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RATIONAL ROLLING BALL BLENDING OF NATURAL QUADRICS

K. KARČIAUSKAS[1], R. KRASAUSKAS [1]

Vilnius University
Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, 2600 Vilnius, Lithuania

E-mail: kestutis.karciauskas@maf.vu.lt, rimvydas.krasauskas@maf.vu.lt

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ABSTRACT

We construct a blending surface of two natural quadrics using rational variable rolling ball approach, i.e. as a canal surface with a rational spine curve and a rational radius. All general positions of the given quadric surfaces are considered. The proposed construction is Laguerre invariant. In particular, the blending surface has rational offset of the same degree.

1. INTRODUCTION

Rolling ball blends with fixed radius between two surfaces are frequently used in geometric modeling. Though in literature one can find satisfactory solutions in simple cases (see e.g. [8]), in general blending surfaces occur to be irrational. Hence they cannot be exactly represented as Bézier or B-spline surfaces.

Our idea is to allow a variable radius of the rolling ball in order to obtain a rational blending surface. We consider a blending problem between two natural quadrics (namely: sphere, circular cylinder or circular cone) in close general position. Our blending surface is a ring-shaped piece of a canal surface with a rational spine curve and radius. This was done earlier only for exceptional situations when Dupin cyclides were used (see, e.g. [7; 9]).

In section 2 we introduce preliminaries of Laguerre geometry. We sketch the proposed blending construction and classify general positions (table 4, fig. 1) in sections 3 and 4. Actual blendings in four canonical cases are explained in

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section 5.

2. LAGUERRE GEOMETRIC PRELIMINARIES

It is convenient to consider natural quadrics in the framework of Laguerre geometry [6; 3]. In so-called cyclographic model of Laguerre geometry oriented spheres in euclidean space $\mathbb{R}^3$ are represented by points in $\mathbb{R}^4$: first three coordinates are for the center and the last coordinate is for the signed radius. The euclidean metric in $\mathbb{R}^3$ is extended to the pseudo- euclidean metric in $\mathbb{R}^4$. It is called a *pe metric* and is defined via the following inner product with the associated norm:

$$\langle \vec{v}, \vec{w} \rangle_{\text{pe}} = v_1 w_1 + v_2 w_2 + v_3 w_3 - v_4 w_4, \tag{2.1}$$

$$\|\vec{v}\|_{\text{pe}} = \sqrt{\langle \vec{v}, \vec{v} \rangle_{\text{pe}}}\ , \quad \text{if } \langle \vec{v}, \vec{v} \rangle_{\text{pe}} \geq 0. \tag{2.2}$$

A pe distance between two points has a clear geometric meaning: it is equal to a tangential distance between both corresponding spheres which do not contain each other. In fact the space $\mathbb{R}^4$ is well-known Minkowski space $\mathbb{R}^4_{1,3}$ Its projective extension $\mathbb{P}^4$ contains the absolute quadric Ω: $x_0 = 0$, $x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$, which plays an important role.

In the rest of this section we sketch some definitions and facts from Laguerre geometry. All details can be found in [3].

em Laguerre transformations are special affine transformations of $\mathbb{R}^4$ keeping the absolute quadric Ω invariant. They preserve tangential distances between oriented spheres. Lines in $\mathbb{R}^4$ are classified from the Laguerre point of view depending on their directional vectors $\vec{l}$: they are called *elliptic* (resp. *parabolic*, *hyperbolic*) if $\langle \vec{l}, \vec{l} \rangle_{\text{pe}} < 0$ (resp. $= 0$, > 0).

Any surface $V \subset \mathbb{R}^3$ has its associated *isotropic hypersurface* $\Gamma(V) \subset \mathbb{R}^4$ consisting of all points corresponding to spheres touching V. It is important that a d-offset of V is directly obtained from $\Gamma(V)$ as the hyperplane $x_4 = d$ section:

$$d\text{-offset } V = \Gamma(V) \cap \{x_4 = d\}. \tag{2.3}$$

In case when V is a natural quadric $\Gamma(V)$ is easy calculated (see Example 4 and Lemma 5 in [3]). Let S and R be a sphere and a cone (or cylinder) then

$$\Gamma(S) = s * \Omega, \quad \Gamma(R) = L * (L^{\perp}_{\text{pe}} \cap \Omega). \tag{2.4}$$

Here $*$ means linear join (i.e., a union of all lines going through both sets surrounding a symbol $*$), s—a point corresponding to the sphere S, L—a hyperbolic line, which defines a family of spheres with the envelop R, $L^{\perp}_{\text{pe}}$—any hyperplane pe orthogonal to L.

3. BLENDING CONSTRUCTION SCHEME

Consider two natural quadrics Q_1 and Q_2 in euclidean space $\mathbb{R}^3$. All spheres touching both given quadrics define a 2D surface $T = \Gamma(Q_1) \cap \Gamma(Q_2)$ in $\mathbb{R}^4$. Any rational curve $\gamma \subset T$ represents a family of spheres with rational variable radius. Its envelope is some rational canal surface $C \subset \mathbb{R}^3$. By the construction the surface C touches both quadrics Q_1 and Q_2, so it can be used for their blending. Hence the initial problem is reduced to an appropriate choice of the curve γ which should be rational.

Suppose such rational curve $\gamma(t)$ is already given. Denote by $\gamma_i(t)$ the curves of contact between the canal surface C and corresponding quadrics Q_i, $i = 1, 2$. A calculation of $\gamma_i(t)$ is easy but depends on quadric type. If Q_i is a sphere S then γ_i is just a central projection of γ with the center s (see (2.4)) to the hyperplane $\mathbb{R}^3$: $x_4 = 0$. If Q_i is a cone/cylinder R then γ_i is obtained in two steps: at first γ is projected from the 2D center on infinity $L^{\perp}_{\mathrm{pe}} \setminus \mathbb{R}^4$ (see (2.4)) to the line L and then the line connecting a point $\gamma(t)$ and its image is intersected with the same hyperplane $\mathbb{R}^3$. We see from the whole construction that both curves γ_i are rational, since they inherit their parameterization from the given rational curve γ.

Now it remains to find a parameterization of a 'ring shaped' patch of the canal surface C bounded by curves γ_1 and γ_2. For $i = 1, 2$ denote by $\delta_i(t)$ the infinite point of a line connecting points $\gamma_i(t)$ and $\gamma(t)$. Then $\delta_i \subset \Omega$. Here we treat Ω as a sphere (see Remark 18 in [3]) in the infinite 3D hyperplane $\mathbb{P}^4 \setminus \mathbb{R}^4$. At first parameterize a spherical patch of Ω via a variable circle arcs with endpoints $b_0 = \delta_1(t)$ and $b_2 = \delta_2(t)$. A middle control point of the circle is determined via intersection of three planes: two tangent planes to Ω in endpoints and an infinite plane of $\dot{\gamma}(t)^{\perp}_{\mathrm{pe}}$. Denote this parameterization by $F(t, u)$ (u is a circle parameter). Finally the parameterization $G(t, u)$ of the canal surface C is obtained via intersecting a line going through points $F(t, u)$ and $\gamma(t)$ with the hyperplane $x_4 = 0$.

4. CLASSIFICATION OF DIFFERENT POSITIONS

Since all previous constructions are Laguerre invariant, we can essentially reduce our considerations to some canonical positions (see fig. 1) of given natural quadrics: all other cases are Laguerre equivalent to them. From (2.4) follows that spheres and cones/cylinders are encoded by points and hyperbolic lines in $\mathbb{R}^4$ respectively. Hence we need to consider positions of pairs point–line and line–line in $\mathbb{R}^4$ and classify them from the Laguerre point of view.

Let p and L be a point and a hyperbolic line. Their affine span is a 2D plane P which can be hyperbolic, parabolic and elliptic (see Laguerre classification of planes in [3]). We skip parabolic and elliptic cases, as non-generic and too simple (Dupin cyclides can be used) respectively. In the hyperbolic case we can suppose the plane $P \subset \mathbb{R}^3$.

Now let L_1, L_2 be skew hyperbolic lines. (If they intersect then Dupin

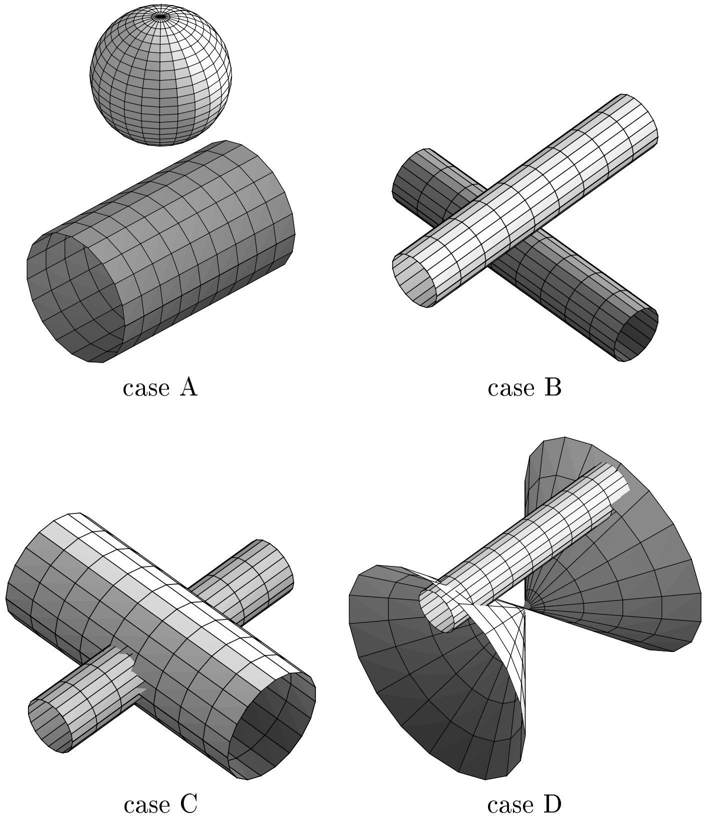

Figure 1. Four cases of two quadrics.

cyclides can be used.) Similar considerations as above leads to three different cases B, C, D. They all are shown in the table 1. All these cases are not Laguerre equivalent as we see from the last column, where a signature of an affine span of $L_1 \cup L_2$ is shown. Here a signature of 2D subspace associated with line directional vectors is enclosed in brackets. We skip various parabolic cases (i.e. with degenerated metrics) as non-generic.

Every canonical case corresponds via offsetting construction (2.3) to special position of pairs of natural quadrics in $\mathbb{R}^3$ (see fig. 1). Some of positions have additional parameters: for example, in case B an angle between lines is Laguerre invariant. Fortunately, different values of these parameters give essentially the same situations.

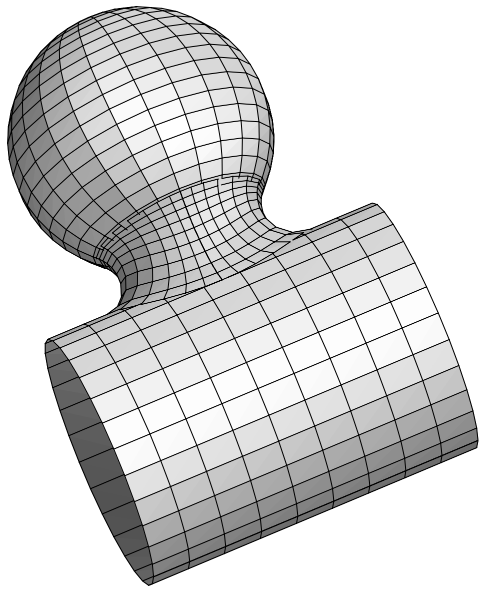

Figure 2. Blending in case A.

5. BLENDING IN FOUR CANONICAL POSITIONS

5.1. Case A

Consider a sphere Q_1 with a center in the origin and a cylinder Q_2 with an axis parallel to the x_1-axis and in distance h from it. They both are of the same diameter d (remember d-offset!). Equations of isotropic hyperquadrics $\Gamma(Q_i)$ in homogeneous coordinates of $\mathbb{P}^4$ are easy derived (here we applied $(-d)$-offsetting for simplicity):

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= x_4^2, \\ (x_2 - hx_0)^2 + x_3^2 &= x_4^2. \end{aligned} \tag{5.1}$$

Table 1.
Different positions of two natural quadrics

Case	Types of quadrics	Representation in 4D	Signature
A	sphere and cylinder	point and line	++
B	two cylinders	two lines	(++)+
C	two cylinders	two lines	(++)−
D	cylinder and cone	two lines	(+−)+

Both these equations defines an intersection surface $T = \Gamma(Q_1) \cap \Gamma(Q_2)$. They are equivalent to the following system of equations

$$\begin{aligned} x_1^2 + x_2^2 &= (x_2 - hx_0)^2, \\ x_4^2 - x_3^2 &= (x_2 - hx_0)^2. \end{aligned} \tag{5.2}$$

Consider a biquadratic parameterization of the surface T:

$$\begin{aligned} x_0 &= -2t_1^2(u_0^2 - u_1^2)/h, \\ x_1 &= 2t_0t_1(u_0^2 - u_1^2), \\ x_2 &= (t_0^2 - t_1^2)(u_0^2 - u_1^2), \\ x_3 &= 2(t_0^2 + t_1^2)u_0u_1, \\ x_4 &= (t_0^2 + t_1^2)(u_0^2 + u_1^2). \end{aligned} \tag{5.3}$$

Define a curve $\gamma(t)$ of degree 8 by substituting

$$t_0 = a(1 - t^2), \quad t_1 = 2bt, \quad u_0 = u_1 = 1 + t^2.$$

to (5.3). It is the smallest possible degree of a rational curve on T with the desired topology of its position. Indeed, T is isomorphic to the spindle torus: see [2; 1], where rational curves on such surfaces are investigated in details. Taking $h = 2$, $d = 0.9$, $a = b = 0.4$ and applying the procedure of section 3 we get fig. 2.

5.2. Case B

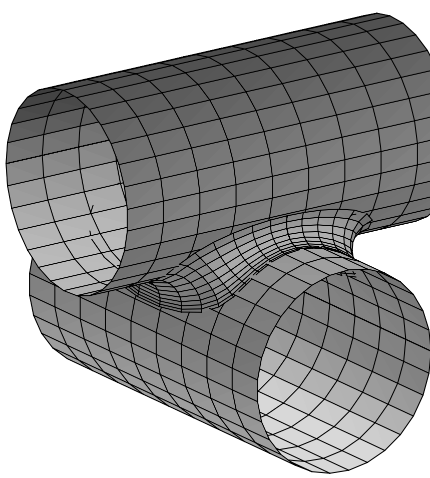

Figure 3. Blending in case B.

Consider two skew cylinders Q_1 and Q_2 with orthogonal axes (in distance h from each other), both of the same radius d. Equations of isotropic hyperquadrics $\Gamma(Q_i)$ in homogeneous coordinates of $\mathbb{P}^4$ are the following (here we applied $(-d)$-offsetting for simplicity):

$$\begin{aligned} x_2^2 + (x_3 - hx_0)^2 &= x_4^2, \\ x_1^2 + (x_3 + hx_0)^2 &= x_4^2. \end{aligned} \tag{5.4}$$

Both these equations defines an intersection surface $T = \Gamma(Q_1) \cap \Gamma(Q_2)$. Consider its biquadratic parameterization

$$\begin{aligned} x_0 &= (t_0^2 u_0^2 - t_1^2 u_1^2)/h, \\ x_1 &= 2(t_0^2 + t_1^2) u_0 u_1, \\ x_2 &= 2(u_0^2 + u_1^2) t_0 t_1, \\ x_3 &= t_1^2 u_0^2 - t_0^2 u_1^2, \\ x_4 &= (t_0^2 + t_1^2)(u_0^2 + u_1^2). \end{aligned} \tag{5.5}$$

Similarly to case A define a curve $\gamma(t)$ of degree 8 by substituting

$$t_0 = a(1 - t^2), \quad t_1 = 2bt, \quad u_0 = u_1 = 1 + t^2.$$

to (5.5). It is also the smallest possible degree of a rational curve on T with the desired topology of its position. Now T is isomorphic to the ring torus (see [2; 1]). Taking $h = 1.3$, $d = 1.6$, $a = b = 0.7$ and applying the procedure of section 3 we get fig. 3.

5.3. Case C

Consider two cylinders Q_1 and Q_2 with axes crossing each other orthogonally and with radii $h + d$ and d. After $(-d)$-offsetting equations of isotropic hyperquadrics $\Gamma(Q_i)$ in homogeneous coordinates of $\mathbb{P}^4$ are the following

$$\begin{aligned} x_2^2 + x_3^2 &= (x_4 - hx_0)^2, \\ x_1^2 + x_3^2 &= x_4^2. \end{aligned} \tag{5.6}$$

We simplify this system of equations (which defines $T = \Gamma(Q_1) \cap \Gamma(Q_2)$) applying the following substitution

$$\begin{aligned} x_0 &= (-y_1 - y_2 + y_3 + y_4)/h, \\ x_1 &= y_3 - y_4, \\ x_2 &= y_1 - y_2, \\ x_3 &= 2y_0, \\ x_4 &= y_3 + y_4. \end{aligned}$$

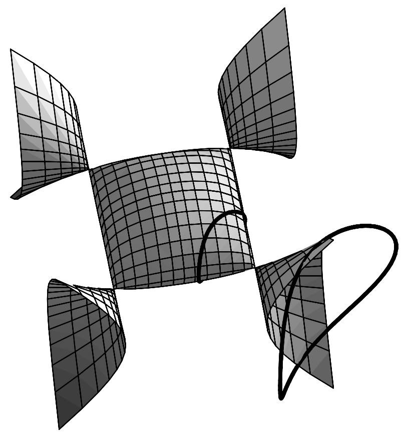

Figure 4. The surface T and two variants of γ projected to $\mathbb{R}^3$.

we get the system

$$\begin{aligned} y_1 y_2 &= y_0^2, \\ y_3 y_4 &= y_0^2. \end{aligned} \tag{5.7}$$

The latter can be parameterized by twisted projective plane $\mathbb{P}(1,1,2)$ (see [2]):

$$y_0 = t_0 t_1 u, \quad y_1 = t_0^2 t_1^2, \quad y_2 = u^2, \quad u_3 = t_0^2 u, \quad u_4 = t_1^2 u. \tag{5.8}$$

In order to define a curve $\gamma(t)$ of degree 4 we substitute $t_0 = t$, $t_1 = 1$, $u = 2(1+t^2)$. This curve is on the 'antenna' of T as it is shown in a projection to $\mathbb{R}^3$ in fig. 4. In fact this is a bisector of two cylinders (cf. [4]). Taking $h = 2$, $d = 1$ and applying the procedure of section 3 we get fig. 5.

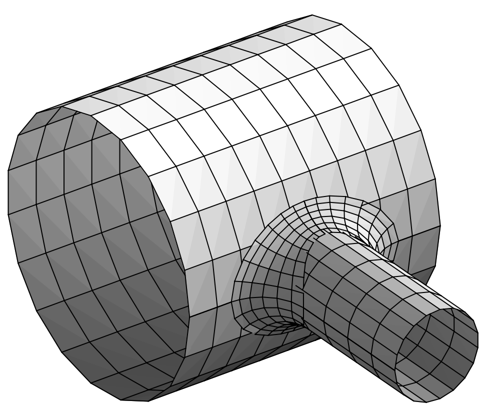

Figure 5. Exterior blending in case C.

Consider slightly different case. Let radii of the given cylinders Q_i be $h-d$ and $-d$. Define $\gamma(t)$ by $t_0 = t$, $t_1 = 1$, $u = -1.5(1+t^2)$. It is shown on the 'pillow' in fig. 4. Taking $h = 2$, $d = 0.5$ we get an interior blending (see fig. 6).

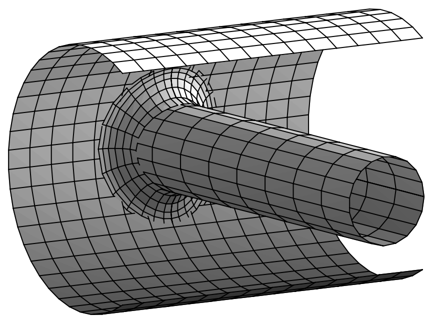

Figure 6. Interior blending in case C.

5.4. Case D

Consider a cylinders Q_1 and a cone Q_2 with parallel axes in distance h from each other (see fig. 1D). The situation corresponds to isotropic hyperquadrics $\Gamma(Q_i)$ given by equations

$$\begin{aligned} x_2^2 + x_3^2 &= (x_1 + \sqrt{2}x_4)^2, \\ (x_2 - hx_0)^2 + x_3^2 &= x_4^2. \end{aligned} \tag{5.9}$$

We simplify this system of equations and applying the following substitution

$$\begin{aligned} x_0 &= (-y_1 + y_2 + y_3 - y_4)/h, \\ x_1 &= -y_1 - y_2 + y_3 + y_4, \\ x_2 &= y_3 - y_4, \\ x_3 &= 2y_0, \\ x_4 &= y_1 + y_2, \end{aligned}$$

we obtain exactly (5.7). Hence the surface $T = \Gamma(Q_1) \cap \Gamma(Q_2)$ is the same (i.e., projectively equivalent) as in case C. Then similar methods leads to the blending in fig. 7.

6. CONCLUSIONS AND FUTURE WORK

We constructed a G^1-blending between two natural quadrics in close general position. The blending surface is a ring-shaped patch of a rational canal surface with rational boundary curves and rational offset. An additional advantage of this construction is its Laguerre invariance. This allows to classify

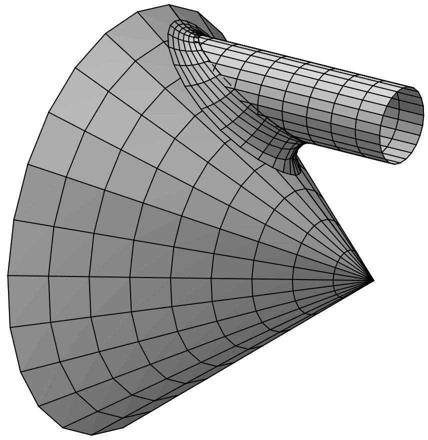

Figure 7. Blending in case B.

all possible positions of a pair of natural quadrics from the Laguerre point of view and to obtain blending solutions only from several canonical cases.

Parabolic cases (so non-generic) were skipped here for simplicity. They will be investigated in the forthcoming paper. Also using B-spline curve γ we will get lower degree blending surfaces.

REFERENCES

[1] R.Krasauskas. Rational Bézier Surface Patches on Quadrics and the Torus. *Preprint 95-25, Vilnius University*, 1995.

[2] R.Krasauskas. Universal parameterizations of some rational surfaces. *In:* Curves and Surfaces with Applications in CAGD, A. Le Méhauté, C. Rabut,C. and L.L. Schumaker (Eds.), Vanderbilt Univ. Press, Nashville, 1997, 231–238.

[3] R.Krasauskas and C.Mäurer. Studying cyclides with Laguerre geometry. *Computer Aided Geometric Design,* 1999. (to appear)

[4] M.Peternell. Geometric Properties of Bisector Surface. *Technical Report, Institut für Geometrie, Technische Universität Wien*, **57**, 1998.

[5] M.Peternell and H.Pottmann. Computing rational parametrizations of canal surfaces. *Journal of Symbolic Computation*, **23**, 1997, 255–266.

[6] M.Peternell and H.Pottmann. A Laguerre geometric approach to rational offsets. *Computer Aided Geometric Design*, **15**, 1998, 223 – 249.

[7] M.J.Pratt. Cyclides in computer aided design. *Computer Aided Geometric Design*, **7**, 1990, 221–242.

[8] M.A.Sanglikar, P.Koparkar and V.N.Joshi. Modelling rolling ball blends for computer aided design. *Computer Aided Geometric Design*, **7**, 1990, 399 – 414.

[9] Y.L.Srinivas and D.Dutta. Blending and joining using cyclides. *ASME J. Mech. Design*, **116**, 1994, 1034 – 1041.

NATŪRALIŲ KVADRIKIŲ JUNGIMAS RACIONALAUS APRIEDANČIO RUTULIUKO METODU

K. KARČIAUSKAS, R. KRASAUSKAS

Natūralios kvadrikos (sferos, apskritiminiai cilindrai ir kūgiai) dažnai naudojamos geometriniame modeliavime. Šiame darbe siūlomas naujas dviejų natūralių kvadrikių glodaus jungimo metodas, naudojant kintamo racionalaus spindulio apriedančio rutuliuko metodą, t.y. jungiamasis paviršius – tai kanalinis paviršius, kuris turi racionalią ašinę kreivę ir racionalų spindulį. Metodas tinka visiems dviejų kvadrikių bendrų pozicijų atvejams. Konstrukcija yra invariantiška Laguerre geometrijos atžvilgiu: pavyzdžiui, jungiamasis paviršius turi to paties laipsnio racionalų ofsetą.