THE ESTIMATES OF ACCURACY OF DIFFERENCE SCHEMES FOR THE NONLINEAR HEAT EQUATION WITH WEAK SOLUTION

B.S. JOVANOVIĆ[1], P.P. MATUS[2] and V.S. SHCHEHLIK[3]

[1] *University of Belgrade, Faculty of Mathematics,*

Studentski trg 16, 11001, Belgrade, Yugoslavia,

[2,3] *Institute of Mathematics, NAS of Belarus,*

Surganov St. 11, 220072, Minsk, Belarus,

E-mail: [1]*bosko@matf.bg.ac.yu,* [2] matus@im.bas-net.by,

E-mail: [3] schehlik@im.bas-net.by

ABSTRACT

In this paper we study the convergence of explicit and implicit finite difference scheme for the first initial–boundary value problem for one dimensional quasilinear heat–conduction equation with "unbounded nonlinearity".

INTRODUCTION

In recent twenty years a great interest was devoted to the construction and analysis of finite difference schemes for approximation of boundary value problems with generalized solutions. In particular, finite difference schemes with convergence rate estimates consistent with the smoothness of data were of the major interest [7].

First results on the study of the convergence of discrete methods for problems with solution from Sobolev spaces were obtained in the theory of finite element methods. However, the methods of constructing difference schemes and obtaining consistent estimates differ from those applied in the finite element method.

For a wide variety of linear problems convergence rate estimates consistent

with the smoothness of data are presented in [7; 6; 9]. However, as a rule, the most of actual problems are nonlinear, and the nature of the nonlinearities is diverse. For quasilinear elliptic equations with bounded nonlinearity, consistent convergence rate estimates have been obtained in [1; 3]. However, the requirements on coefficients of equations, such as positive definiteness, boundedness of partial derivatives over all values $u \in R^n$, narrows down the class of admissible input data of the problem.

The aim of this paper is to construct convergence rate estimates consistent with the smoothness of data for finite difference schemes approximating a nonlinear parabolic equation with generalized solution. Only minimal assumptions on the coefficients of equation are used.

1. INITIAL–BOUNDARY VALUE PROBLEM

In the rectangle $Q_T = \{(x, t) : x \in \Omega = (0, 1), 0 < t < T\}$ for some $T > 0$ we consider the initial–boundary value problem for quasilinear heat–conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right), \quad (x, t) \in Q_T, \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad u(0, t) = u(1, t) = 0, \quad 0 < t < T. \tag{1.2}$$

Let us introduce the Sobolev space $H^s(\Omega)$ of functions $u = u(x)$, defined on Ω, and the anisotropic Sobolev space $H^{s,r}(Q_T)$ of functions $u = u(x, t)$, defined on Q_T. Let $H_0^s(\Omega)$ $(H_0^{s,r}(Q_T))$ denotes the closure of the set of infinitely smooth functions which are equal to 0 in the neighbourhood of $x = 0$ and $x = 1$ in the space $H^s(\Omega)$ $(H^{s,r}(Q_T))$ [2].

DEFINITION 1.1. The element u of the space $H^{2,1}(Q_T) \cap H_0^{1,0}(Q_T)$, satisfying (1.1) almost everywhere and equal $u_0(x)$ for $t = 0$ is called generalized solution of the problem (1.1), (1.2).

Let us define the region of the exact solution $\mathcal{M}_u$ and its neighbourhood $\mathcal{D}_u$:

$$\mathcal{M}_u = \left\{u : u_1 \leq u(x, t) \leq u_2, (x, t) \in \overline{Q}_T\right\}, \tag{1.3}$$

$$\mathcal{D}_u = \left\{\tilde{u} : u_1 - \delta \leq \tilde{u}(x, t) \leq u_2 + \delta, \quad (x, t) \in K \subseteq \overline{Q}_T\right\}. \tag{1.4}$$

We assume, that there exists the unique solution $u(x, t)$ of the problem (1.1), (1.2) in $H^{2,1}(Q_T)$, $u_0 \in H^1(0, 1)$, and $k \in C(\mathcal{D}_u)$, $0 < k_1 \leq k(v) \leq k_2$ for $v \in \mathcal{D}_u$. Note that similar problems are considered in [10; 4].

Let us define the new function $\varphi(u) = \int_0^u k(w)\,dw$ and transform the equation (1.1) in the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \varphi(u)}{\partial x^2}, \quad (x, t) \in Q_T. \tag{1.5}$$

2. EXPLICIT FINITE DIFFERENCE SCHEME

Let us introduce uniform meshes

$$\bar{\omega}_\tau = \left\{ t_j = j\tau, j = \overline{0, j_0}, j_0\tau = T \right\}, \quad \bar{\omega}_h = \left\{ x_i = ih, i = \overline{0, N}, Nh = 1 \right\},$$

and set $\omega_\tau = \bar{\omega}_\tau \cap [0, T)$, $\omega_h = \bar{\omega}_h \cap (0, 1)$ and $\omega_{h\tau} = \omega_h \times \omega_\tau$.

We define the discrete inner product $(v, w)_h = \sum\limits_{x \in \omega_h} v(x)w(x)h$ and norms $\|v\|_h = (v, v)_h^{1/2}$, $\|v\|_{C(\omega_h)} = \max_{x \in \bar{\omega}_h} |v(x)|$, $\|v\|_{h\tau}^2 = \sum\limits_{t \in \omega_\tau} \tau\|v(\cdot, t)\|_h^2$. We use the standard notation of the theory of difference schemes [5].

We define the Steklov averaging operators [7]

$$S_t f(x, t) = \frac{1}{\tau} \int\limits_t^{t+\tau} f(x, t')dt', \quad S_x f(x, t) = \frac{1}{h} \int_{x-h/2}^{x+h/2} f(x', t)dx',$$

$$S_x^2 f(x, t) = \frac{1}{h} \int\limits_{x-h}^{x+h} \left(1 - \frac{|x' - x|}{h} \right) f(x', t)dx',$$

and we approximate the problem (1.5), (1.2) with the explicit finite difference scheme

$$y_t = (\varphi(y))_{\bar{x}x}, \quad (x, t) \in \omega_{h\tau}, \tag{2.1}$$

$$y(x, 0) = S_x^2 u_0(x), \quad x \in \omega_h, \qquad y(0, t) = y(1, t) = 0, \quad t \in \bar{\omega}_\tau. \tag{2.2}$$

Let us consider the convergence of finite difference scheme (2.1), (2.2) in the discrete norm $\|\cdot\|_{h\tau}$. Let us denote $\bar{u} = S_x^2 u$ supposing that the solution $u(x, t)$ of the problem (1.5), (1.2) is oddly extended outside Q_T, i.e.

$$\tilde{u}(x, t) = \begin{cases} -u(-x, t), & x \in (-1, 0], \\ u(x, t), & x \in (0, 1), \\ -u(2 - x, t), & x \in [1, 2). \end{cases}$$

It is easy to see [8], that

$$\|\tilde{u}\|_{H^{2,1}(\bar{Q}_T)}^2 = 3\|u\|_{H^{2,1}(Q_T)}^2, \quad \tilde{Q}_T = \{(x, t) : x \in (-1, 2), 0 < t < T\}.$$

Consequently, $\tilde{u} \in H^{2,1}(\tilde{Q}_T)$.

We define the error in the following manner $z = y - \bar{u}$. This error satisfies the finite difference scheme:

$$z_t = (\varphi(\bar{u} + z) - \varphi(\bar{u}))_{\bar{x}x} - \eta_{\bar{x}x}, \quad (x, t) \in \omega_{h\tau}, \quad \eta = S_t\varphi(u) - \varphi(\bar{u}), \tag{2.3}$$

$$z(x,0) = 0, \quad x \in \omega_h, \qquad z(0,t) = z(1,t) = 0, \quad t \in \overline{\omega}_\tau. \qquad (2.4)$$

Let us define the space H of functions defined on the mesh $\overline{\omega}_h$ which vanish for $x = 0$ and $x = 1$. In the space H with inner product $(\cdot, \cdot)_h$ and norm $\|\cdot\|_h$ we define the linear operator $Av = -v_{\bar{x}x}$.

It follows from $\varphi(0) = 0$, $\varphi(u(0,t)) = \varphi(u(1,t)) = 0$ and the definition of η that $\eta(0,t) = \eta(1,t) = 0$. Consequently, the problem (2.3), (2.4) can be represented in operator form

$$z_t + A\kappa z = A\eta, \quad t \in \overline{\omega}_\tau, \qquad z(0) = 0, \qquad (2.5)$$

where $\varphi(\bar{u} + z) - \varphi(\bar{u}) = \int\limits_{\bar{u}}^{\bar{u}+z} k(w)dw = k(\bar{u} + \theta z)z = \kappa z$, $\theta \in (0,1)$.

Theorem 2.1. *Suppose there exists the unique solution of the problem (1.1), (1.2) in $H^{2,1}(Q_T)$, $k \in C(\mathcal{D}_u)$, $0 < k_1 \leq k(v) \leq k_2$ for $v \in \mathcal{D}_u$. Then, for sufficiently small $h < h_0$, $\tau < \tau_0$ and $\tau \leq \dfrac{k_1}{16k_2^2}h^2$ there exists the unique solution of the finite difference scheme (2.1), (2.2), which converges to the solution of the problem (1.1), (1.2) as $h, \tau \to 0$ and for every $t \in \omega_\tau$ the following estimate*

$$\|z\|_{C(\omega_h)} \leq M\frac{h^2 + \tau}{h^{3/2}}\|k\|_{C(\mathcal{D}_u)}\|u\|_{H^{2,1}(Q_T)} \leq \delta \qquad (2.6)$$

holds.

In the sequel, M denotes a positive generic constant, independent of h and τ.

Proof We shall prove the assertion by means of mathematical induction. Evidently, for $t = 0$ the estimate (2.6) holds. Suppose that for all $t' = 0, \tau, \ldots, t$ the solution $y(x, t')$ of finite difference scheme (2.1), (2.2) exists and estimate (2.6) holds. We will prove the same for $t' = t + \tau$.

Firstly, we show that $\bar{u} + \theta z \in \mathcal{D}_u$. Obviously,

$$\bar{u} - u_1 = \frac{1}{h}\int\limits_{x-h}^{x+h}\left(1 - \frac{|x' - x|}{h}\right) \times (u(x', t) - u_1)dx' \geq 0,$$

$$\bar{u} - u_2 = \frac{1}{h}\int\limits_{x-h}^{x+h}\left(1 - \frac{|x' - x|}{h}\right)(u(x', t) - u_2)dx' \leq 0.$$

Consequently $\bar{u} \in \mathcal{M}_u$. Since $\|z(\cdot, t)\|_{C(\omega_h)} \leq \delta$, it follows that $\bar{u} + \theta z \in \mathcal{D}_u$, and the value $\kappa = k(\bar{u} + \theta z)$ is defined and $0 < k_1 \leq \kappa \leq k_2$. In such a manner the values z (and y) for $t' = t + \tau$, are defined.

We estimate $\|z(\cdot, t+\tau)\|_{C(\omega_h)}$ by the energy method. Multiplying (2.5) in a scalar way with $2\tau A^{-1}z$, we obtain

$$\|\hat{z}\|_{A^{-1}}^2 - \|z\|_{A^{-1}}^2 - \tau^2\|z_t\|_{A^{-1}}^2 + 2\tau(\kappa z, z) = 2\tau(\eta, z).$$

Substituting z_t from equation (2.5), we obtain

$$\|\hat{z}\|_{A^{-1}}^2 - \|z\|_{A^{-1}}^2 + 2\tau(\kappa z, z) = \tau^2\|\kappa z + \eta\|_A^2 + 2\tau(\eta, z). \qquad (2.7)$$

Using inequalities

$$\|\kappa z + \eta\|_A^2 \le 2\|\kappa z\|_A^2 + 2\|\eta\|_A^2 \le \frac{8k_2^2}{h^2}\|z\|_h^2 + \frac{8}{h^2}\|\eta\|_h^2,$$

$$2\tau(\eta, z) \le k_1\tau\|z\|_h^2 + \tau\frac{1}{k_1}\|\eta\|_h^2, \quad 2\tau(\kappa z, z) \ge 2\tau k_1\|z\|_h^2,$$

we obtain

$$\|\hat{z}\|_{A^{-1}}^2 - \|z\|_{A^{-1}}^2 + \tau\left(k_1 - \frac{8\tau k_2^2}{h^2}\right)\|z\|_h^2 \le \tau\left(\frac{1}{k_1} + \frac{8\tau}{h^2}\right)\|\eta\|_h^2. \qquad (2.8)$$

Omitting positive the term with $\|z\|_h$ in the left side of (2.8), we obtain

$$\|\hat{z}\|_{A^{-1}}^2 \le \|z\|_{A^{-1}}^2 + \tau\left(\frac{1}{k_1} + \frac{8\tau}{h^2}\right)\|\eta\|_h^2 \le \left(\frac{1}{k_1} + \frac{k_1}{2k_2^2}\right)\sum_{t'=0}^{t}\tau\|\eta(\cdot, t')\|_h^2. \qquad (2.9)$$

Using representation

$$\eta = \eta_1 + \eta_2 = (S_t\varphi(u) - S_t\varphi(\bar{u})) + (S_t\varphi(\bar{u}) - \varphi(\bar{u})),$$

and inequalities $\|v\|_{A^{-1}} \ge 0.5h\|v\|_h$ and $\|v\|_{C(\omega_h)} \le h^{-1/2}\|v\|_h$, from (2.9) we obtain

$$\|\hat{z}\|_{C(\omega_h)}^2 \le \frac{4}{h^3}\left(\frac{1}{k_1} + \frac{k_1}{2k_2^2}\right)2\tau\sum_{t'=0}^{t}\left(\|\eta_1(\cdot, t')\|_h^2 + \|\eta_2(\cdot, t')\|_h^2\right). \qquad (2.10)$$

Further

$$\begin{aligned}
\eta_1(x, t') &= \frac{1}{\tau}\int_{t'}^{t'+\tau}\left[\varphi(u(x, t'')) - \varphi(\bar{u}(x, t''))\right]dt'' \\
&= \frac{1}{\tau}\int_{t'}^{t'+\tau}\int_{\bar{u}(x, t'')}^{u(x, t'')}\varphi'(w)dwdt'' = \frac{1}{\tau}\int_{t'}^{t'+\tau}\int_{\bar{u}(x, t'')}^{u(x, t'')}k(w)dwdt''.
\end{aligned}$$

From here it follows

$$
\begin{aligned}
|\eta_1(x,t')| &\leq \frac{1}{\tau}\|k\|_{C(\mathcal{D}_u)} \int_{t'}^{t'+\tau} |u(x,t'') - \bar{u}(x,t'')|dt'' \\
&= \frac{1}{h\tau}\|k\|_{C(\mathcal{D}_u)} \int_{t'}^{t'+\tau} \left| \int_{x-h}^{x+h} \left(1 - \frac{|x'-x|}{h}\right)[u(x,t'') - u(x',t'')]dx' \right| dt'' \\
&= \frac{1}{h\tau}\|k\|_{C(\mathcal{D}_u)} \int_{t'}^{t'+\tau} \left| \int_{x-h}^{x+h} \left(1 - \frac{|x'-x|}{h}\right) \right. \\
&\qquad \times \left. \int_{x'}^{x}\int_{x}^{x''} \frac{\partial^2 u(x''',t'')}{\partial x^2}dx'''dx''dx' \right| dt'',
\end{aligned}
$$

and

$$
|\eta_1(x,t')| \leq \frac{Mh^2}{\sqrt{h\tau}}\|k\|_{C(\mathcal{D}_u)} \left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L_2(e)}, \tag{2.11}
$$

where $e = (x-h, x+h) \times (t', t'+\tau)$. In an analogous manner we obtain

$$
|\eta_2(x,t')| \leq \frac{M\tau}{\sqrt{h\tau}}\|k\|_{C(\mathcal{D}_u)} \left\|\frac{\partial u}{\partial t}\right\|_{L_2(e)}. \tag{2.12}
$$

It follows from (2.10), (2.11) and (2.12) that

$$
\|z(\cdot, t+\tau)\|_{C(\omega_h)} = \|\hat{z}\|_{C(\omega_h)} \leq M\frac{h^2+\tau}{h^{3/2}}\|k\|_{C(\mathcal{D}_u)}\|u\|_{H^{2,1}(Q_T)}.
$$

Choosing sufficiently small h and τ we finally obtain $\|z(\cdot, t+\tau)\|_{C(\omega_h)} \leq \delta$.
∎

Theorem 2.2. *Let the assumptions of theorem 2.1 are satisfied. Then the following estimate*

$$
\|y - \bar{u}\|_{h\tau} \leq M(h^2+\tau)\|k\|_{C(\mathcal{D}_u)}\|u\|_{H^{2,1}(Q_T)} \tag{2.13}
$$

holds.

Proof From (2.8), summing over the mesh ω_τ and using inequality $k_1 - 8\tau k_2^2/h^2 \geq \frac{k_1}{2}$, we obtain

$$
\|z(\cdot, T)\|_{A^{-1}}^2 + \frac{k_1}{2}\tau\sum_{t=0}^{T-\tau} \|z\|_h^2 \leq \tau\left(\frac{1}{k_1} + \frac{k_1}{2k_2^2}\right)\sum_{t=0}^{T-\tau} \|\eta\|_h^2. \tag{2.14}
$$

Applying to (2.14) estimates (2.11) and (2.12) and omitting the term $\|z(\cdot, T)\|_{A^{-1}}^2$ we obtain (2.13). ∎

Since the solution u is a continuous function it is interesting to estimate the difference $y - u$.

Theorem 2.3. *Let the assumptions of theorem 2.1 are satisfied. Then the following estimate holds*

$$\|y - u\|_{h\tau} \leq M(h^2 + \tau)\Big(\|k\|_{C(\mathcal{D}_u)} + 1\Big)\|u\|_{H^{2,1}(Q_T)}. \tag{2.15}$$

3. IMPLICIT FINITE DIFFERENCE SCHEME

We assume, that there exists a unique solution $u(x, t)$ of the problem (1.1), (1.2) in $H^{3,3/2}(Q_T)$, and $u_0 \in H^2(0,1)$, $k \in C(\mathcal{D}_u)$, $0 < k_1 \leq k(v) \leq k_2$ for $v \in \mathcal{D}_u$.

We approximate the problem (1.5), (1.2) with implicit finite difference scheme

$$y_t = (a(y)\hat{y}_{\bar{x}})_x, \ (x, t) \in \omega_{h\tau}, \tag{3.1}$$

$$y(0, t) = y(1, t) = 0, \ t \in \overline{\omega}_\tau, \quad y(x, 0) = S_x u_0(x), \ x \in \omega_h, \tag{3.2}$$

where $a(y(x)) = k\left((0.5(y(x) + y(x - h)))\right)$.

Let us consider the convergence of finite difference scheme (3.1), (3.2). Having the solution $u(x, t)$ of the problem (1.5), (1.2) oddly extended outside of Q_T, suppose that $\bar{u} = S_x^2 u$. The following identity takes place

$$\bar{u}_t = S_t \left(k(u(x - 0.5h, t))\frac{\partial u}{\partial x}(x - 0.5h, t) \right)_x. \tag{3.3}$$

Using the identity (3.3), we can write a problem for $z = y - \bar{u}$ in the form

$$z_t = (a(y)\hat{z}_{\bar{x}})_x + \Big((a(y) - a(\bar{u}))\,\hat{\bar{u}}_{\bar{x}}\Big)_x + \eta_x(x, t), \ (x, t) \in \omega_{h\tau}, \tag{3.4}$$

$$z(0, t) = z(1, t) = 0, \ t \in \overline{\omega}_\tau, \quad z(x, 0) = 0, \ x \in \omega_h, \tag{3.5}$$

where

$$\eta(x, t) = (a(\bar{u})\hat{\bar{u}}_{\bar{x}})(x, t) - S_t k(u(x - 0.5h, t))\frac{\partial u}{\partial x}(x - 0.5h, t).$$

For our convenience, let us represent $\eta(x, t)$ as below

$$\eta(x, t) = \frac{1}{\tau} \int\limits_t^{t+\tau} \Big\{ \left[a(\bar{u}(x, t)) - k(u(x - 0.5h, t'))\right] \bar{u}_{\bar{x}}(x, t+\tau) + k(u(x - 0.5h, t'))$$

$$\times \left[\bar{u}_{\bar{x}}(x, t + \tau) - \frac{\partial u}{\partial x}(x - 0.5h, t') \right] \Big\} dt'$$

$$= \frac{1}{\tau} \int\limits_{t}^{t+\tau} \left\{ \eta_1(x, t, t') \bar{u}_{\bar{x}}(x, t + \tau) + k(u(x - 0.5h, t')) \eta_2(x, t, t') \right\} dt'.$$

First, we shall estimate $\eta_2(x, t, t')$. For this purpose, we shall take an advantage of the obvious identity

$$\bar{u}_{\bar{x}}(x, t + \tau) = \frac{1}{h^2} \int\limits_{x-h}^{x} \int\limits_{x'-0.5h}^{x'+0.5h} \frac{\partial u}{\partial x}(x'', t + \tau) dx'' dx'. \tag{3.6}$$

Using (3.6), we obtain

$$\begin{aligned}
\eta_2(x, t, t') &= \bar{u}_{\bar{x}}(x, t + \tau) - \frac{\partial u}{\partial x}(x - 0.5h, t') \\
&= \frac{1}{h^2} \int\limits_{x-h}^{x} \int\limits_{x'-0.5h}^{x'+0.5h} \left(\frac{\partial u}{\partial x}(x'', t + \tau) - \frac{\partial u}{\partial x}(x - 0.5h, t') \right) dx'' dx' \\
&= \frac{1}{h^2} \int\limits_{x-h}^{x} \int\limits_{x'-0.5h}^{x'+0.5h} \int\limits_{t'}^{t+\tau} \frac{\partial^2 u}{\partial x \partial t}(x'', t'') dt'' dx'' dx' nonumber \tag{3.7} \\
&+ \frac{1}{h^2} \int\limits_{x-h}^{x} \int\limits_{x'-0.5h}^{x'+0.5h} \int\limits_{x-0.5h}^{x''} (x'' - x''') \frac{\partial^2 u}{\partial x^2}(x''', t') dx''' dx'' dx'. \tag{3.8}
\end{aligned}$$

Integrating $|\eta_2(x, t, t')|$ over $t' \in (t, t + \tau)$, applying the Cauchy–Schwartz–Bunyakovskii inequality and taking into account the estimate (3.8), we obtain

$$\frac{1}{\tau} \int\limits_{t}^{t+\tau} |\eta_2(x, t + \tau, t')| dt' \le \frac{\sqrt{\tau}}{\sqrt{h}} \left\| \frac{\partial^2 u}{\partial t \partial x} \right\|_{L_2(e)} + \frac{h\sqrt{h}}{\sqrt{\tau}} \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{L_2(e)}, \tag{3.9}$$

where $e = (x - 1.5h, x + 0.5h) \times (t, t + \tau)$.

To estimate $\eta_1(x, t, t')$, we use the relation

$$u(x - 0.5h, t') = \frac{u(x - h, t') + u(x, t')}{2} - \frac{1}{2} \int\limits_{x-0.5h}^{x} \int\limits_{x'-0.5h}^{x'} \frac{\partial^2 u}{\partial x^2}(x'', t') dx'' dx'.$$

Then the following expressions are valid

$$
\begin{aligned}
\eta_1(x,t,t') &= a(\bar{u}(x,t)) - k(u(x - 0.5h, t')) \\
&\leq L|0.5(\bar{u}(x - h, t) + \bar{u}(x,t)) - u(x - 0.5h, t')| \\
&\leq \frac{L}{2}\Big|\bar{u}(x - h, t) - u(x - h, t') + \bar{u}(x,t) - u(x,t') \\
&\quad + \int\limits_{x-0.5h}^{x} \int\limits_{x'-0.5h}^{x'} \frac{\partial^2 u}{\partial x^2}(x'', t')dx''dx'\Big| \\
&\leq \frac{L}{2}\Big(\Big|\frac{1}{h} \int\limits_{x-0.5h}^{x+0.5h} \int\limits_{t'}^{t} \frac{\partial u}{\partial t}(x', t')dt'dx'\Big| \\
&\quad + \Big|\frac{1}{h} \int\limits_{x-0.5h}^{x+0.5h} \int\limits_{x}^{x'} (x'' - x')\Big(\frac{\partial^2 u}{\partial x^2}(x'', t') + \frac{\partial^2 u}{\partial x^2}(x'' - h, t')dx''dx'\Big| \\
&\quad + \Big|\int\limits_{x-0.5h}^{x} \int\limits_{x'-0.5h}^{x'} \frac{\partial^2 u}{\partial x^2}(x'', t')dx''dx'\Big|\Big).
\end{aligned}
$$

Integrating $|\eta_1(x,t,t')|$ over $t' \in (t, t+\tau)$ and taking into account the inequality (3.10), we obtain

$$
\frac{1}{\tau}\int\limits_{t}^{t+\tau} |\eta_1(x,t,t')|dt' \leq L\left(\frac{\sqrt{\tau}}{\sqrt{h}}\Big\|\frac{\partial u}{\partial t}\Big\|_{L_2(e)} + \frac{h^{3/2}}{\sqrt{\tau}}\Big\|\frac{\partial^2 u}{\partial x^2}\Big\|_{L_2(e)}\right). \tag{3.10}
$$

It is not easy to show ([2], vol.2, pp.10–11) that

$$
|\bar{u}_{\bar{x}}(x, t + \tau)| \leq \max_{x \in [0,1]} |\bar{u}_{\bar{x}}(x, t + \tau)| \leq M\|u(\cdot, t + \tau)\|_{W_2^{3,3/2}(Q_T)}. \tag{3.11}
$$

Combining the estimates (3.8), (3.10), (3.11), we obtain a priori estimate for $\eta(x,t)$

$$
\begin{aligned}
|\eta(x,t)| &\leq |\bar{u}_{\bar{x}}(x, t + \tau)|\frac{1}{\tau}\int\limits_{t}^{t+\tau} |\eta_1(x,t,t')|dt' \\
&\quad + \|k(u)\|_{C(\mathcal{D}_u)}\frac{1}{\tau}\int\limits_{t}^{t+\tau} |\eta_2(x, t + \tau, t')|dt'. \tag{3.12}
\end{aligned}
$$

Then summing $\eta(x,t)$ over the grid points $\{\overline{\omega}_\tau \backslash \{t = T\}\} \times \{\overline{\omega}_h \backslash \{x = 1\}\}$, we

obtain

$$\left(\sum_{t=0}^{T-\tau}\tau\,\|\eta\|^2\right)^{1/2} \leq M\left(\tau\left\|\frac{\partial u}{\partial t}\right\|_{L_2(Q_T)} + h^2\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{L_2(Q_T)}\right)$$
$$\times\left(\|u(\cdot,t+\tau)\|_{W_2^{3,3/2}(Q_T)}+1\right). \qquad (3.13)$$

Theorem 3.1. *Suppose there exists the unique solution of the problem (1.1), (1.2) in $H^{3,3/2}(Q_T)$, $k \in C(\mathcal{D}_u)$, $0 < k_1 \leq k(v) \leq k_2$ for $v \in \mathcal{D}_u$. Then, for sufficiently small $h < h_0$, $\tau < \tau_0$ there exists the unique solution of the finite difference scheme (3.1), (3.2), which converges to the solution of the problem (1.1), (1.2) as $h, \tau \to 0$ and for any $t \in \omega_\tau$ the following estimates*

$$\|z\|_* \leq M(h^2 + \tau)\|u\|_{W_2^{3,3/2}(Q_T)}, \quad \|z\|_*^2 = \|z\|^2 + \tau k_1\,\|z_{\bar x}]\|^2, \qquad (3.14)$$

$$\|z\|_{C(\omega_h)} \leq M(h + \sqrt{\tau})\|u\|_{W_2^{3,3/2}(Q_T)} \leq \delta \qquad (3.15)$$

hold.

Proof   We shall prove the assertion by means of mathematical induction. Obviously, that for $t = 0$ the estimates (3.15), (3.14) hold. Suppose that for all $t' = 0, \tau, \ldots, t$ the solution $y(x, t')$ of the finite difference scheme (3.1), (3.2) exists and the estimate (3.15) holds. To prove the same for $t' = t + \tau$, we shall use the energy method. Considering an inner product of (3.4) and $2\tau\hat z$, we obtain the energy identity

$$\|\hat z\|^2 - \|z\|^2 + \tau\,\|z_t\|^2 + 2\tau\,(a(y)\hat z_{\bar x}, \hat z_{\bar x}] + 2\tau\,((a(y) - a(\bar u))\hat{\bar u}_{\bar x}, \hat z_{\bar x}] = -2\tau(\eta, \hat z_{\bar x}).$$

Then
$$2\tau\,(a(y)\hat z_{\bar x}, \hat z_{\bar x}) \geq 2\tau k_1\,\|\hat z_{\bar x}]\|^2,$$

$$2\tau((a(y) - a(\bar u))\hat{\bar u}_{\bar x}, \hat z_{\bar x}] \leq \frac{\tau k_1}{2}\,\|\hat z_{\bar x}]\|^2 + \frac{\tau}{2k_1}M^2L^2\|u\|_{W_2^{3,3/2}(Q_T)}^2\,\|z\|^2,$$

$$2\tau(\eta, \hat z_{\bar x}) \leq \frac{\tau k_1}{2}\,\|\hat z_{\bar x}]\|^2 + \frac{\tau}{2k_1}\,\|\eta\|^2.$$

Summing over these estimates, we get the energy inequality

$$\|\hat z\|_*^2 \leq (1 + \tau M)\|z\|_*^2 + \frac{\tau}{2k_1}\,\|\eta\|^2$$
$$\leq M\sum_{t'=0}^{t}\tau\,\|\eta(t')]\|^2 \leq M\sum_{t'=0}^{T-\tau}\tau\,\|\eta(t')]\|^2. \qquad (3.16)$$

Substituting the estimate (3.13) into (3.16), we obtain (3.14). The inequality (3.15) follows from Theorem 1 ([11]). ∎

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BAIGTINIŲ SKIRTUMŲ SCHEMŲ, APROKSIMUOJANČIŲ SILPNUOSIUS NETIESINĖS ŠILUMOS LYGTIES SPRENDINIUS, TIKSLUMO ĮVERČIAI

B.S. JOVANOVIČ, P.P. MATUS, V.S. SHCHEHLIK

Nagrinėjami vienmatės netiesinės šilumos laidumo lygties skaitiniai sprendimo metodai, kai lygtis turi tik silpnąjį sprendinį. Lygties koeficientai yra stipriai netiesiniai. Ištirtos išreikštinė ir neišreikštinė baigtinių skirtumų schemos. Aproksimuojant funkcijas panaudoti Steklovo vidurkinimo operatoriai. Gautieji paklaidų įverčiai yra suderinti su diferencialinio uždavinio sprendinio glodumu.