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THE BÄCKLUND TRANSFORMATIONS OF THE FIFTH PAINLEVÉ EQUATION AND THEIR APPLICATIONS

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ABSTRACT

The main objective of this paper is to study the properties of the Bäcklund transformations and the auto-Bäcklund transformations, to construct the nonlinear superposition formulas for the solutions of the fifth Painlevé equation and to obtain the values of the parameters after the successive application of the Bäcklund transformations.

1. INTRODUCTION

The present paper is devoted to the study of the properties of the Bäcklund transformations and the construction of the nonlinear superposition formulas for the solutions of the fifth Painlevé equation

$$w'' = \frac{3w-1}{2w(w-1)}w'^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \quad (1.1)$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary complex parameters.

Informally, a Bäcklund transformation is defined as a system of equations relating one solution of a given equation either to another solution of the same equation, possibly with different values of the parameters, or to a solution of another equation. In case the Bäcklund transformation links one solution of a given equation to another solution of the same equation with the same values of the parameters, it is usually referred to as the auto-Bäcklund transformation.

The Bäcklund transformations of the equation (1.1) which, in particular, allow us to build a hierarchy of solutions using the known seed solution are obtained and the conditions of the existence of the rational and one-parameter solutions can be found in papers [3], [6], [5] (for overview see [4] – [2]). The repeated application of the Bäcklund transformation has been studied in [6]. However, in this paper the relations presented in [6] will be extended and modified.

2. THE BÄCKLUND TRANSFORMATIONS

It is well-known that there exist [3] the Bäcklund transformations which establish the correspondence between solutions of equation (1.1) with different values of the parameters in case $\delta \neq 0$.

Theorem 2.1. *Let* $w = w(z, \alpha,\ \beta,\ \gamma,\ \delta)$ *be a solution of* (1.1) *with the parameters* $\alpha,\ \beta,\ \gamma,\ \delta \neq 0$, *such that*

$$F_1\ (w) = zw' - \varepsilon_1 c w^2 + (\varepsilon_1 c - \varepsilon_2 a + \varepsilon_3 k z) w + \varepsilon_2 a \not\equiv 0, \tag{2.1}$$

where $c^2 = 2\alpha,\ a^2 = -2\beta,\ k^2 = -2\delta$. *Then the transformation*

$$T_{\varepsilon_1, \varepsilon_2, \varepsilon_3}:\ w(z, \alpha, \beta, \gamma, \delta) \to w_1(z, \alpha_1, \beta_1, \gamma_1, \delta_1) = 1 - 2\varepsilon_3 k z w F_1^{-1}(w) \tag{2.2}$$

defines another solution $w_1(z, \alpha_1, \beta_1, \gamma_1, \delta_1)$ *of (1.1) with the values of the parameters*

$$\alpha_1 = -\frac{1}{16\delta}[\gamma + \varepsilon_3 k(1 - \varepsilon_2 a - \varepsilon_1 c)]^2,\ \beta_1 = \frac{1}{16\delta}[\gamma - \varepsilon_3 k(1 - \varepsilon_2 a - \varepsilon_1 c)]^2,$$

$$\gamma_1 = \varepsilon_3 k(\varepsilon_2 a - \varepsilon_1 c),\ \delta_1 = \delta, \tag{2.3}$$

where $\varepsilon_i^2 = 1,\ i \in \{1, 2, 3\}$.

Solution $w(z, \alpha,\ \beta,\ \gamma,\ \delta)$, for which inequality (2.1) holds, i.e., $F_1(w) \not\equiv 0$, and which is called the solution of the zero level (or the seed solution), generates eight different solutions of (1.1) of the first level $w_i(z, \alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i)$, where $i \in \{1, 2, 3, \ldots, 8\}$, according to the choice of the branches $c = \sqrt{2\alpha},\ a = \sqrt{-2\beta},\ k = \sqrt{-2\delta}$ determined by the choice of $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$ respectively. Solutions generated after the $n-$th fold application of the transformation of Theorem 2.1 are called the solutions of the $n-$th level with the assigned indices $i_1, \ldots, i_n$, where $i_j \in \{1, 2, \ldots, 8\},\ j \in \{1, \ldots n\},\ n > 1$.

Let us show that the value of the parameter ε_3 may be fixed, and, consequently, we may consider solutions of (1.1) of the $n-$th level according to the choice of the branches of parameters ε_1 and ε_2 when $\varepsilon_3 = 1$. The following lemmas will essentially be used to prove this statement and the other theorems.

Lemma 2.1. *The branching of the parameter ε_3 for the solutions of the first level is equivalent to the inversion $T_{\varepsilon_1,\,\varepsilon_2,\,1}\;w(z) = (T_{\varepsilon_1,\,\varepsilon_2,\,-1}\;w(z))^{-1}$.*

Proof. The statement of Lemma 2.1 follows immediately from transformation (2.2), (2.3). Actually, let w_i and $w_{i+4},\ i \in \{1,2,3,4\}$ be the solutions of (1.1) generated from the solution $w(z)$ when $\varepsilon_3 = 1$ and $\varepsilon_3 = -1$ respectively by means of the transformation of Theorem 2.1. Using formula (2.2) it is easy to show the validity of the relation $w_i(z,\alpha_i,\,\beta_i,\,\gamma_i,\,\delta_i) = w_{i+4}^{-1}(z,-\beta_i,\,-\alpha_i,\,-\gamma_i,\,\delta_i)$. ■

Lemma 2.2. *The following relation is true: $T_{\sigma_1,\,\sigma_2,\,\sigma_3}\circ T_{\varepsilon_1,\,\varepsilon_2,\,\varepsilon_3}\;w(z) = T_{\sigma_1,\,\sigma_2,\,-\sigma_3}\circ T_{\varepsilon_1,\,\varepsilon_2,\,-\varepsilon_3}\;w(z)$, where $\sigma_i^2 = \varepsilon_i^2 = 1,\ i \in \{1,2,3\}$.*

Proof. From Lemma 2.1 and transformation (2.2), (2.3) we get immediately that if $\tilde{w}(z) = T_{\varepsilon_1,\,\varepsilon_2,\,\varepsilon_3}w(z)$ and $\breve{w} = T_{\varepsilon_1,\,\varepsilon_2,\,-\varepsilon_3}w(z)$, then $\breve{w}(z) = 1/\tilde{w}(z)$. Thus, $T_{\sigma_1,\,\sigma_2,\,\sigma_3}\tilde{w}(z) = T_{\sigma_1,\,\sigma_2,\,-\sigma_3}\tilde{w}^{-1}(z)$. ■

Note, that the relation of Lemma 2.2 demonstrates the commutativity of the construction of solutions according to the choice of branches of the parameters when transformation (2.2), (2.3) is applied.

Thus, the solutions of the second level generated with the help of the double application of transformation (2.2), (2.3) with the fixed branch of ε_3 coincide when the parameters $\alpha,\ \beta,\ \gamma,\ \delta$ are the same.

Lemma 2.3. *The double application of the transformation of Theorem* 2.1 *with the following choice of the branches of the parameters $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$ is equivalent to the inversion of the seed solution*

$$T_{1,\,-1,\,\varepsilon_3}\circ T_{\varepsilon_1,\,\varepsilon_2,\,\varepsilon_3} : w(z) \to w^{-1}(z).$$

Proof. It suffices to remark that Lemma 2.3 easily follows from transformation (2.2), (2.3) applied twice. ■

Using Lemmas 2.1 – 2.3, we immediately infer the following statement.

Theorem 2.2. *The repeated applications of the transformation of Theorem* 2.1 *with the following choice of the parameter branches $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$ are equivalent to the identical transformation*

$$\begin{aligned} I &= T_{1,\,-1,\,-\varepsilon_3}\circ T_{\varepsilon_1,\,\varepsilon_2,\,\varepsilon_3} : w(z) \to w(z),\\ I &= T_{1,\,-1,\,1}\circ T_{\varepsilon_1,\,\varepsilon_2,\,1}\circ T_{1,\,-1,\,1}\circ T_{\varepsilon_1,\,\varepsilon_2,\,1} : w(z) \to w(z). \end{aligned}$$

Thus, by virtue of the statements of Theorem 2.2 the value of the parameter ε_3 may be fixed and below we may take $\varepsilon_3 = 1$ without loss of generality.

Moreover, in case $\delta \neq 0$ we may always take $\delta = -1/2$ without loss of generality as it can be achieved by applying the gauge transformation.

Let us study the nonlinear superposition formulas which link solutions of (1.1) after the repeated application of transformation (2.2), (2.3). Note, that the superposition formulas obtained below may be regarded as the alternative forms of the discrete Painlevé equations [1]. When the parameter ε_3 is fixed ($\varepsilon_3 = 1$), the seed solution $w(z, \alpha, \beta, \gamma, -1/2)$ for which (2.1) holds generates four different solutions of the first level $w_i(z, \alpha_i, \beta_i, \gamma_i, -1/2)$, $i \in \{1, 2, 3, 4\}$, of (1.1) according to the choice of the branches ε_1, ε_2 :

$$T_{1,1,1}\, w = w_1 = 1 - 2zw\left(zw' - cw^2 + (c - a + z)w + a\right)^{-1}, \tag{2.4}$$

$$T_{1,-1,1}\, w = w_2 = 1 - 2zw\left(zw' - cw^2 + (c + a + z)w - a\right)^{-1}, \tag{2.5}$$

$$T_{-1,1,1}\, w = w_3 = 1 - 2zw\left(zw' + cw^2 + (-c - a + z)w + a\right)^{-1}, \tag{2.6}$$

$$T_{-1,-1,1}\, w = w_4 = 1 - 2zw\left(zw' + cw^2 + (-c + a + z)w - a\right)^{-1}. \tag{2.7}$$

Note, that further we shall consider the general case when $a \neq 0$, $c \neq 0$. When $a = 0$ we have $w_1 = w_2$ and $w_3 = w_4$ and the superposition formula $1/(1 - w_1) - 1/(1 - w_3) = c(1 - w)/z$ is valid. In case $c = 0$ we get $w_1 = w_3$, $w_2 = w_4$ and $1/(1 - w_1) - 1/(1 - w_2) = a(1 - w)/(zw)$. If the parameters a and c are equal to zero simultaneously, all solutions of the first level coincide provided $\varepsilon_3 = 1$ and the subsequent application of the transformation (2.2), (2.3) is possible if $w \neq const\ \exp(-\sqrt{-2\delta} z)$ when $\gamma = \sqrt{-2\delta}$.

Theorem 2.3. *The seed solution and any two solutions of the first level are algebraically dependent.*

Proof. By eliminating $w'(z)$ between equalities (2.4) — (2.7), we find the explicit superposition formulas connecting the seed solution $w(z)$ and the first level solutions $w_i(z)$, where $i \in \{1, 2, 3, 4\}$. ■

From Theorem 2.3 we get immediately by direct computation

$$\frac{1}{1 - w_1} - \frac{1}{1 - w_2} = \frac{1}{1 - w_3} - \frac{1}{1 - w_4}. \tag{2.8}$$

Theorem 2.4. *Any three solutions of the first level are algebraically dependent.*

Proof. From Theorem 2.3 we get immediately by direct computations

$$w_4 = \frac{c(w_1 - w_2)(w_2 - 1) + w_2(a(w_1 - 1)(w_2 - 1) + (w_1 - w_2)z)}{a(w_1 - 1)(w_2 - 1) + (w_1 - w_2)(c(w_2 - 1) + z)}, \tag{2.9}$$

$$w_3 = \frac{a(w_1-1)w_1(w_2-1)+(w_1-w_2)(c(w_1-1)+w_1 z)}{a(w_1-1)(w_2-1)+(w_1-w_2)(c(w_1-1)+z)}, \tag{2.10}$$

$$w_4 = \frac{a(w_1-w_3)(w_3-1)+w_3(c(w_1-1)(w_3-1)+(w_3-w_1)z)}{c(w_1-1)(w_3-1)+(w_1-w_3)(a(w_3-1)-z)}, \tag{2.11}$$

$$w_3 = \frac{a(w_2-w_4)(w_4-1)+w_4(c(1-w_2)(w_4-1)+(w_2-w_4)z)}{c(1-w_2)(w_4-1)+(w_2-w_4)(a(w_4-1)+z)}. \tag{2.12}$$

Relations (2.9) – (2.12) link three arbitrary solutions of the first level and that proves the statement of Theorem 2.4. ■

It is necessary to point out that the question of the deduction of the nonlinear superposition formulas linking solutions of two adjacent levels is partially considered in [8].

Let $w_{i,\,j}(z),\ i,\,j \in \{1,2,3,4\}$, be the solutions of the second level obtained after the repeated application of the transformation (2.2), (2.3) to the seed solution $w(z)$, i.e., $w_{i,\,j}(z,\alpha_{i,\,j},\,\beta_{i,\,j},\,\gamma_{i,\,j},\,\delta_{i,\,j}) = T_{\sigma_1,\,\sigma_2,1} \circ T_{\varepsilon_1,\,\varepsilon_2,\,1} w(z,\alpha,\beta,\gamma,\delta)$, where $\varepsilon_k^2 = \sigma_k^2 = 1,\ k \in \{1,2\}$.

Taking into account (2.8) and Lemma 2.3, it is not difficult to get the following nonlinear superposition formula relating solutions of the zero and second levels:

$$\frac{1}{1-w_{i,\,1}} + \frac{1}{1-w_{i,\,4}} + \frac{1}{w_{i,\,3}-1} = \frac{w}{w-1},$$

where $i \in \{1,2,3,4\}$.

Let us deduce the nonlinear superposition formulas linking the seed solution to the solutions of the first and second levels after the repeated application of the transformation (2.2), (2.3). Assume that $w_i = T_{\varepsilon_1,\,\varepsilon_2,\,1} w(z)$ and $w_{i,\,j} = T_{\sigma_1,\,\sigma_2\,,1} w_i(z)$. Calculating directly, we get that if $w_{i,\,1} = T_{1,\,1\,,1} w_i(z),\ w_{i,\,2} = T_{1,\,-1\,,1} w_i(z),\ w_{i,\,3} = T_{-1,\,1\,,1} w_i(z)$, and $w_{i,\,4} = T_{-1,-\,1\,,1} w_i(z)$, then the following relations are true:

$$\frac{w}{w-1} - \frac{1}{1-w_{i,\,1}} = -a_1\frac{(1-w_i)}{zw_i}, \quad w_{i,\,2} = \frac{1}{w},$$

$$\frac{w}{w-1} - \frac{1}{1-w_{i,\,3}} = \frac{(1-w_i)(c_1 w_i - a_1)}{zw_i}, \quad \frac{w}{w-1} - \frac{1}{1-w_{i,\,4}} = \frac{c_1(1-w_i)}{z},$$

where the choice of the branches $c_1,\ a_1$ is fixed and $c_1 = \sqrt{2\alpha_1},\ a_1 = \sqrt{-2\beta_1}$.

3. THE REPEATED APPLICATION OF THE BÄCKLUND TRANSFORMATION

In this section we prove the following theorem on the general structure of the parameters after the repeated application of the Bäcklund transformation (2.2), (2.3).

Theorem 3.1. *Successive applications of the Bäcklund transformation* $T_{\varepsilon_1,\varepsilon_2,1}$ *to the seed solution* $w(z,\alpha,\beta,\gamma,\delta)$ *of* (1.1), *where* $\delta=-1/2$, *lead to the solution* $w_1(z,\alpha_1,\beta_1,\ \gamma_1,\ \delta_1=-1/2)$, *where new parameters have one of the following forms:*

$$1)\ \alpha_1=(p+n_1)^2/2,\quad \beta_1=-(q+n_2)^2/2,\quad \gamma_1=\gamma+n_3,\quad s=2n_4,\qquad (3.1)$$

$$2)\ \alpha_1=(q+n_2)^2/2,\quad \beta_1=-(p+n_1)^2/2,\quad \gamma_1=-\gamma+n_3,\quad s=2n_4,\qquad (3.2)$$

$$3)\ \alpha_1=(-\varepsilon p-vq+\varepsilon_1\gamma+2n_1+1)^2/8,\quad \beta_1=-(-\varepsilon p-vq-\varepsilon_1\gamma+2n_2+1)^2/8,$$

$$\gamma_1=(-\varepsilon p+vq)\varepsilon_1+n_3,\quad s=2n_4,\qquad (3.3)$$

$$4)\ \alpha_1=(-\varepsilon p+vq-\varepsilon_1\gamma+2n_1+1)^2/8,\ \beta_1=-(\varepsilon p-vq-\varepsilon_1\gamma+2n_2+1)^2/8,$$

$$\gamma_1=(\varepsilon p+vq-1)\varepsilon_1+n_3,\quad s=2n_4,\qquad (3.4)$$

where

$$p^2=2\alpha,\ \ q^2=-2\beta,\ \ \nu^2=\varepsilon^2=\varepsilon_1^2=1,\ \ s=n_1+n_2+n_3,\ \ n_j\in Z,\ j=1..4.$$

Proof. This theorem is best proved by using induction.

Note, that we may apply a sequence of the Bäcklund transformations if $w\neq c\exp(-kz)$ when $\alpha=0,\beta=0,\gamma=-k=-\sqrt{-2\delta}$. But let us assume here that neither $\eta\gamma+\sqrt{2\alpha}+\sqrt{-2\beta}-2n+1$ nor $(2\alpha-n^2)(2\beta+n^2)$ are equal to zero and $\eta^2=1,\ n\in N$. These assumptions mean that $w(z)$ does not belong to any one-parameter family of solutions.

Let $\Lambda_{\varepsilon1,\varepsilon2},\ \varepsilon_1^2=\varepsilon_2^2=1$ denote a transformation $(\alpha,\beta,\gamma)\to(\alpha_1,\beta_1,\gamma_1)$, where $\alpha_1,\beta_1,\gamma_1$ are defined by (2.3). The notation $\Lambda^{k1}_{\varepsilon1,\varepsilon2}$ means that $\Lambda_{\varepsilon1,\varepsilon2}$ has been applied k_1 times. At first we prove by induction that specific combinations of the transformation give us the following result:

a) $\Lambda^{k1}_{1,1}\circ\Lambda_{\varepsilon1,\varepsilon2}:\ (\alpha,\beta,\gamma)\to(\alpha_1,\beta_1,\gamma_1)$, where $\alpha_1,\beta_1,\gamma_1$ take one of the following forms:

$$(\alpha,\ \beta,\ \gamma),\ \ (-\beta,\ -\alpha,\ -\gamma),$$

$$((1-q\varepsilon_2-p\varepsilon_1-\gamma)^2/8,\ -(1-q\varepsilon_2-p\varepsilon_1+\gamma)^2/8,\ p\varepsilon_1-q\varepsilon_2),$$

$$((1-q\varepsilon_2-p\varepsilon_1+\gamma)^2/8,\ -(1-q\varepsilon_2-p\varepsilon_1-\gamma)^2/8,\ -p\varepsilon_1+q\varepsilon_2),$$

b) $\Lambda^{k2}_{1,-1}\circ\Lambda_{\varepsilon1,\varepsilon2}:\ (\alpha,\beta,\gamma)\to(\alpha_1,\beta_1,\gamma_1)$, where $\alpha_1,\beta_1,\gamma_1$ take one of the following forms:

$$(\alpha,\ -(n_1-q\varepsilon_2)^2/2,\ \gamma-n_1),\ n_1\in N,$$

$$((1+q\varepsilon_2-p\varepsilon_1-\gamma)^2/8,\ -(1-q\varepsilon_2+p\varepsilon_1-\gamma+2n_1)^2/8,\ p\varepsilon_1+q\varepsilon_2-n_1-1),$$

$$((1-q\varepsilon_2-p\varepsilon_1+\gamma)^2/8,\ -(1-q\varepsilon_2-p\varepsilon_1-\gamma+2n_1)^2/8,\ -p\varepsilon_1+q\varepsilon_2-n_1),$$

c) $\Lambda^{k3}_{-1,1}\circ\Lambda_{\varepsilon 1,\varepsilon 2}:(\alpha,\beta,\gamma)\to(\alpha_1,\beta_1,\gamma_1)$, where $\alpha_1,\beta_1,\gamma_1$ take one of the following forms:

$$((n_1-p\varepsilon_1)^2/2,\ \beta,\ \gamma+n_1),\ n_1\in N,$$

$$((1+q\varepsilon_2-p\varepsilon_1+\gamma+2n_1)^2/8,\ -(1-q\varepsilon_2+p\varepsilon_1+\gamma)^2/8,\ p\varepsilon_1-q\varepsilon_2+n_1+1),$$

$$((1-q\varepsilon_2-p\varepsilon_1+\gamma+2n_1)^2/8,\ -(1-q\varepsilon_2-p\varepsilon_1-\gamma)^2/8,\ -p\varepsilon_1+q\varepsilon_2+n_1),$$

d) $\Lambda^{k4}_{-1,-1}\circ\Lambda_{\varepsilon 1,\varepsilon 2}:(\alpha,\beta,\gamma)\to(\alpha_1,\beta_1,\gamma_1)$, where $\alpha_1,\beta_1,\gamma_1$ take one of the following forms:

$$((n_1-p\varepsilon_1)^2/2,\ -(n_1-q\varepsilon_2)^2/2,\ \gamma),\ n_1\in N,$$

$$((1-q\varepsilon_2-p\varepsilon_1+\gamma+2n_1)^2/8,\ -(1-q\varepsilon_2-p\varepsilon_1-\gamma+2n_1)^2/8,\ -p\varepsilon_1+q\varepsilon_2).$$

Thus, combining a) – d), we get that successive applications of the Bäcklund transformation $T_{\varepsilon_1,\varepsilon_2,1}$ to the solution $w(z,\alpha,\beta,\gamma,\delta)$ of (1.1), where $\delta=-1/2$, lead to the solution $w_1(z,\alpha_1,\beta_1,\ \gamma_1,\ \delta_1=-1/2)$, $(k=1)$, where new parameters take one of the following forms:

$$1)\ \alpha_1=(p+n_1)^2/2,\ \ \beta_1=-(q+n_2)^2/2,\ \gamma_1=\gamma+n_3,\ s=2n_4,$$

$$2)\ \alpha_1=(q+n_2)^2/2,\ \beta_1=-(p+n_1)^2/2,\ \gamma_1=-\gamma+n_3,\ s=2n_4,$$

$$3)\ \alpha_1=(-\varepsilon p-vq+\gamma+2n_1+1)^2/8,\ \beta_1=-(-\varepsilon p-vq-\gamma+2n_2+1)^2/8,$$

$$\gamma_1=-\varepsilon p+vq+n_3,\ \ s=2n_4,$$

$$4)\ \alpha_1=(-\varepsilon p-vq-\gamma+2n_1+1)^2/8,\ \beta_1=-(-\varepsilon p-vq+\gamma+2n_2+1)^2/8,$$

$$\gamma_1=\varepsilon p-vq+n_3,\ \ s=2n_4,$$

$$5)\ \alpha_1=(-\varepsilon p+vq-\gamma+2n_1+1)^2/8,\ \beta_1=-(\varepsilon p-vq-\gamma+2n_2+1)^2/8,$$

$$\gamma_1=\varepsilon p+vq-1+n_3,\ \ s=2n_4,$$

$$6)\ \alpha_1=(-\varepsilon p+vq+\gamma+2n_1+1)^2/8,\ \beta_1=-(\varepsilon p-vq+\gamma+2n_2+1)^2/8,$$

$$\gamma_1=-(\varepsilon p+vq-1)+n_3,\ \ s=2n_4,$$

where

$$p^2=2\alpha,\ q^2=-2\beta,\ \nu^2=\varepsilon^2=1,\ s=n_1+n_2+n_3,\ n_j\in Z,\ \ j=1..4.$$

To end the proof of theorem, it is sufficient to make different combinations of transformations, e. g., $\Lambda^{k3}_{-1,1}\circ\Lambda^{k2}_{1,-1}\circ\Lambda_{\varepsilon 1,\varepsilon 2}$, $\Lambda^{k4}_{-1,-1}\circ\Lambda^{k2}_{1,-1}\circ\Lambda_{\varepsilon 1,\varepsilon 2}$, $\Lambda^{k2}_{1,-1}\circ$

$\Lambda_{1,1} \circ \Lambda_{\varepsilon 1,\varepsilon 2}$, $\Lambda^{k3}_{-1,1} \circ \Lambda_{1,1} \circ \Lambda_{\varepsilon 1,\varepsilon 2}$ and to show the completeness of 1) – 6) with respect to Λ-transformation. The application of Λ-transformation to 1) – 6) yields the following: $\Lambda_{\delta 1,\delta 2} : 1) \to 3)$ This notation means that if we apply $\Lambda_{\delta 1,\delta 2}$ to the parameters which are written in the general form 1), then we obtain new parameters which may be written in the general form 3). Next, $\Lambda_{\delta 1,\delta 2} : 2) \to 4)$, $\Lambda_{1,1} : 3) \to 2)$, $\Lambda_{1,-1} : 3) \to 5)$, $\Lambda_{-1,1} : 3) \to 6)$, $\Lambda_{-1,-1} : 3) \to 1)$, $\Lambda_{1,1} : 4) \to 1)$, $\Lambda_{1,-1} : 4) \to 5)$, $\Lambda_{-1,1} : 4) \to 6)$, $\Lambda_{-1,-1} : 4) \to 2)$, $\Lambda_{1,1} : 5) \to 3)$, $\Lambda_{1,-1} : 5) \to 1)$, $\Lambda_{-1,1} : 5) \to 2)$, $\Lambda_{-1,-1} : 5) \to 4)$, $\Lambda_{1,1} : 6) \to 1)$, $\Lambda_{1,-1} : 6) \to 3)$, $\Lambda_{-1,1} : 6) \to 4)$, $\Lambda_{-1,-1} : 6) \to 2)$. The proof is completed. Note, that parameters of the current theorem may be rewritten in more compact form (3.1) – (3.4). ■

4. THE AUTO-BÄCKLUND TRANSFORMATIONS

Theorem 3.1 allows us to deduce the auto-Backlund transformations that link different solutions of (1.1) with the same parameter values. There exist several trivial auto-Backlund transformations for (1.1), namely:

$$\begin{aligned} S_0 &: \quad w(z,\alpha,\beta,0,\delta) \to w(-z,\alpha,\beta,0,\delta), \\ S_1 &: \quad w(z,\alpha,-\alpha,0,\delta) \to w^{-1}(z,\alpha,-\alpha,0,\delta), \\ S_2 &: \quad w(z,\alpha,-\alpha,\gamma,\delta) \to w^{-1}(-z,\alpha,-\alpha,\gamma,\delta), \\ S_3 &: \quad w(z,\alpha,\beta,0,0) \to w(\varepsilon i z,\alpha,\beta,0,0), \varepsilon^2 = 1. \end{aligned}$$

We shall not consider the last one because (1.1) may be solved in the case when $\gamma = \delta = 0$. In general, the application of $S_0 - S_2$ yields new solutions, as it is clear from the following examples:

1) $w = z - a$, $\alpha = 1/2$, $\gamma = a + 2$, $a^2 = -2\beta$, $a \neq 1$, $\delta = -1/2$;
2) $w = kz + 1$, $\gamma \neq 0$, $\alpha = -\beta = 1/2$, $k^2 = -2\delta$.

Now we may apply a composition of (T,Λ)-transformations and $S_0 - S_2$ in the following way: $w_1(z,\alpha,\beta,\gamma,\delta) = T^{-k} \circ S_i \circ T^k w(z,\alpha,\beta,\gamma,\delta)$, $k \in Z$, $i = 1, 2, 3$. Thus, we may obtain all parameter values when (1.1) has the auto-Backlund transformations, using the general parameter structure after the repeated application of the Backlund transformation. Therefore, the following statement is valid:

Theorem 4.1. *Equation* (1.1) *has the auto-Backlund transformations of the form* $T^{-k} \circ S_i \circ T^k$, $i = 0, 2$ *when either* $\gamma = n$, $n \in Z$ *or* $\sqrt{2\alpha} + \sqrt{-2\beta} = n$, $n \in Z$, *and the auto-Backlund transformations of the form* $T^{-k} \circ S_1 \circ T^k$, *when either* $\sqrt{2\alpha} + \sqrt{-2\beta} = 2n_1 + n_2$, $\gamma = n_2$, $n_1, n_2 \in Z$ *or* $8\alpha = (2n_1+1)^2$, $8\beta = -(2n_2+1)^2$, $n_1, n_2 \in Z$.

Proof. The deduction of the formulas is based on Theorem 3.1 on the general structure of the parameters after the substitution of the parameters of $S_0 - S_2$ into (3.1) – (3.4) ■

Next we derive the explicit form of the auto-Backlund transformation S_1 of the solutions of the fifth Painleve equation. As it has been noted, S_1 links different solutions with the same values of the parameters. Further we obtain the explicit form of the auto-Backlund transformation $T_{\varepsilon_1,\varepsilon_2,\varepsilon_3} \circ S \circ (T_{\varepsilon_1,\varepsilon_2,\varepsilon_3})^{-1}$, where $(T_{\varepsilon_1,\varepsilon_2,\varepsilon_3})^{-1}$ is given by

$$(T_{\varepsilon_1,\varepsilon_2,\varepsilon_3})^{-1} : \tilde{w}(z,\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta}) \to w(z,\alpha,\beta,\gamma,\delta) = 1 - 8\delta z\tilde{w}F_2^{-1}(\tilde{w}),$$

where $F_2(\tilde{w}) = (1-\varepsilon_2 a-\varepsilon_1 c)\varepsilon_3 k - \gamma + (2\gamma+4z\delta)\tilde{w} + +((-1+\varepsilon_2 a+\varepsilon_1 c)\varepsilon_3 k - \gamma)\tilde{w}^2 + 2\varepsilon_3 kz\tilde{w}'$,

$$\tilde{\alpha} = -(\gamma + \varepsilon_3 k(1-\varepsilon_2 a - \varepsilon_1 c))^2/(16\delta),\ \tilde{\beta} = (\gamma - \varepsilon_3 k(1-\varepsilon_2 a - \varepsilon_1 c))^2/(16\delta),$$

$$\tilde{\gamma} = \varepsilon_3 k(\varepsilon_2 a - \varepsilon_1 c),\quad \tilde{\delta} = \delta,\ \varepsilon_i^2 = 1,\ i \in \{1,2,3\}.$$

Let $w = w(z,\alpha,-\alpha,0,\delta)$, $w_1 = T_{1,1,\varepsilon_3}w$, $w_2 = T_{1,-1,\varepsilon_3}w, w_3 = T_{-1,1,\varepsilon_3}w$, $w_4 = T_{-1,-1,\varepsilon_3}w$, $\tilde{w}_1 = T_{1,1,\varepsilon_3}(Sw), \tilde{w}_2 = T_{1,-1,\varepsilon_3}(Sw)$, $\tilde{w}_3 = T_{-1,1,\varepsilon_3}(Sw)$, $\tilde{w}_4 = T_{-1,-1,\varepsilon_3}(Sw)$ be the solutions of the fifth Painleve equation obtained with the help of transformations (2.1) – (2.3). Note that solutions of (1.1) $\tilde{w}_i$ are the functions of w_i and w_i', $i \in \{1,2,3,4\}$. However, according to (2.1), we can find the algebraic relations between solutions w, w_i, $\tilde{w}_i$. Thus, the following statement is valid.

Theorem 4.2. *Solutions of the fifth Painleve equation listed above are connected by means of the following nonlinear superposition formulas:*

$$\tilde{w}_1 = 1/w_1,\quad \tilde{w}_4 = 1/w_4,$$

$$\tilde{w}_2 = \frac{2\delta zw + \varepsilon_3 kc(w-1)^2(w_2-1)}{2\delta zww_2 + \varepsilon_3 kc(w-1)^2(w_2-1)},$$

$$\tilde{w}_3 = \frac{-2\delta zw + \varepsilon_3 kc(w-1)^2(w_3-1)}{-2\delta zww_3 + \varepsilon_3 kc(w-1)^2(w_3-1)}.$$

Proof. These relations are obtained by direct computations. Note that the above formulas are not valid for the rational so lutions of 1.1). ■

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Penktosios Penlevė lygties Bäcklund'o transformacijos ir jų pritaikymai

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Darbe nagrinėjamos Backlund'o transformacijų siejamų su penktąja Penlevė lygtimi, savybės. Remiantis šiomis savybėmis sukonstruotos Penlevė lygties sprendinių netiesioginės superpozicijos. Tai suteikia galimybę sudaryti nagrinėjamos lygties sprendinių herarchiją ir susieti ją su kitos lygties (su kitais parametrais) sprendiniais.