EXTENSIONS OF LINMAP MODEL FOR MULTI CRITERIA DECISION MAKING WITH GREY NUMBERS

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Abstract. The linear programming technique for multidimensional analysis of preference, known as LINMAP is one of the existing well-known ideal seeking methods for multi attribute decision making problems. This method originally is proposed under crisp and deterministic circumstances. However, uncertainty is an indubitable property of decision making problems. In this paper, a new version of LINMAP-G is proposed where the decision maker's judgments are expressed as grey numbers. Like original LINMAP method, the grey ideal solution and attributes weight vector is determined and alternatives are ranked according to their weighted distance from determined ideal point. Application of the proposed method is illustrated in two numerical examples.

Keywords: MCDM, LINMAP-G, Uncertain Decision making, Grey data, Positive ideal solution, Grey Euclidean Distance.

1. Introduction

Decision makers always seek a criterion to appraise their decisions. In this context, decision-making methods arise when decision maker simultaneously envisages various criteria for evaluating his or her decisions favorite (Kuo *et al.* 2008). Such a problem is the subject of multiple criteria decision making (MCDM) methods (Zavadskas, Turskis 2011; Peng *et al.* 2011; Kou *et al.* 2012). This class is further divided into multi objective decision making (MODM) and multi attribute decision making (MADM) (Climaco 1997). A formalized definition of MADM problems can be stated as follows.

Let we have a nonempty and finite set of decision alternatives, i.e. $A_1, A_2, \ldots, A_m$, and there are a finite set of goals, attributes or criteria, i.e. $C_1, C_2, \ldots, C_n$, according to which the desirability of an alternative is to be judged. The aim of MADM is to determine the optimal alternative with the highest degree of desirability with respect to all relevant goals (Zimmerman 1987). An optimal alternative in MADM problems can be defined as an alternative $A^\star$ that has the highest value in all decision-making criteria. Usually, MADM problems do not have an optimal solution in practice and current methods seek an alternative with the highest degree of satisfaction for decision makers. In last decades, MADM techniques have a wide application in different areas that concern with selection.

Multi attribute decision making methods require decision makers judgments and evaluations about alternatives performance regard to multiple attributes. These judgments are a subject of uncertainty. Indeed, decision makers do not have complete information about alternatives or their conditions regard to a certain attribute. Therefore, it will be so difficult for them to express their evaluations based on exact numbers. Hence, uncertainty contexts are widely applied in MADM problems.

Fuzzy set theory (FST) was developed by Zadeh (Zadeh 1965) as a generalized form of the classical set theory that assigns a membership degree to each element of a given set in a universe. FST is one of the well-known paradigms in studying systems with uncertainty. Bellman and Zadeh (Bellman, Zadeh 1970) have introduced the concept of decision making under fuzzy environment. Afterward, MADM techniques have been extended under fuzzy environment (Aouam *et al.* 2003; Yazdani *et al.* 2011; Xu 2004; Li, Yang 2004; Wang, Chuu 2004; Hu *et al.* 2004; Antucheviciene *et al.* 2011; Kersuliene, Turskis 2011; Brauers *et al.* 2011; Fouladgar *et al.* 2011; Balezentis, Balezentis 2011).

Another paradigm of uncertainty is developed such that the crisp numbers are substituted with grey numbers (Deng 1982). Interval numbers also have a wide application in decision making field with TOPSIS (Chen, Tzeng 2004; Jahanshahloo *et al.* 2006; Lin *et al.* 2008; Zavadskas *et al.* 2010a; Tsaur 2011; Yue 2011), with PROMETHEE (Le Teno, Mareschal 1998), with ELECTRE (Vahdani *et al.* 2010; Özcan *et al.* 2011), with COPRAS-G (Zavadskas *et al.* 2010b; Hashemkhani Zolfani *et al.* 2011), with ARAS-G (Turskis, Zavadskas 2010; Zavadskas *et al.* 2010c), with SAW-G (Zavadskas *et al.* 2010a), with MOORA (Stanujkic *et al.* 2012). An interval number can be considered as a number whose exact value is unknown, but a range within which the value lies is known (Moore 1966).

The Linear Programming Technique for Multidimensional Analysis of Preference (LINMAP) was developed by Srinivasan and Schocker (Srinivasan, Shocker 1973) as one of the MADM tech-

niques that determines the preference order among a set of alternatives by determination of weight vector w and positive ideal solution (PIS) vector. However, the LINMAP can only deal with MADM problems in crisp environments. Xia *et al.* (Xia *et al.* 2006) developed LINMAP method for MADM problems under fuzzy environment. Li and Sun (Li, Sun 2007) developed LINMAP method for MADM problems with linguistic variables and incomplete preference information. Li (2008) also developed this method under intuitionistic fuzzy environment. Considering the simplicity and clearness of grey numbers in expressing the uncertainty and lack of knowledge, in this paper, a LINMAP method is extended with grey data. The rest of the paper is organized as follows: Section 2 briefly introduces the grey numbers and their operations. MADM problems with grey data are expressed in section 3. The extended grey LINMAP model and proposed decision process is introduced in section 4. The proposed method is illustrated with an example in section 5. Finally, the paper is concluded in section 6.

2. Grey numbers

As stated in previous section, a grey number can be indicated as a range. In fact, a number x is called an grey number when its exact value is unknown and only it is known that $x \in \left[\underline{x}, \overline{x} \right]$, where $\underline{x}$ is the lower bound and $\overline{x}$ is the upper bound, such that $\underline{x} \prec \overline{x}$. Arithmetic operations on interval numbers are introduced by Moore (Moore 1966). If $x = \left[\underline{x}, \overline{x} \right]$ and $y = \left[\underline{y}, \overline{y} \right]$ are two grey numbers, then:

$$x + y = \left[\underline{x} + \underline{y}, \overline{x} + \overline{y} \right], \tag{1}$$

$$x - y = \left[\underline{x} - \overline{y}, \overline{x} - \underline{y} \right], \tag{2}$$

$$x \times y = \left[\min\left(\underline{x}\,\underline{y}, \underline{x}\overline{y}, \overline{x}\,\underline{y}, \overline{x}\overline{y} \right), \max\left(\underline{x}\,\underline{y}, \underline{x}\overline{y}, \overline{x}\,\underline{y}, \overline{x}\overline{y} \right) \right], \tag{3}$$

$$x \div y = \left[\underline{x}, \overline{x} \right] \times \left[\frac{1}{\overline{y}}, \frac{1}{\underline{y}} \right]. \tag{4}$$

The center, x_C, and width, x_W of a grey number x are defined as follows (Ishibuchi, Tanaka 1990):

$$x_C = \frac{1}{2}\left(\underline{x} + \overline{x} \right), \tag{5}$$

$$x_W = \frac{1}{2}\left(\overline{x} - \underline{x} \right). \tag{6}$$

3. Grey MADM problem definition

Consider an MADM problem that consist evaluation of a set of m alternatives regard to a set of n attributes. In classic form, ratings of alternatives regard to attributes are stated with crisp data. However in many situations, and due to uncertainty or lack of knowledge, the crispness of ratings is an unfair assumption. Therefore, suppose that the ratings values are expressed

in form of grey numbers. If $A_1, A_2, \ldots, A_m$ are m possible alternatives and $C_1, C_2, \ldots, C_n$ criteria over which alternatives performance are measured, and $x_{ij} \in \left[\underline{x}_{ij}, \overline{x}_{ij} \right]$ is the rating of alternative A_i, with respect to criterion C_j, then a grey MADM problem can be concisely defined in a decision matrix as follows.

	C_1	C_2	$\cdots$	C_n
A_1	$\left[\underline{x}_{11}, \overline{x}_{11} \right]$	$\left[\underline{x}_{12}, \overline{x}_{12} \right]$	$\cdots$	$\left[\underline{x}_{1n}, \overline{x}_{1n} \right]$
A_2	$\left[\underline{x}_{21}, \overline{x}_{21} \right]$	$\left[\underline{x}_{22}, \overline{x}_{22} \right]$	$\cdots$	$\left[\underline{x}_{2n}, \overline{x}_{2n} \right]$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_m	$\left[\underline{x}_{m1}, \overline{x}_{m1} \right]$	$\left[\underline{x}_{m2}, \overline{x}_{m2} \right]$	$\cdots$	$\left[\underline{x}_{mn}, \overline{x}_{mn} \right]$

In addition, the weigh vector of criteria is defined as $W = \left(w_1, w_2, \ldots, w_n \right)$ that w_j is the importance weight of criterion j. The problem here is to rank the alternative set's elements.

4. Grey LINMAP

In this section, the proposed method of LINMAP-G (Srinivasan, Shocker 1973; Hwang, Yoon 1981) with grey data is developed. It is noted that the LINMAP-G method seeks a positive ideal solution (PIS) and a weight vector w that minimizes the distance of a set of preference relations among alternatives that are expressed priori by decision makers from unknown PIS.

4.1. Normalization of Grey decision matrix

An intrinsic aspect of MADM problems is that different attributes have different dimensions that make their comparison impossible. Therefore, an initial step before the decision making process, is to normalize the grey decision matrix, defined in previous section. Different procedures are introduced to normalize grey decision matrix. In this paper the method proposed in Jahanshahloo *et al.* (Jahanshahloo *et al.* 2006) is applied by modifications. If attribute j is as profit (maximization) type, its normalized values are calculated as:

$$n_{ij} = \left[\frac{\underline{x}_{ij}}{\sqrt{\sum_{i=1}^{m} \underline{x}_{ij}^2 + \overline{x}_{ij}^2}}, \frac{\overline{x}_{ij}}{\sqrt{\sum_{i=1}^{m} \underline{x}_{ij}^2 + \overline{x}_{ij}^2}} \right]. \tag{7}$$

Otherwise, if attribute j is as cost (minimization) type, its normalized values are calculated as:

$$n_{ij} = \left[1 - \frac{\overline{x}_{ij}}{\sqrt{\sum_{i=1}^{m} \underline{x}_{ij}^2 + \overline{x}_{ij}^2}}, 1 - \frac{\underline{x}_{ij}}{\sqrt{\sum_{i=1}^{m} \underline{x}_{ij}^2 + \overline{x}_{ij}^2}} \right]. \tag{8}$$

4.2. Grey LINMAP modeling process

The main idea of LINMAP-G method is to determine an unknown PIS vector, like $PIS = A^* = \left(x_1^*, x_2^*, \ldots, x_n^* \right)$, where $x_i^* = \left[\underline{x}_i^*, \overline{x}_i^* \right]$. Then, the best alternative is chosen as the nearest one to this PIS vector. A note here is to define the distance between two grey vectors.

If $A_i = \left(x_{i1}, x_{i2}, \ldots, x_{in} \right)$ and $A^* = \left(x_1^*, x_2^*, \ldots, x_n^* \right)$ are two grey vectors, and $W = \left(w_1, w_2, \ldots, w_n \right)$ is a weight vector, the weighted grey numbers Euclidean distance is defined as follows:

$$d_i = \sqrt{\sum_{j=1}^{n} w_j \left[\left(\underline{x}_{ij} - \underline{x}_j^* \right)^2 + \left(\overline{x}_{ij} - \overline{x}_j^* \right)^2 \right]}. \tag{9}$$

Now, the variable $s_i = d_i^2$ is defined. Suppose that decision maker specified an order relations set between alternatives as Ω, where each $(k, l) \in \Omega$ means that decision maker preferred alternative A_k to alternative A_l.

For a given PIS and weight vector w, alternative A_k is closer to PIS than alternative A_l, if $s_k \leq s_l$. In this case, the ranking obtained by (w, PIS) is consistent with decision maker's preference. Otherwise, if $s_k \succ s_l$ then the ranking obtained by (w, PIS) is inconsistent with decision maker's preference.

The inconsistency between alternatives A_k and A_l ranking based on s_k and s_l with preference relations that are determined by decision maker is measured by an inconsistency index $\left(s_l - s_k \right)^-$:

$$\left(s_l - s_k \right)^- = \begin{cases} s_k - s_l & \text{if } s_k \succ s_l \\ 0 & \text{if } s_k \leq s_l \end{cases}. \tag{10}$$

In fact, the alternatives A_k and A_l ranking is consistent with decision maker's preferences if $s_k \leq s_l$ and $\left(s_l - s_k \right)^-$ will be equal to zero. Otherwise, if $s_k \succ s_l$ the rankings are not consistent and their inconsistency will be equal to $\left(s_l - s_k \right)^- = s_k - s_l$. Therefore:

$$\left(s_l - s_k \right)^- = \max \left(0, s_k - s_l \right). \tag{11}$$

Now, the total inconsistency index is defined as follows:

$$I = \sum_{(k,j) \in \Omega} \left(s_k - s_l \right)^- = \sum_{(k,j) \in \Omega} \max \left(0, s_k - s_l \right). \tag{12}$$

Similarly, the consistency index is defined as follows:

$$\left(s_l - s_k \right)^+ = \begin{cases} s_l - s_k & \text{if } s_k \leq s_l \\ 0 & \text{if } s_k \succ s_l \end{cases}. \tag{13}$$

Eq. (13) can be written as follows:

$$\left(s_l - s_k \right)^+ = \max \left(0, s_l - s_k \right). \tag{14}$$

Therefore, total consistency index is defined as follows:

$$C = \sum_{(k,l) \in \Omega} \left(s_l - s_k \right)^+ = \sum_{(k,l) \in \Omega} \max \left(0, s_l - s_k \right). \tag{15}$$

Note that whether $s_k \leq s_l$ or $s_k \succ s_l$, the following relation is hold:

$$\left(s_l - s_k\right)^+ - \left(s_l - s_k\right)^- = s_l - s_k. \tag{16}$$

The grey LINMAP-G model to determine the PIS $A^* = \left(x_1^*, x_2^*, \ldots, x_n^*\right)$ and weight vector $W = \left(w_1, w_2, \ldots, w_n\right)$ can be constructed as follows:

$$\begin{aligned} &Max \quad C \\ &S.T. \quad C - B \geq h \\ &\qquad w_j \geq \varepsilon \quad j = 1, 2, \ldots, n, \end{aligned} \tag{17}$$

where, h is a constant that is determined by decision maker. Also, $\varepsilon \succ 0$ is a sufficiently small real value that guarantees that obtained weights are greater than zero. The objective of model (17) is to maximize the consistency index C, while it will be greater than I at least as pre-determined value of h.

Using Eq. (15) and (16), the model (17) is translated to:

$$\begin{aligned} &Max \quad \sum_{(k,l)\in\Omega} \max\left(0, s_l - s_k\right) \\ &S.T. \quad \sum_{(k,l)\in\Omega} \left(s_l - s_k\right) \geq h \\ &\qquad w_j \geq \varepsilon \quad j = 1, 2, \ldots, n. \end{aligned} \tag{18}$$

The variable λ_{kl} is introduced as follows:

$$\lambda_{kl} = \max\left(0, s_l - s_k\right). \tag{19}$$

For each pair $(k,l) \in \Omega$ the following relations are hold:

$$\lambda_{kl} \geq 0, \tag{20}$$

and

$$\lambda_{kl} \geq s_l - s_k. \tag{21}$$

Substituting Eq. (19) – (21), the model (18) is transformed to the following model:

$$\begin{aligned} &Max \quad \sum_{(k,l)\in\Omega} \lambda_{kl}, \\ &S.T. \quad h + \sum_{(k,l)\in\Omega} \left(s_k - s_l\right) \leq 0, \\ &\qquad s_k - s_l + \lambda_{kl} \geq 0 \quad \forall (k,l) \in \Omega, \\ &\qquad \lambda_{kl} \geq 0 \qquad\qquad \forall (k,l) \in \Omega, \\ &\qquad w_j \geq \varepsilon \quad j = 1, 2, \ldots, n. \end{aligned} \tag{22}$$

The final LINMAP-G model is achieved by acquisition of a corresponding relation for $s_k - s_l$. Using Eq. (9) and the definition of variable s_i, this relation is obtained as follows:

$$s_k - s_l = \sum_{j=1}^{n} w_j \left[\left(\underline{x}_{kj} - \underline{x}_j^*\right)^2 + \left(\overline{x}_{kj} - \overline{x}_j^*\right)^2\right] - \sum_{j=1}^{n} w_j \left[\left(\underline{x}_{lj} - \underline{x}_j^*\right)^2 + \left(\overline{x}_{lj} - \overline{x}_j^*\right)^2\right].$$

The extended form of the above equation after calculation of squares and factorization is as follows:

$$s_k - s_l = \sum_{j=1}^{n} w_j \left[\left(\underline{x}_{kj}^2 + \overline{x}_{kj}^2 - \underline{x}_{lj}^2 - \overline{x}_{lj}^2 \right) \right] + \sum_{j=1}^{n} 2 w_j \underline{x}_j^* \left(\underline{x}_{lj} - \underline{x}_{kj} \right)$$

$$+ \sum_{j=1}^{n} 2 w_j \overline{x}_j^* \left(\overline{x}_{lj} - \overline{x}_{kj} \right). \tag{23}$$

The model (22) is now transformed into the following model, which is called grey LINMAP-G model.

$$Max \sum_{(k,l) \in \Omega} \lambda_{kl},$$

$$S.T. \quad h + \sum_{j=1}^{n} w_j \sum_{(k,l) \in \Omega} \left(\underline{x}_{kj}^2 + \overline{x}_{kj}^2 - \underline{x}_{lj}^2 - \overline{x}_{lj}^2 \right) + \sum_{j=1}^{n} 2 \underline{v}_j \sum_{(k,l) \in \Omega} \left(\underline{x}_{lj} - \underline{x}_{kj} \right)$$

$$+ \sum_{j=1}^{n} 2 \overline{v}_j \sum_{(k,l) \in \Omega} \left(\overline{x}_{lj} - \overline{x}_{kj} \right) \leq 0,$$

$$\sum_{j=1}^{n} w_j \left[\left(\underline{x}_{kj}^2 + \overline{x}_{kj}^2 - \underline{x}_{lj}^2 - \overline{x}_{lj}^2 \right) \right] + \sum_{j=1}^{n} 2 \underline{v}_j \left(\underline{x}_{lj} - \underline{x}_{kj} \right) \tag{24}$$

$$+ \sum_{j=1}^{n} 2 \overline{v}_j \left(\overline{x}_{lj} - \overline{x}_{kj} \right) + \lambda_{kl} \geq 0 \quad \forall (k,l) \in \Omega,$$

$$\lambda_{kl} \geq 0 \qquad \forall (k,l) \in \Omega,$$

$$\underline{v}_j \leq \overline{v}_j \quad j = 1, 2, \ldots, n,$$

$$w_j \geq \varepsilon \quad j = 1, 2, \ldots, n,$$

where,

$$\underline{v}_j = w_j \underline{x}_j^*, \tag{25}$$

and

$$\overline{v}_j = w_j \overline{x}_j^*. \tag{26}$$

Note that constraints $\underline{v}_j \leq \overline{v}_j \; j = 1, 2, \ldots, n$ are added to guarantee the grey property of obtained PIS. By solving the model (24), the optimal values of $\underline{v}_j$, $\overline{v}_j$ and w_j are determined. Then, the PIS solution $A^* = \left(x_1^*, x_2^*, \ldots, x_n^* \right)$ and weight vector $W = \left(w_1, w_2, \ldots, w_n \right)$ are determined. The optimal weights of attributes can be determined after normalization of weight vector W. Finally, the ranking of alternatives are specified by calculation of s_i variables for all alternatives and ascending sort of these values.

Figure 1 shows an algorithm about the decision making process with LINMAP-G method. It is possible that decision maker has some viewpoints regard to weight vector, such that he/she do not want none of the attribute's weights be greater than other's weights. This set of constraints can be added to model as $u_{kj} \leq w_k / w_j \leq l_{kj}, k, j = 1, 2, \ldots, n, k \neq j$.
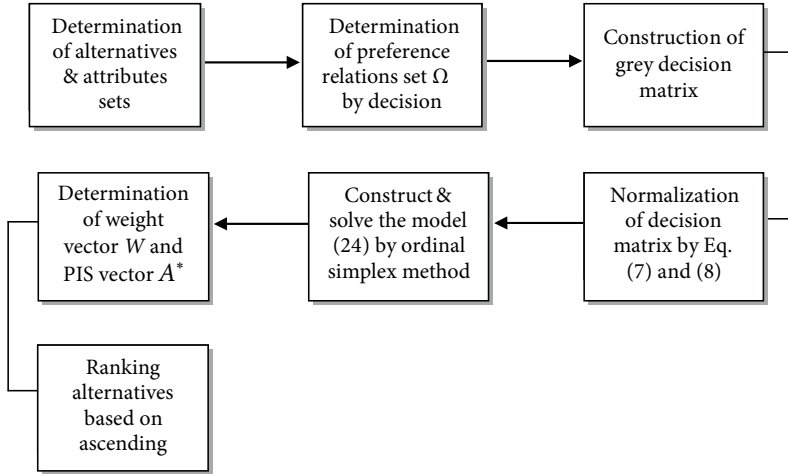
Fig. 1. Decision making process by LIMAP-G model

5. Numerical example

In this section, two numerical examples are solved by proposed LINMAP-G decision making process.

5.1. Ranking of constructing projects

The first study is done on a relatively small instance. This example is about a company that wants to rank its target market sectors. The company's market is divided into five different sectors A, B, C, D, and E that are evaluated based on four attributes: three attributes include (1) market size, (2) market growth and (3) consistency with company's mission as profit attributes and a (4) structural risk attribute as cost attributes. The grey decision matrix is constructed as follows (see Table 1).

Table 1. Grey decision matrix (example 1)

	1	2	3	4
A	[6, 7]	[3, 4]	[4, 5]	[6, 7]
B	[4, 5]	[5, 6]	[5, 6]	[6, 7]
C	[5, 7]	[6, 7.5]	[4, 5]	[3, 4]
D	[7, 8]	[4, 5]	[6, 8]	[5, 6]
E	[6, 8]	[5, 6]	[7, 9]	[7, 8]

Assume that decision makers have specified their preferences between alternatives as $\Omega = \left\{ (2,1), (3,2), (4,3), (5,4) \right\}$. The first step is to normalize the decision matrix. Attributes 1-3 are normalized based on Eq. (7) and the attribute 4 by Eq. (8). The normalized decision matrix is shown in Table 2.

Table 2. Normalized grey decision matrix (example 1)

	1	2	3	4
A	[0.295, 0.344]	[0.179, 0.239]	[0.207, 0.259]	[0.649, 0.699]
B	[0.197, 0.246]	[0.299, 0.358]	[0.259, 0.311]	[0.598, 0.649]
C	[0.246, 0.344]	[0.358, 0.448]	[0.207, 0.259]	[0.799, 0.849]
D	[0.344, 0.394]	[0.299, 0.311]	[0.311, 0.414]	[0.699, 0.749]
E	[0.295, 0.394]	[0.358, 0.362]	[0.362, 0.466]	[0.598, 649]

In the next step, the grey LINMAP-G method is developed according to Eq. (24) as follows. Assume that decision maker wants that none of the attributes weights be more than three times greater than the others. Note that $h = 1$ and $\varepsilon = 0.001$.

$$Max \ \lambda_{21} + \lambda_{32} + \lambda_{34} + \lambda_{54}$$

$$1 + 0.0363w_1 + 0.1285w_2 + 0.2386w_3 - 0.1302w_4$$

$$+2\left(-0.1195\underline{v}_2 - 0.1553\underline{v}_3\right)$$

$$+2\left(-0.0492\overline{v}_1 - 0.1195\overline{v}_2 - 0.2071\overline{v}_3 + 0.0502\overline{v}_4\right) \leq 0$$

$$-0.1065w_1 + 0.1285w_2 + 0.0536w_3 - 0.1302w_4$$

$$+2\left(0.098\underline{v}_1 - 0.119\underline{v}_2 - 0.052\underline{v}_3 + 0.05\underline{v}_4\right)$$

$$+2\left(-0.098\overline{v}_1 - 0.119\overline{v}_2 - 0.052\overline{v}_3 + 0.05\overline{v}_4\right) + \lambda_{21} \leq 0$$

$$0.0799w_1 + 0.1115w_2 - 0.0536w_3 + 0.5814w_4$$

$$+2\left(-0.049\underline{v}_1 - 0.06\underline{v}_2 + 0.052\underline{v}_3 - 0.201\underline{v}_4\right)$$

$$+2\left(-0.098\overline{v}_1 - 0.09\overline{v}_2 + 0.0521\overline{v}_3 - 0.201\overline{v}_4\right) + \lambda_{32} \leq 0$$

$$-0.0944w_1 + 0.1829w_2 - 0.1582w_3 + 0.3108w_4$$

$$+2\left(0.098\underline{v}_1 - 0.119\underline{v}_2 + 0.104\underline{v}_3 - 0.1\underline{v}_4\right)$$

$$+2\left(0.049\overline{v}_1 - 0.149\overline{v}_2 + 0.155\overline{v}_3 - 0.1\overline{v}_4\right) + \lambda_{34} \leq 0$$

$$0.0315w_1 + 0.0714w_2 + 0.0804w_3 - 0.2705w_4$$

$$+2\left(0.049\underline{v}_1 - 0.6\underline{v}_2 - 0.052\underline{v}_3 + 0.1\underline{v}_4\right)$$

$$+2\left(-0.06\overline{v}_2 - 0.052\overline{v}_3 + 0.1\overline{v}_4\right) + \lambda_{54} \leq 0$$

$$w_k/w_j \leq 3, \ k, j = 1, 2, \ldots, n, k \neq j$$

$$\lambda_{21}, \lambda_{32}, \lambda_{34}, \lambda_{54} \geq 0$$

$$\underline{v}_j \leq \overline{v}_j \ \ j = 1, 2, 3, 4$$

$$w_j \geq 0.001 \ \ \ \ j = 1, 2, 3, 4$$

The optimal solution of the above model is as follows:

$$W^* = \left(w_1, w_2, w_3, w_4\right) = \left(0.35, 1.05, 1.05, 0.35\right),$$

$$V^* = \left(\left(0.299, 0.299\right), \left(0, 0\right), \left(0, 3.54\right), \left(0.713, 0.713\right)\right).$$

Now, the PIS A^* can be derived as $A^* = V^*/W^*$

$$A^* = \left((0.854, 0.854), (0,0), (0, 3.37), (2.037, 2.037) \right).$$

Now, the square distances s_i from PIS A^* are $s_1 = 11.813$, $s_2 = 11.816$, $s_3 = 11.813$, $s_4 = 10.810$ and $s_5 = 10.813$. Therefore, the ranking order of alternatives are $D \succ E \succ A = C \succ B$.

5.2. Contractor ranking

This example is solved in (Jahanshahloo *et al.* 2006) through grey TOPSIS method. The problem is to rank 15 bank branches based on four financial attributes. Grey decision matrix is shown in Table 3. Assume that the decision maker determines his/her preferences between branches as follows:

$$\Omega = \left\{ \begin{array}{l} (1,2), (3,2)(1,3)(3,4), (4,5), (6,4), (6,7), (8,7), \\ (9,8), (9,10), (10,11), (12,11), (13,12), (13,14), (15,14) \end{array} \right\}.$$

Table 3. Grey decision matrix (example 2)

	C_1	C_2	C_3	C_4
A_1	[500.37, 961.37]	[2696995, 3126798]	[26364, 38254]	[965.97, 6957.33]
A_2	[873.7, 1775.5]	[1027546, 1061260]	[3791, 50308]	[2285.03, 3174]
A_3	[95.93, 196.39]	[1145235, 1213541]	[22964, 26846]	[207.98, 510.93]
A_4	[848.07, 1752.66]	[390902, 395241]	[492, 1213]	[63.32, 92.3]
A_5	[58.69, 120.47]	[144906, 165818]	[18053, 18061]	[176.58, 370.81]
A_6	[464.39, 955.61]	[408163, 416416]	[40539, 48643]	[4654.71, 5882.53]
A_7	[155.29, 342.89]	[335070, 410427]	[33797, 44933]	[560.26, 2506.67]
A_8	[1752.31, 3629.54]	[700842, 768593]	[1437, 1519]	[58.89, 86.86]
A_9	[244.34, 495.78]	[641680, 696338]	[11418, 24108]	[1070.81, 2283.08]
A_{10}	[730.27, 1417.11]	[453170, 481943]	[2719, 2955]	[375.07, 559.85]
A_{11}	[454.75, 931.24]	[309670, 642598]	[2016, 2617]	[936.62, 1468.45]
A_{12}	[303.58, 630.01]	[286149, 317186]	[14918, 27070]	[1203.79, 4335.24]
A_{13}	[658.81, 1345.58]	[321435, 347848]	[6616, 8045]	[200.36, 399.8]
A_{14}	[420.18, 860.79]	[618105, 835839]	[24425, 40457]	[2781.24, 4555.42]
A_{15}	[144.68, 292.15]	[119948, 120208]	[1494, 1749]	[282.73, 471.22]

The next step according to Figure 1 is to normalize the grey decision matrix. Table 4 shows the normalized decision matrix (while all attributes are as maximizing type, Eq. (7) is used here). Then, the grey model (24) is constructed and solved. In this model $h = 1$ and $\varepsilon = 0.01$ are considered. The obtained solution is as follows:

$$W^* = (0.086, 0.086, 0.173, 0.173), V^* = \left((0,0), (0.1332, 0.385), (0, 0.1), (0,0) \right).$$

Table 4. Normalized grey decision matrix (example 2)

	C_1	C_2	C_3	C_4
A_1	[0.0856, 0.1645]	[0.5176, 0.6001]	[0.1974, 0.2865]	[0.0706, 0.5086]
A_2	[0.1495, 0.3038]	[0.1972, 0.2037]	[0.0283, 0.3768]	[0.1670, 0.2320]
A_3	[0.0164, 0.0336]	[0.2198, 0.2329]	[0.1720, 0.2010]	[0.0152, 0.0373]
A_4	[0.1451, 0.2999]	[0.0750, 0.0758]	[0.0036, 0.0090]	[0.0046, 0.0067]
A_5	[0.0100, 0.0206]	[0.0278, 0.0318]	[0.1352, 0.1352]	[0.0129, 0.0271]
A_6	[0.0794, 0.1635]	[0.0783, 0.0799]	[0.3036, 0.3643]	[0.3403, 0.4300]
A_7	[0.0265, 0.0586]	[0.0643, 0.0787]	[0.2531, 0.3365]	[0.0409, 0.1832]
A_8	[0.2999, 0.6211]	[0.1345, 0.1475]	[0.0107, 0.0113]	[0.0043, 0.0063]
A_9	[0.0418, 0.0848]	[0.1231, 0.1336]	[0.0855, 0.1805]	[0.0782, 0.1669]
A_{10}	[0.1249, 0.2425]	[0.0869, 0.0925]	[0.0203, 0.0221]	[0.0274, 0.0409]
A_{11}	[0.0788, 0.1593]	[0.0594, 0.0657]	[0.0151, 0.0196]	[0.0684, 0.1073]
A_{12}	[0.0519, 0.1078]	[0.0549, 0.0608]	[0.1117, 0.2027]	[0.0880, 0.3169]
A_{13}	[0.1127, 0.2302]	[0.0616, 0.0667]	[0.0495, 0.0602]	[0.0146, 0.0292]
A_{14}	[0.0719, 0.1473]	[0.1186, 0.1604]	[0.1829, 0.3030]	[0.2033, 0.3330]
A_{15}	[0.0247, 0.0500]	[0.0230, 0.0230]	[0.0111, 0.0131]	[0.0206, 0.0344]

Now, the PIS A^* can be derived as $A^* = \big((0,0), (1.54, 4.47), (0.0.578), (0,0) \big)$. Table 5 shows the square distances and ranking of alternatives by proposed.

Table 5. Square distances and ranking of alternatives (example 2)

	S_i	ranking
A_1	1.4479	1
A_2	1.7515	3
A_3	1.7240	2
A_4	1.9107	11
A_5	1.9279	14
A_6	1.9199	13
A_7	1.8733	7
A_8	1.8732	6
A_9	1.8251	5
A_{10}	1.8899	8
A_{11}	1.9162	12
A_{12}	1.9081	9.5
A_{13}	1.9081	9.5
A_{14}	1.8185	4
A_{15}	1.9544	15

6. Conclusions

Constructing project selection is an important issue, according to high risks and costs of mistakes. Therefore decision making based on multiple attributes eases it to prevent these likely problems. In construction evaluation problems, the decision maker selects some alternatives among different ones and it is necessary to consider different qualitative and quantitative criteria.

Evaluation of a set of alternatives regard to a set of quantitative or qualitative attributes is the main concentration of multi attribute decision making problems. In crisp MADM algorithms, subjective judgments and qualitative measures are translated into crisp numbers. This transformation means that decision maker ignores the uncertainty and ambiguity of his/her thinking and believes. Therefore some frameworks are presented to handle these uncertainties. According to uncertainty in the real world, we tried to calculate these parameters by uncertain data and in this paper, a new version of LINMAP method, originally presented in (Srinivasan, Shocker 1973), where decision maker's judges are expressed as grey number is proposed. The proposed method ranks alternatives by solving a linear programming that determines the attributes weight vector and an ideal solution. Then, the alternatives are ranked regard to their distances from PIS by specified weights. Application of the developed method is shown in two constructing examples that one of them was about ranking a set of various constructing projects for a developer company and another example was about ranking a set of constructing contractors. This suggests that the proposed method can be applied in different multi attribute decision making problems which contain uncertainty and ill-defined data and decision maker has not determined attributes weights priori.

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