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Some Tauberian Remainder Theorems for Hölder Summability

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Abstract. In this paper, we prove some Tauberian remainder theorems that generalize the results given by Meronen and Tammeraid [*Math. Model. Anal.*, **18**(1):97–102, 2013] for Hölder summability method using the notion of the general control modulo of the oscillatory behaviour of nonnegative integer order.

Keywords: Tauberian remainder theorem, λ-bounded series, general control modulo, Hölder summability.

AMS Subject Classification: 40E05; 40G05.

1 Introduction

Let $u = (u_n)$ be a sequence of real numbers. Throughout this paper, the symbol $u_n = O(1)$ means that (u_n) is bounded for large enough n. The sequence of the backward differences of (u_n) is denoted by (Δu_n), where $\Delta u_n = u_n - u_{n-1}$ for $n \geq 1$, and $\Delta u_0 = u_0$ for $n = 0$.

For a sequence (u_n),

$$u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u), \tag{1.1}$$

where $\sigma_n^{(1)}(u) = \frac{1}{n+1}\sum_{k=0}^{n} u_k$ and $V_n^{(0)}(\Delta u) = \frac{1}{n+1}\sum_{k=0}^{n} k\Delta u_k$.The identity (1.1) is called Kronecker identity.

Let the sequence $V^{(m)}(\Delta u) = (V_n^{(m)}(\Delta u))$ be defined as follows: For each integer $m \geq 1$ and for all nonnegative integers n,

$$V_n^{(m)}(\Delta u) = \sigma_n^{(1)}\big(V^{(m-1)}(\Delta u)\big).$$

The classical control modulo of the oscillatory behaviour of (u_n) is denoted by $\omega_n^{(0)}(u) = n\Delta u_n$. The general control modulo of the oscillatory behaviour of integer order $m \geq 1$ of a sequence (u_n) is defined in [9] by

$$\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}\big(\omega^{(m-1)}(u)\big).$$

Notice that the notion of the general control modulo of the oscillatory behavior of integer order $m \geq 1$ of a sequence (u_n) is a generalization of the classical control modulo of (u_n). For instance, if the sequence $(\omega_n^{(0)}(u))$ is bounded then the sequence $(\omega_n^{(1)}(u))$ is bounded from the definition of general control modulo. However, boundedness of the sequence $(\omega_n^{(1)}(u))$ does not imply boundedness of the sequence $(\omega_n^{(0)}(u))$. For example, if we take the sequence $(\omega_n^{(1)}(u))$ as a constant sequence, then $(\omega_n^{(0)}(u))$ is not bounded.

The concept of general control modulo of the oscillatory behavior of a sequence is used in some articles [6,7,8,20] in Tauberian theory. Actually, Çanak and Dik [6] obtained some Tauberian conditions in terms of the general control modulo of integer order to retrieve subsequential convergence of (u_n) from the boundedness of (u_n). Çanak and Totur [7] gave some sufficient conditions for the usual convergence and subsequential convergence of regularly generated sequences. Çanak and Totur [8] proved a Tauberian theorem for Cesàro summability methods, and Totur and Dik [20] gave some one-sided Tauberian conditions for a general summability method using the general control modulo of integer order. Moreover, various Tauberian theorems have been demonstrated by Çanak [1,2,3,4] and Çanak et al. [10].

Summability theory and Tauberian theorems have been applied to the sequence of fuzzy real numbers. In the recent papers, some results have been obtained for Nörlund and Riesz summability methods of sequences of fuzzy real numbers (see [5,21]).

A sequence (u_n) is said to be $(C,1)$ summable to s if the limit

$$\lim_{n\to\infty} \sigma_n^{(1)}(u) = s$$

exists.

The Hölder means of integer order of a sequence (u_n) are defined by

$$\sigma_n^{(k)}(u) = \begin{cases} \frac{1}{n+1}\sum_{j=0}^{n} \sigma_j^{(k-1)}(u), & k \geq 1 \;, \\ u_n, & k = 0 \end{cases}$$

for each integer $k \geq 0$ and for all nonnegative integers n and the sequence (u_n) is said to be Hölder, in short (H,k) summable to s if the limit

$$\lim_{n\to\infty} \sigma_n^{(k)}(u) = s$$

exists. It can be verified that $(H,0)$ summable of (u_n) means that (u_n) converges ordinary, and the $(H,1)$ method of summability is equivalent to the $(C,1)$ method of summability.

The Hölder summability method is regular, more generally, if a sequence (u_n) is (H,k) summable to s, where $k \geq 0$ and $k' > k$ for integer k, k', then (u_n) is also (H,k') summable to s. However, the converse is not necessarily true. For example, the sequence $(u_n) = (\sum_{j=0}^{n}(j+1)(-1)^j)$ is not $(H,1)$ summable, but

$$\lim_{n\to\infty} \sigma_n^{(2)}(u) = \frac{1}{n+1}\sum_{i=0}^{n} \sigma_i^{(1)}(u) = \frac{1}{4}.$$

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers such that $\lambda_n \to \infty$. A sequence (u_n) is called bounded with the rapidity (λ_n) (in short λ-bounded), if

$$\lambda_n(u_n - s) = O(1)$$

with $\lim\limits_{n\to\infty} u_n = s$. Let

$$m^\lambda = \left\{u = (u_n) \;\middle|\; \lim_{n\to\infty} u_n = s \text{ and } \lambda_n(u_n - s) = O(1)\right\}.$$

A sequence (u_n) is called λ-bounded by (H,k) method of summability, if

$$\lambda_n\big(\sigma_n^{(k)}(u) - s\big) = O(1)$$

with $\lim_{n\to\infty} \sigma_n^{(k)}(u) = s$. Shortly, we write $u \in ((H,k), m^\lambda)$.

It is known that a λ-bounded sequence is also λ-bounded by (H,k) method of summability. An example can be constructed to show that λ-boundedness by (H,k) method is not sufficient for λ-boundedness of a sequence. Let $(u_n) = (\sum_{k=0}^{n}(-1)^k)$, $\lambda_n = n+1$. Therefore $\lim_{n\to\infty} \sigma_n^{(1)}(u) = \frac{1}{2}$ and we have $\lambda_n(\sigma_n^{(1)}(u) - \frac{1}{2}) = O(1)$. That means $u \in ((H,1), m^\lambda)$. However the sequence $(\sum_{k=0}^{n}(-1)^k)$ is not convergent and this implies that $u \notin m^\lambda$.

G. Kangro [11] introduced the concepts of Tauberian remainder theorems using summability with given rapidity λ. Tammeraid [18] introduced the concept of λ-convergent sequence, and showed the relationship between some spaces of λ-convergent sequences. Tammeraid [17, 19] proved some Tauberian remainder theorems for several summability method, such as Cesàro, Hölder, Euler-Knopp methods. Moreover, a number of authors represented some Tauberian remainder theorems (see [12, 13, 14]). Recently, Sezer and Çanak [16] have obtained several Tauberian remainder theorems for the weighted mean method of summability using the weighted general control modulo of integer order 1 and 2.

Meronen and Tammeraid [15] proved the following Tauberian remainder theorems:

Theorem 1. *Let the condition*

$$\lambda_n V_n^{(0)}(\Delta u) = O(1)$$

is satisfied. If $u \in ((H,1), m^\lambda)$, *then* $u \in m^\lambda$.

Theorem 2. *Let the conditions*

$$\lambda_n \omega_n^{(0)}(u) = O(1), \quad \lambda_n \omega_n^{(2)}(u) = O(1), \quad \lambda_n\big(\sigma_n^{(2)}(u) - s\big) = O(1)$$

are satisfied. If $u \in ((H,1), m^\lambda)$, *then* $u \in m^\lambda$.

In this paper, we generalize Theorem 1 and Theorem 2 given by Meronen and Tammeraid [15] to Tauberian remainder theorems for (H,k) summability method. In our results, we use general control modulo order any integer $m \geq 0$ of the sequence (u_n).

2 Preliminary Results

We need the following lemma to be used in the proofs of main theorems.

Lemma 1. *For each integer $m \geq 2$ and for all nonnegative integers n,*

$$\omega_n^{(m)}(u) = \omega_n^{(0)}(u) - u_n + \sigma_n^{(1)}(u) + \sum_{j=1}^{m-1} (-1)^j \binom{m-1}{j} \big(\sigma_n^{(j-1)}(u) - 2\sigma_n^{(j)}(u) + \sigma_n^{(j+1)}(u)\big),$$

where $\binom{m-1}{j} = (m-1)(m-2)\dots(m-j)/j!$.

Proof. We establish the proof by the method of induction. For $m = 2$, we have

$$\begin{aligned}\omega_n^{(2)}(u) &= \omega_n^{(1)}(u) - \sigma_n^{(1)}\big(\omega^{(1)}(u)\big)\\ &= \omega_n^{(0)}(u) - \sigma_n^{(1)}\big(\omega^{(0)}(u)\big) - \sigma_n^{(1)}\big(\omega^{(0)}(u) - \sigma^{(1)}\big(\omega^{(0)}(u)\big)\big)\\ &= \omega_n^{(0)}(u) - 2V_n^{(0)}(\Delta u) + V_n^{(1)}(\Delta u)\\ &= \omega_n^{(0)}(u) - u_n + \sigma_n^{(1)}(u) + \big(-u_n + 2\sigma_n^{(1)}(u) - \sigma_n^{(2)}(u)\big)\\ &= \omega_n^{(0)}(u) - u_n + \sigma_n^{(1)}(u) + \sum_{j=1}^{1} (-1)^j \binom{1}{j} \big(\sigma_n^{(j-1)}(u) - 2\sigma_n^{(j)}(u) + \sigma_n^{(j+1)}(u)\big).\end{aligned}$$

Assume that the assertion is true for $m = k$. That is,

$$\omega_n^{(k)}(u) = \omega_n^{(0)}(u) - u_n + \sigma_n^{(1)}(u) + \sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j} \big(\sigma_n^{(j-1)}(u) - 2\sigma_n^{(j)}(u) + \sigma_n^{(j+1)}(u)\big). \tag{2.1}$$

We must show that the assertion is true for $m = k + 1$. Namely, we establish that

$$\omega_n^{(k+1)}(u) = \omega_n^{(0)}(u) - u_n + \sigma_n^{(1)}(u) + \sum_{j=1}^{k} (-1)^j \binom{k}{j} \big(\sigma_n^{(j-1)}(u) - 2\sigma_n^{(j)}(u) + \sigma_n^{(j+1)}(u)\big).$$

By the definition of the general control modulo of (u_n), we get

$$\omega_n^{(k+1)}(u) = \omega_n^{(k)}(u) - \sigma_n^{(1)}\big(\omega^{(k)}(u)\big). \tag{2.2}$$

By the identity (2.1), we have

$$
\begin{aligned}
\omega_n^{(k+1)}(u) &= \omega_n^{(0)}(u) - u_n + \sigma_n^{(1)}(u) - V_n^{(0)}(\Delta u) + \sigma_n^{(1)}(u) - \sigma_n^{(2)}(u) \\
&\quad + \sum_{j=1}^{k-1}(-1)^j \binom{k-1}{j}\big(\sigma_n^{(j-1)}(u) - 2\sigma_n^{(j)}(u) + \sigma_n^{(j+1)}(u)\big) \\
&\quad - \sum_{j=1}^{k-1}(-1)^j \binom{k-1}{j}\big(\sigma_n^{(j)}(u) - 2\sigma_n^{(j+1)}(u) + \sigma_n^{(j+2)}(u)\big) \\
&= \omega_n^{(0)}(u) - u_n + \sigma_n^{(1)}(u) \\
&\quad + \sum_{j=1}^{k-1}(-1)^j \binom{k-1}{j}\big(\sigma_n^{(j-1)}(u) - 2\sigma_n^{(j)}(u) + \sigma_n^{(j+1)}(u)\big) \\
&\quad - \sum_{j=0}^{k-1}(-1)^j \binom{k-1}{j}\big(\sigma_n^{(j)}(u) - 2\sigma_n^{(j+1)}(u) + \sigma_n^{(j+2)}(u)\big).
\end{aligned}
$$

For the last sum on the right-hand side of the last identity, we have

$$
\begin{aligned}
&- \sum_{j=0}^{k-1}(-1)^j \binom{k-1}{j}\big(\sigma_n^{(j)}(u) - 2\sigma_n^{(j+1)}(u) + \sigma_n^{(j+2)}(u)\big) \\
&\quad = - \sum_{j=1}^{k}(-1)^{j-1} \binom{k-1}{j-1}\big(\sigma_n^{(j-1)}(u) - 2\sigma_n^{(j)}(u) + \sigma_n^{(j+1)}(u)\big) \\
&\quad = \sum_{j=1}^{k}(-1)^{j} \binom{k-1}{j-1}\big(\sigma_n^{(j-1)}(u) - 2\sigma_n^{(j)}(u) + \sigma_n^{(j+1)}(u)\big) \\
&\quad = \sum_{j=1}^{k-1}(-1)^{j} \binom{k-1}{j-1}\big(\sigma_n^{(j-1)}(u) - 2\sigma_n^{(j)}(u) + \sigma_n^{(j+1)}(u)\big) \\
&\qquad + (-1)^k \binom{k-1}{k-1}\big(\sigma_n^{(k-1)}(u) - 2\sigma_n^{(k)}(u) + \sigma_n^{(k+1)}(u)\big).
\end{aligned}
$$

Since $\binom{k-1}{j} + \binom{k-1}{j-1} = \binom{k}{j}$, the identity (2.2) can be written as

$$
\begin{aligned}
\omega_n^{(k+1)}(u) &= \omega_n^{(0)}(u) - u_n + \sigma_n^{(1)}(u) \\
&\quad + \sum_{j=1}^{k}(-1)^j \binom{k}{j}\big(\sigma_n^{(j-1)}(u) - 2\sigma_n^{(j)}(u) + \sigma_n^{(j+1)}(u)\big).
\end{aligned}
$$

Thus, we conclude that Lemma 1 is true for each integer $m \geq 2$. □

3 Main Results

Theorem 3. *Let the conditions*

$$
\lambda_n \omega_n^{(0)}(u) = O(1), \quad \lambda_n \omega_n^{(m)}(u) = O(1) \tag{3.1}
$$

and

$$\lambda_n\big(\sigma_n^{(j)}(u)-s\big)=O(1)\quad \text{for each integer } j \text{ such that } 2\le j\le m \tag{3.2}$$

are satisfied. If $u\in((H,1),m^{\lambda})$, then $u\in m^{\lambda}$.

Proof. From Lemma 1, we have

$$\begin{aligned}\lambda_n\omega_n^{(m)}(u)&=\lambda_n\Bigg(\omega_n^{(0)}(u)-u_n+\sigma_n^{(1)}(u)\\&\quad+\sum_{j=1}^{m-1}(-1)^j\binom{m-1}{j}\big(\sigma_n^{(j-1)}(u)-2\sigma_n^{(j)}(u)+\sigma_n^{(j+1)}(u)\big)\Bigg)\\&=\lambda_n\Bigg(\omega_n^{(0)}(u)-(u_n-s)+\big(\sigma_n^{(1)}(u)-s\big)\\&\quad+\sum_{j=1}^{m-1}(-1)^j\binom{m-1}{j}\Big(\big(\sigma_n^{(j-1)}(u)-s\big)-2\big(\sigma_n^{(j)}(u)-s\big)+\big(\sigma_n^{(j+1)}(u)-s\big)\Big)\Bigg).\end{aligned}$$

Rewriting the above equation, we have

$$\begin{aligned}\lambda_n(u_n-s)&=\lambda_n\omega_n^{(0)}(u)-\lambda_n\omega_n^{(m)}(u)\\&\quad+\lambda_n\big(\sigma_n^{(1)}(u)-s\big)+\lambda_n\sum_{j=1}^{m-1}(-1)^j\binom{m-1}{j}\big(\sigma_n^{(j-1)}(u)-s\big)\\&\quad-2\lambda_n\sum_{j=1}^{m-1}(-1)^j\binom{m-1}{j}\big(\sigma_n^{(j)}(u)-s\big)+\lambda_n\sum_{j=1}^{m-1}(-1)^j\binom{m-1}{j}\big(\sigma_n^{(j+1)}(u)-s\big).\end{aligned}$$

Using (3.1) and (3.2) we get

$$\lambda_n(u_n-s)=O(1)+O(1)+O(1)+O(1)+O(1)+O(1)=O(1).$$

Therefore we obtain $u\in m^{\lambda}$. □

Theorem 4. *Let the condition*

$$\lambda_n V_n^{(j)}(\Delta u)=O(1)\quad \text{for each integer } j \text{ such that } 0\le j\le k-1, \tag{3.3}$$

be satisfied. If $u\in((H,k),m^{\lambda})$, then $u\in m^{\lambda}$.

Proof. Suppose that $u\in((H,k),m^{\lambda})$. Taking $j=k-1$ in (3.3), it follows from the identity

$$\sigma_n^{(k-1)}(u)-\sigma_n^{(k)}(u)=V_n^{(k-1)}(\Delta u),$$

that

$$\begin{aligned}\lambda_n\big(\sigma_n^{(k-1)}(u)-s\big)&=\lambda_n\big(\sigma_n^{(k)}(u)-s\big)+\lambda_n\big(\sigma_n^{(k-1)}(u)-\sigma_n^{(k)}(u)\big)\\&=O(1)+O(1)=O(1).\end{aligned}$$

This implies $u \in \big((H,k-1), m^{\lambda}\big)$. Taking $j = k-2$ in (3.3), we get

$$\lambda_n\big(\sigma_n^{(k-2)}(u) - s\big) = \lambda_n\big(\sigma_n^{(k-2)}(u) - \sigma_n^{(k-1)}(u)\big) + \lambda_n\big(\sigma_n^{(k-1)}(u) - s\big) \\ = O(1) + O(1) = O(1)$$

from the identity

$$\sigma_n^{(k-2)}(u) - \sigma_n^{(k-1)}(u) = V_n^{(k-2)}(\Delta u).$$

Hence we have $u \in \big((H,k-2), m^{\lambda}\big)$. Continuing in this way, we obtain that $u \in \big((H,1), m^{\lambda}\big)$. Taking $j = 0$ in (3.3), we get $\lambda_n(u_n - \sigma_n^{(1)}(u)) = O(1)$ from the Kronecker identity. Thus we have

$$\lambda_n(u_n - s) = \lambda_n\big(u_n - \sigma_n^{(1)}(u)\big) + \lambda_n\big(\sigma_n^{(1)}(u) - s\big) = O(1) + O(1) = O(1).$$

This completes the proof. □

Theorem 5. *Let the condition*

$$\lambda_n \omega_n^{(j)} = O(1) \quad \text{for each integer } j \text{ such that } 0 \le j \le k \tag{3.4}$$

be satisfied. If $u \in ((H,k), m^{\lambda})$, *then* $u \in m^{\lambda}$.

Proof. By the definition of the general control modulo order 1 of (u_n), it follows

$$\lambda_n \omega_n^{(1)}(u) = \lambda_n \omega_n^{(0)}(u) - \lambda_n \sigma_n^{(1)}\big(\omega^{(0)}(u)\big) = \lambda_n \omega_n^{(0)}(u) - \lambda_n V_n^{(0)}(\Delta u).$$

Taking $j = 0$ and $j = 1$ in (3.4), we get $\lambda_n V_n^{(0)}(\Delta u) = O(1)$. From the definition of the general control modulo order 2 of (u_n), we obtain

$$\lambda_n \omega_n^{(2)}(u) = \lambda_n \omega_n^{(1)}(u) - \lambda_n \sigma_n^{(1)}\big(\omega^{(1)}(u)\big).$$

Taking $j = 0$ and $j = 2$ in (3.4), we obtain $\lambda_n V_n^{(1)}(\Delta u) = O(1)$. Continuing in this way, by Lemma 1, we obtain

$$\begin{aligned} \lambda_n \omega_n^{(k)}(u) &= \lambda_n \Bigg(\omega_n^{(0)}(u) - u_n + \sigma_n^{(1)}(u) \\ &\quad + \sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j} \big(\sigma_n^{(j-1)}(u) - 2\sigma_n^{(j)}(u) + \sigma_n^{(j+1)}(u)\big)\Bigg) \\ &= \lambda_n \omega_n^{(0)}(u) - \lambda_n V_n^{(0)}(\Delta u) \\ &\quad + \lambda_n \sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j} \Big(\big(\sigma_n^{(j-1)}(u) - \sigma_n^{(j)}(u)\big) - \big(\sigma_n^{(j)}(u) - \sigma_n^{(j+1)}(u)\big)\Big). \end{aligned}$$

From the last identity, we get

$$\lambda_n \omega_n^{(k)}(u) = \lambda_n \omega_n^{(0)}(u) - \lambda_n V_n^{(0)}(\Delta u) + \lambda_n \left(\sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j} V_n^{(j-1)}(\Delta u) \right) \\ - \lambda_n \left(\sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j} V_n^{(j)}(\Delta u) \right).$$

If taking $j = 0, 1, \ldots, k$ in (3.4), we obtain $\lambda_n V_n^{(k-1)}(\Delta u) = O(1)$. The conditions in Theorem 4 holds. Hence the proof is completed. □

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