, and we will consider other types of HFHMs.

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