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SIMPLE ALGORITHM'S FOR THE CALCULATION OF HEAT TRANSPORT PROBLEM IN PLATE

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ABSTRACT

The approximations of some heat transport problem in a thin plate are based on the finite volume and conservative averaging methods [1,2]. These procedures allow one to reduce the two dimensional heat transport problem described by a partial differential equation to an initial-value problem for a system of two ordinary differential equations (ODEs) of the first order or to an initial-value problem for one ordinary differential equations of the first order with one algebraic equation. Solution of the corresponding problems is obtained by using MAPLE routines "gear","mgear" and "lsode".

1. THE MATHEMATICAL MODEL

A plate of small thickness $\tilde{l}$ is heated in the furnace chamber. We assume that the plate occupies the region $\Omega = \{0 \leq x \leq \tilde{l}, -\infty \leq y \leq +\infty, -\infty \leq z \leq +\infty\}$ in a furnace. The heat is provided at the top $(x > \tilde{l})$ and on the bottom $(x < 0)$.

We shall consider the dimensionless initial-boundary value problem for temperature $T(t,x)$ distributions on the plate in the following form:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad x \in (0,l), t > 0, \tag{1.1}$$

$$\partial T(t,0)/\partial x = f_1\big(T(t,0)\big), \tag{1.2}$$
$$\partial T(t,l)/\partial x = f_2\big(T(t,l)\big), \tag{1.3}$$
$$T(0,x) = T_0(x), \tag{1.4}$$

where $f_1\big(T(t,0)\big) = \mathsf{Bi}_1\big(T^4(t,0)-\theta_b^4\big)+\mathsf{B}_1\big(T(t,0)-\theta_1\big)$, $f_2\big(T(t,l)\big) = \mathsf{Bi}_2\big(\theta_t^4 - T^4(t,l)\big) + \mathsf{B}_2\big(\theta_2 - T(t,l)\big)$, $\mathsf{B}_j = \alpha_j\tilde{l}/k$ $(j=1,2)$ are Bio numbers, $\mathsf{Bi}_j = \epsilon_j\sigma\tilde{l}T_{\max}^3/\ k$ $(j=1,2)$ are Bio radiation numbers, $T_0(x)$ is the dimensionless initial temperature in the plate, α_1,α_2 are convective heat transfer coefficients, θ_1,θ_2 are the dimensionless temperatures of air in the furnace, θ_b,θ_t denote, respectively, the dimensionless temperature of the heater at the furnace top and on the furnace bottom, ϵ_1,ϵ_2 are coefficients of emissivity $(0 \le \epsilon_1,\epsilon_2 \le 1)$, k is thermal conductivity of the plate, $\sigma = 5.6703{\cdot}10^{-8}W/(m^2K^4)$ is the Stefan-Boltzmann constant, $T_{\max}$ is the maximum of dimensional temperature in the furnace. The dimensionless time $t \ge 0$ and coordinate $x \in (0,l)$ have the following multipliers: $\tilde{l}^2\varrho c_p/k,\ \tilde{l}$, where $\varrho,\ c_p$, are, respectively, the density and the specific heat of the plate. Generally the parameter $l=1$, but in the special symmetrical case ($\mathsf{Bi}_1 = \mathsf{Bi}_2 = \mathsf{Bi}, \mathsf{B}_1 = \mathsf{B}_2 = \mathsf{B}, \theta_b = \theta_t, \theta_1 = \theta_2$) we have $l = 0.5$ and the boundary condition (1.3) in the form

$$\partial T(t,0.5)/\partial x = 0. \tag{1.5}$$

Boundary conditions (1.2) – (1.3) describe the radiation from heaters and the convection. We assume that all parameters in expressions (1.1) – (1.4) are constant. The temperature distribution $T_0(x)$ in initial condition (1.4) is consistent with boundary conditions (1.2), (1.3) in the form

$$T_0'(0) = f_1\big(T_0(0)\big), \qquad T_0'(l) = f_2\big(T_0(l)\big), \tag{1.6}$$

where $T_0' = dT_0/dx$.

For boundary conditions (1.2), (1.5) we have

$$T_0'(0) = f_1\big(T_0(0)\big), \qquad T_0'(0.5) = 0.$$

In the symmetrical case we can consider also boundary conditions (1.2), (1.3) by $l=1, f_2 = -f_1, T(t,1) = T(t,0)$. In this case in the initial condition (1.4) we can consider

$$T_0(x) = T_* - (x^2 - x)f_1(T_*), \tag{1.7}$$

where $T_* = T_0(0)$ is constant temperature.

2. TWO POINT FINITE-DIFFERENCE SCHEME(TWO ODES)

The approximation of differential problem (1.1) – (1.5) is based on the application of the method of finite volumes [2]. We consider only two grid points in the x-direction: $x_1 = 0, x_2 = l$. To derive a finite-difference equation associated with the first grid point $x_1 = 0$ we integrate the differential equation (1.1) from x_1 to $x = l/2$. Thus we obtain

$$W_{0.5} - W_0 = \int_0^{l/2} G(t,x)dx, \tag{2.1}$$

where $W(t,x) = \partial T/\partial x$ is the flux-function, $W_{0.5} = W(t,l/2), W_0 = W(t,0) = f_1\big(T(t,0)\big), G(t,x) = \partial T/\partial t = \dot{T}$. Expression (2.1) presents the integral form of the conservation law within the interval $x \in (0,l/2)$. In the classical formulation of the finite volumes method it is assumed that the flux terms $W_{0.5}$ in (2.1) are approximated by the finite difference expressions. Then the corresponding difference scheme is not exact for a given function G. Here we have the possibility to make the exact difference scheme. Having denoted $T(t,0) = T_1, T(t,l) = T_2$, we integrate equation (1.1) from $x = l/2$ to $x \in (0,l)$, and then from x_1 to x_2. This yields

$$T_2 - T_1 = lW_{0.5} + D_1, \quad D_1 = \int\limits_0^l dx \int\limits_{l/2}^x G(t,\xi)d\xi. \tag{2.2}$$

Next, from the integral form of the conservation law (2.1) there follows a 2-point difference equation associated with the grid point x_1 in the form

$$\big(T_2(t) - T_1(t)\big)/l - f_1\big(T_1(t)\big) = R_1, \quad R_1 = \int\limits_0^l (1 - x/l)G(t,x)dx. \tag{2.3}$$

Similarly as before by integrating equation (1.1) from $x = l/2$ to x_2 we obtain the integral form of the conservation law within the interval $x \in (l/2,l)$ in the form

$$W_1 - W_{0.5} = \int_{l/2}^{l} G(t,x)dx, \quad W_1 = W(t,l) = f_2\big(T(t,l)\big). \tag{2.4}$$

We determine $W_{0.5}$ and obtain a difference equation associated with the grid point x_2 as

$$\big(T_1(t) - T_2(t)\big)/l + f_2\big(T_2(t)\big) = R_2, \quad R_2 = \frac{1}{l}\int_0^l xG(t,x)dx. \tag{2.5}$$

When the function $G(t,x)$ is given and the functions R_1, R_2 are known, then the difference equations (2.3), (2.5) are exact approximations for unknown functions $T_1(t), T_2(t)$. To approximate the right-side integrals R_1, R_2 we consider different quadrature formulae.

2.1. The quadrature rule of interpolating type

For the approximation of the dimensionless integral $I_1 = R_1/l = \int_0^1 (1-\xi)G(\xi)d\xi$ with the weight function $1-\xi$ we consider the two point integration formula

$$I_1 = A_1 G_1 + A_2 G_2 + r(\eta_1), \tag{2.6}$$

where $\xi = x/l, G(\xi) = G(t,\xi)$, $r(\eta_1) = 0.5E_0 \partial^2 G(\eta_1)/\partial\xi^2$ is the error term $\eta_1 \in (0,1)$, A_1, A_2, E_0 are the undetermined coefficients, $G_1 = G(0) = \dot{T}_1$, $G_2 = G(1) = \dot{T}_2$. By using the powers functions $G(\xi) = \xi^j$, $j = 0,1,2$ we find the linear system of three algebraic equations in the following form

$$\begin{cases} 1/2 = A_1 + A_2 \\ 1/6 = A_2 \\ 1/12 = A_2 + E_0. \end{cases} \tag{2.7}$$

Therefore $A_1 = 1/3, A_2 = 1/6, E_0 = -(1/12)$, and the quadrature formula in dimensional form

$$R_1 = l\Big(\frac{1}{3}G_1 + \frac{1}{6}G_2 - \frac{l^2}{24}\partial^2 G(t,\eta_1)/\partial x^2\Big) \tag{2.8}$$

is precise for the linear functions $(\eta_1 \in (0,l), G_1 = G(t,0), G_2 = G(t,l))$. Similarly,

$$R_2 = l\Big(\frac{1}{6}G_1 + \frac{1}{3}G_2 - \frac{l^2}{24}\partial^2 G(t,\eta_2)/\partial x^2\Big), \quad \eta_2 \in (0,l). \tag{2.9}$$

The degree of precision from quadrature rules (2.8), (2.9) is only 1. For the approximate solutions $T_1(t), T_2(t)$, deleting the error terms r, we have from (2.3), (2.5) a system of two ODEs of the first order in the following form:

$$\begin{cases} \dot{T}_1 = 6(T_2 - T_1)/l^2 - 4f_1(T_1)/l - 2f_2(T_2)/l, \\ \dot{T}_2 = -6(T_2 - T_1)/l^2 + 4f_2(T_2)/l + 2f_1(T_1)/l, \end{cases} \tag{2.10}$$

where $f_1(T_1) = \mathsf{Bi}_1(T_1^4 - \theta_b^4) + \mathsf{B}_1(T_1 - \theta_1)$, $f_2(T_2) = -\mathsf{Bi}_2(T_2^4 - \theta_t^4) - \mathsf{B}_2(T_2 - \theta_2)$.

The system (2.10) of ODEs should be solved with initial conditions

$$T_1(0) = T_0(0), \qquad T_2(0) = T_0(l). \tag{2.11}$$

In the symmetrical case $(l = 1, f_2 = -f_1, T_1 = T_2)$ we have from (1.7), (2.10), (2.11) the following initial-value problem:

$$\dot{T}_1 = -2f_1(T_1), \qquad T_1(0) = T_*. \tag{2.12}$$

In the other symmetrical case ($l = 0.5, f_2 = 0, T_2 = T(t, 0.5)$) from (2.10) it follows

$$\begin{cases} \dot{T}_1 = 24(T_2 - T_1) - 8f_1(T_1), \\ \dot{T}_2 = 24(T_1 - T_2) + 4f_1(T_1), \end{cases} \tag{2.13}$$

with $T_1(0) = T_*, \quad T_2(0) = T_* + 0.25f_1(T_*)$.

2.2. The quadrature formulae with derivatives

We consider the non classical quadrature formulae with derivatives of the first order for the approximation of a dimensionless integral I_1 in the following form:

$$I_1 = A_1G_1 + A_2G_2 + C_1G'_1 + C_2G'_2 + r(\eta_1), \tag{2.14}$$

where $r(\eta_1) = \frac{1}{4!}E_0\partial^4G(\eta_1)/\partial\xi^4$ is the error term $\eta_1 \in (0,1)$, A_1, A_2, C_1, C_2, E_0 are the undetermined coefficients, $G'_1 = \partial G(0)/\partial\xi, \quad G'_2 = \partial G(1)/\partial\xi$. The linear system of 5 algebraic equations ($G = \xi^j,\ j = 0,1,2,3,4$) is in the form

$$\begin{cases} 1/2 = A_1 + A_2, \\ 1/6 = A_2 + C_1 + C_2, \\ 1/12 = A_2 + 2C_2, \\ 1/20 = A_2 + 3C_2, \\ 1/30 = A_2 + 4C_2 + E_0. \end{cases} \tag{2.15}$$

Therefore, $A_1 = 7/20, A_2 = 3/20, C_1 = 1/20,\ C_2 = -(1/30), E_0 = 1/60$, and the quadrature formula

$$R_1 = l\Big(\frac{7}{20}G_1 + \frac{3}{20}G_2 + l\big(\frac{1}{20}G'_1 - \frac{1}{30}G'_2\big) + \frac{l^4}{1440}\partial^4G(t,\eta_1)/\partial x^4\Big),\ \eta_1 \in (0,l). \tag{2.16}$$

has to integrate any polynomial of degree 3 or less precisely. From boundary conditions (1.2), (1.3) it follows that $G'_1 = f'_1(T_1)\dot{T}_1, G'_2 = f'_2(T_2)\dot{T}_2$, where $f'_1(T_1) = 4\mathsf{Bi}_1T_1^3 + \mathsf{B}_1, f'_2(T_2) = -(4\mathsf{Bi}_2T_2^3 + \mathsf{B}_2)$. For the approximation of the integral R_2 in a similar way we can obtain a formula

$$R_2 = l\Big(\frac{3}{20}G_1 + \frac{7}{20}G_2 + l\big(\frac{1}{30}G'_1 - \frac{1}{20}G'_2\big) + \frac{l^4}{1440}\partial^4G(t,\eta_2)/\partial x^4\Big),\ \eta_2 \in (0,l). \tag{2.17}$$

The degree of precision of quadrature rules (2.16) – (2.17) is 3. Therefore, deleting the error terms r we have for the approximate solutions $T_1(t), T_2(t)$ from (2.3), (2.5) the following system of two ODEs of the first order:

$$\begin{cases} \dot{T}_1 = \Big((T_2 - T_1)(10 - \frac{5}{3}lf'_2)/l^2 - (7 - lf'_2)f_1/l - (3 - \frac{2}{3}lf'_2)f_2/l\Big)/A, \\ \dot{T}_2 = \Big((T_1 - T_2)(10 + \frac{5}{3}lf'_1)/l^2 + (7 + lf'_1)f_2/l + (3 + \frac{2}{3}lf'_1)f_1/l\Big)/A, \end{cases} \tag{2.18}$$

where $A = 2 + l(f_1' - f_2')/4 - l^2 f_1' f_2'/36$. The initial conditions has the form (2.11).

In the symmetrical case ($l = 1, f_2 = -f_1, T_1 = T_2$) we have from (2.18) the following initial–value problem:

$$\dot{T}_1 = -(4 + \frac{1}{3} f_1') f_1(T_1)/A, \qquad T_1(0) = T_*,\ A = 2 + f_1'/2 + (f_1'/6)^2. \quad (2.19)$$

In the other symmetrical case ($l = 0.5, f_2 = 0, T_2 = T(t, 0.5)$) from (2.18) follows ODEs

$$\begin{cases} \dot{T}_1 = \big(40(T_2 - T_1) - 14 f_1(T_1)\big)/A, \\ \dot{T}_2 = \big((T_1 - T_2)(40 + \frac{10}{3} f_1') + (6 + \frac{2}{3} f_1') f_1(T_1)\big)/A, \end{cases} \quad (2.20)$$

where $A = 2 + f_1'/8$. From the estimations of the error terms r in (2.16), (2.17) it follows that the truncation error of the methods (2.18) – (2.20) is proportional $\tilde{l}^4$. Hence the local error for method (2.20) is smaller than the error for method (2.19).

2.3. The system of three ODEs

We add one more point to the grid and consider the following three points $x_1 = 0, x_2 = l/2, x_3 = l$ in the interval $(0, l)$. To denoted $T_j = T(t, x_j), (j = 1, 2, 3)$ we have from (2.18) the system of two ODEs in the form

$$\begin{cases} \dot{T}_1 = \big((T_3 - T_1)(10 - \frac{5}{3} l f_2')/l^2 - (7 - l f_2') f_1/l - (3 - \frac{2}{3} l f_2') f_2/l\big)/A, \\ \dot{T}_3 = \big((T_1 - T_3)(10 + \frac{5}{3} l f_1')/l^2 + (7 + l f_1') f_2/l + (3 + \frac{2}{3} l f_1') f_1/l\big)/A, \end{cases} \quad (2.21)$$

where $f_1 = f_1(T_1), \quad f_2 = f_2(T_3)$. To derive a finite–difference equation associated with the grid point x_2 we integrate the differential equation (1.1) from $l/4$ to $3l/4$ and obtain

$$W_{0.75} - W_{0.25} = \int_{l/4}^{3l/4} G(t, x) dx, \quad (2.22)$$

where $W_{0.75} = W(t, 3l/4), W_{0.25} = W(t, l/4)$.
Determined the values of flux-function W we integrate equation (1.1):

1. from $l/4$ to $x \in (0, l/2)$, and from x_1 to x_2,
2. from $3l/4$ to $x \in (l/2, l)$, and from x_2 to x_3.

Thus we derive the 3-point difference equation in the form

$$2(T_1 - 2T_2 + T_3)/l = R_*, \quad (2.23)$$

where $R_* = \frac{2}{l}(\int_0^{l/2} xG(t,x)dx + \int_{l/2}^{l}(l-x)G(t,x)dx)$.
For the approximation of the dimensionless integral

$$I_* = 2R_*/l = \int_0^1 \xi G(\xi)d\xi + \int_1^2 (2-\xi)G(\xi)d\xi, \quad (\xi = 2x/l)$$

we consider the quadrature formula in the following form:

$$I_* = A_1G_1 + A_2G_2 + A_3G_3 + C_1G_1' + C_3G_3' + r(\eta), \tag{2.24}$$

where $r(\eta) = \frac{1}{6!}E_0\partial^6G(\eta)/\partial\xi^6$ is the error term ($\eta \in (0,2)$). By repeating arguments of (2.14), we obtain for the unknown coefficients the following values:

$$A_1 = A_3 = \frac{2}{15}, A_2 = \frac{11}{15}, C_1 = \frac{1}{40}, C_3 = -\frac{1}{40}, E_0 = \frac{29}{420},$$

and the quadrature formula

$$R_* = \frac{l}{30}\Big(2G_1 + 11G_2 + 2G_3 + \frac{3l}{16}(G_1' - G_3') + (\frac{l}{4})^6\frac{29}{315}\partial^6G(t,\eta)/\partial x^6\Big) \tag{2.25}$$

is precise up to 5-th degree $\eta \in (0,l)$. Therefore, by deleting the error term, we have from (2.23), (2.25) the following ODE:

$$\Big(\frac{2}{15} + \frac{l}{80}f_1'\Big)\dot{T}_1 + \frac{11}{15}\dot{T}_2 + \Big(\frac{2}{15} - \frac{l}{80}f_2'\Big)\dot{T}_3 = \Big(\frac{2}{l}\Big)^2(T_1 - 2T_2 + T_3). \tag{2.26}$$

In the symmetrical case ($l = 1, f_2 = -f_1, T_1 = T_3$) we have from (2.21), (2.25) the following system of ODEs:

$$\begin{cases} \dot{T}_1 = -(4 + \frac{1}{3}f_1')f_1(T_1)/A, \\ 11\dot{T}_2 + (4 + \frac{3}{8}f_1')\dot{T}_1 = 120(T_1 - T_2), \end{cases} \tag{2.27}$$

where $T_1(0) = T_*,\ T_2(0) = T_* + 0.25f_1(T_*)$ are the initial conditions, $A = 2 + f_1'/2 + (f_1'/6)^2$.

3. THE AVERAGING METHODS

The average temperature $\bar{T}$ is defined by the integral [1] $\bar{T}(t) = \frac{1}{l}\int_0^l T(t,x)dx$. Having integrated heat-conduction equation (1.1), we obtain

$$\dot{\bar{T}}(t) = (\partial T(t,l)/\partial x - \partial T(t,0)/\partial x)/l, \tag{3.1}$$

where $\dot{\bar{T}} = d\bar{T}/dt$. From boundary conditions (1.2), (1.3) it follows into equation (3.1)

$$\dot{\bar{T}}(t) = (f_2(T_2) - f_1(T_1))/l, \quad T_2 = T(t,l),\ T_1 = T(t,0). \tag{3.2}$$

3.1. The simple averaging method

A simple averaging method is based on the assumption that the temperature $T(t,x)$ of the plate is constant in x [1], so that $\bar{T}(t) = T(t,x),\ x \in [0,l]$. Then we find the following initial value problem for ODEs:

$$\dot{\bar{T}} = \big(f_2(\bar{T}) - f_1(\bar{T})\big)/l, \qquad \bar{T}(0) = \bar{T}_0, \quad \bar{T}_0 = \frac{1}{l}\int_0^l T_0(x)dx.xdd \tag{3.3}$$

In the symmetrical case $(l = 1, f_2 = -f_1)$ equations (2.12), (3.3) are equal.

3.2. The quadratic spline and averaging method

For a more accurate approximation of the averaging method we assume that the temperature distribution in the plate is parabolic in x:

$$T(t,x) = \bar{T}(t) + \Delta(t)(x - l/2) + \delta(t)(x^2 - l^2/3), \tag{3.4}$$

where Δ, δ are unknown functions of t. Then from equation (3.2) we have

$$T_2 = \bar{T} + 0.5l\Delta + 2\delta l^2/3, \qquad T_1 = \bar{T} - 0.5l\Delta - \delta l^2/3,$$

$$l\Delta = 6\bar{T} - 4T_1 - 2T_2, l^2\delta = 3(T_1 + T_2 - 2\bar{T}).$$

The unknown functions $T_1(t), T_2(t)$ can be derived from boundary conditions (1.2), (1.3) and expression (3.4) as a set of two nonlinear algebraic equations:

$$\Delta(t) = f_1\big(T_1(t)\big), \quad \Delta(t) + 2l\delta(t) = f_2\big(T_2(t)\big). \tag{3.5}$$

Therefore in this case we have the initial-value problem for a ODEs (3.2) and two algebraic equations (3.5). At each step of integrating the ODEs we have to solve two equations (3.5).

In the symmetrical case $(l = 0.5)$ the parabolic profile (3.4) is described as follows:

$$T(t,x) = \bar{T}(t) + \delta(t)(x^2 - x + 1/6), i \quad \bar{T}(t) = 2\int_0^{0.5} T(t,x)dx, \quad \Delta = -\delta.$$

Then equation (3.2) is written as

$$\dot{\bar{T}} = -2f_1(T_1), \tag{3.6}$$

where $T_1 = \bar{T} + \delta/6, T_2 = T(t, 0.5) = \bar{T} - \delta/12, \bar{T}_0 = T_* + f_1(T_*)/6$. From the boundary condition (1.2) it follows the algebraic equation

$$f_1(T_1) - 6(\bar{T} - T_1) = 0. \tag{3.7}$$

We have the initial-value problem for the ODEs (3.6) and one algebraic equation (3.7).

3.3. The averaging value of temperature

From the approximation of the integral $\bar{T}(t) = \frac{1}{l}\int_0^l T(t,x)dx$ we obtain the following 3-point quadrature formula:

$$\bar{T} = \frac{1}{30}(7T_1 + 16T_2 + 7T_3 + \frac{l}{2}(T_1' - T_3') + \frac{l^6}{20160}\partial^6 T(t,\eta)/\partial x^6),$$

where $\eta \in (0,l), T_1' = f_1(T_1), T_3' = f_2(T_3)$.

In the symmetrical case ($l = 1, f_2 = -f_1, T_1 = T_3$) we have formula:

$$\bar{T} = \frac{1}{30}\big(14T_1 + 16T_2 + f_1(T_1)\big). \tag{3.8}$$

In the case of two grid points ($x_1 = 0, x_2 = l$) we can obtain the following quadrature formula:

$$\bar{T} = \frac{1}{2}\big(T_1 + T_2 + \frac{l}{6}(T_1' - T_2') + \frac{l^4}{360}\partial^4 T(t,\eta)/\partial x^4\big),$$

where $\eta \in (0,l),\ T_2' = f_2(T_2)$.

In the symmetrical case ($l = 1/2, f_2 = 0$) we have formula:

$$\bar{T} = \frac{1}{2}\big(T_1 + T_2 + \frac{1}{12}f_1(T_1)\big).$$

4. SOME EXAMPLES AND NUMERICAL RESULTS

The numerical solutions of stiff systems of ODEs (2.10) – (2.13), (2.18) – (2.20), (2.26), (2.27), (3.3), (3.2), (3.5) and (3.6), (3.7) are obtained by using "MAPLE" routines "gear" (C. W. Gear single step extrapolation method), "mgear" (C. W. Gear multi step methods) or "lsode" (Livermore stiff ODE solver)[3]. With other "MAPLE" routines ("rkf 45", "rkf 78", "classical") the solutions are not obtainable for $\mathsf{Bi} \neq 0$ ($\mathsf{Bi}_1 \neq 0$ or $\mathsf{Bi}_2 \neq 0$).

The approximate values of T are compared with values of T^* obtained by the Fourier series in the linear case ($\mathsf{Bi}_1 = \mathsf{Bi}_2 = 0$) and by the explicit finite difference method with the space step $h = 0.02$ and time step $\tau = h^2/6$ in the nonlinear case ($\mathsf{Bi} \neq 0$).

In the linear symmetrical case ($l = 1, \mathsf{B}_1 = \mathsf{B}_2 = f_1' = \mathsf{B},\ \theta_1 = \theta_2 = 1$, $\mathsf{Bi}_1 = \mathsf{Bi}_2 = 0$) we have the solution of initial boundary-value problem (1.1) – (1.4), (1.7) in the following analytic form of Fourier series:

$$\begin{cases} T_1^* = T(t,0) = 1 - 0.5\mathsf{B}^2(1 - T_*)\sum\limits_{i=1}^{\infty} a_i(t), \\ T_2^* = T(t,0.5) = 1 - 0.5\mathsf{B}^2(1 - T_*)\sum\limits_{i=1}^{\infty} a_i(t)/\cos(\gamma_i), \\ T_{av}^* = \int_0^1 T(t,x)dx = 1 - 0.25\mathsf{B}^3(1 - T_*)\sum\limits_{i=1}^{\infty} a_i(t)/\gamma_i^4, \end{cases} \tag{4.1}$$

Table 1.
Linear case, B=0.9

t	T_1^*	T_2^*	T_{av}^*	$T_1(2.13)$	$T_2(2.13)$	$T_{av}(2.13)$
0.1	0.40324	0.26449	0.31134	0.406	0.268	0.315
0.2	0.48945	0.37069	0.41079	0.495	0.377	0.417
0.3	0.56318	0.46157	0.49589	0.570	0.470	0.504
0.4	0.62626	0.53933	0.56869	0.634	0.549	0.578
0.5	0.68024	0.60586	0.63098	0.689	0.616	0.641
0.6	0.72642	0.66278	0.68427	0.735	0.673	0.694
0.7	0.76593	0.71148	0.72987	0.774	0.722	0.740
0.8	0.79973	0.75315	0.76888	0.808	0.763	0.778
0.9	0.82865	0.78880	0.80226	0.837	0.799	0.811
1.0	0.85340	0.81930	0.83081	0.861	0.829	0.839

where $a_i(t) = \exp(-4\gamma_i^2 t)/(\gamma_i^2 + 0.5B + 0.25B^2)/\gamma_i^2$, γ_i is the solution of following transcendent equation

$$\gamma_i \tan(\gamma_i) = 0.5B \ (i = 1, 2, 3, \ldots).$$

The solutions of ODEs (2.12) and (2.19) are in the following form: $T_1(t) = 1-(1-T_*)\exp(-\beta t)$, where $\beta = 2\mathsf{B}$ for (2.12) and $\beta = \mathsf{B}(4+\mathsf{B}/3)/(2+\mathsf{B}/2+(\mathsf{B}/6)^2)$ for (2.19). Corresponding, $\beta = 2\mathsf{B}$ for (3.3) and $\beta = 2i\mathsf{B}/(1+\mathsf{B}/6)$ for (3.6), (3.7), where β is the parameter in the following solutions of averaging methods: $\bar{T}(t) = 1-(1-\bar{T}_0)\exp(-\beta t)$, $T_1(t) = 1-(1-T_*)\exp(-\beta t)$, $\delta(t) = 0.5\beta(1-\bar{T}_0)\exp(-\beta t)$. The control values for the temperature $(T_1, T_2, \bar{T} = T_{av})$ of $f_2 = -f_1, T_* = 0.3$ are computed

1. for the linear case by $\mathsf{B}_1 = \mathsf{B}_2 = 0.9, \theta_1 = \theta_2 = 1$, $T_2(0) = 0.1425$, $\bar{T}_0 = 0.195$ at the moments of dimensionless time $t = 0.1 * i, \quad i = \overline{1,10}$,
2. for the nonlinear case by $\mathsf{Bi}_1 = \mathsf{Bi}_2 = \mathsf{Bi} = 0.3, \mathsf{B}_1 = \mathsf{B}_2 = 0$, $\theta_b = \theta_t = 1$, $T_2(0) = 0.22561$, $\bar{T}_0 = 0.25040$ at the moments of time $t = 0.2 * i, \quad i = \overline{1,10}$.

Table 2.
Linear case, B=0.9

t	$T_1(2.20)$	$T_2(2.20)$	$T_{av}(2.20)$	$T_1(3.6)$	$T_2(3.6)$	$T_{av}(3.6)$
0.1	0.40313	0.26436	0.31121	0.4014	0.2667	0.3116
0.2	0.48937	0.37059	0.41070	0.4882	0.3730	0.4114
0.3	0.56313	0.46151	0.49583	0.5623	0.4638	0.4967
0.4	0.62623	0.53929	0.56865	0.6257	0.5415	0.5696
0.5	0.68022	0.60584	0.63096	0.6800	0.6079	0.6319
0.6	0.72641	0.66277	0.68426	0.7263	0.6647	0.6853
0.7	0.76593	0.71148	0.72987	0.7660	0.7133	0.7309
0.8	0.79974	0.75316	0.76889	0.7999	0.7549	0.7699
0.9	0.82867	0.78881	0.80227	0.8289	0.7904	0.8032
1.0	0.85342	0.81932	0.83083	0.8537	0.8208	0.8317

Comparison of the values of temperature obtained by different numerical methods can be seen in Tables 1 – 4. The best agreement is observed for averaging methods (3.2), (3.5), (3.6), (3.7) and for difference methods (2.18) – (2.20). The averaging values of temperature $\bar{T} = T_{av}$ for methods (2.13), (2.20) are obtained by formula (3.8). Computing the ODEs (2.27) and (3.6), (3.7), is obtained that the value of T_1 from (2.27) are equal with 6 decimal places comparing with (3.6), (3.7), but the values of T_2 are more precise than (3.6), (3.7) on one decimal place. Comparing the numerical solutions obtained from finite-difference method (2.20) and averaging methods (3.3), (3.6), (3.7), it is visible, that is method (2.20) more precise (accurate to 4 decimal places), but the correction of averaging method (3.6), (3.7) (accurate to 3 decimal places) is more precise than simple averaging method (3.3) (accurate only to one decimal place).

The given methods can be generalized for other nonlinear problems of partial differential equations.

Table 3.
Nonlinear case, Bi=0.3

t	T_1^*	T_2^*	T_{av}^*	$T_1(2.13)$	$T_2(2.13)$	$T_{av}(2.13)$
0.2	0.41693	0.34396	0.36833	0.4172	0.3442	0.3686
0.4	0.52847	0.45886	0.48214	0.5293	0.4597	0.4830
0.6	0.63076	0.56690	0.58830	0.6324	0.5687	0.5900
0.8	0.71999	0.66420	0.68294	0.7226	0.6671	0.6857
1.0	0.79378	0.74753	0.76309	0.7972	0.7515	0.7669
1.2	0.85181	0.81531	0.82762	0.8557	0.8200	0.8320
1.4	0.89555	0.86793	0.87726	0.8996	0.8728	0.8819
1.6	0.92743	0.90723	0.91406	0.9312	0.9119	0.9184
1.8	0.95011	0.93569	0.94057	0.9534	0.9399	0.9444
2.0	0.96594	0.95585	0.95927	0.9687	0.9594	0.9625

Table 4.
Nonlinear case, Bi=0.9

t	$T_1(2.20)$	$T_2(2.20)$	$T_{av}(2.20)$	$T_1(3.6)$	$T_2(3.6)$	$T_{av}(3.6)$
0.2	0.41692	0.34395	0.36830	0.4168	0.3441	0.3683
0.4	0.52845	0.45884	0.48211	0.5282	0.4591	0.4821
0.6	0.63073	0.56687	0.58826	0.6304	0.5673	0.5883
0.8	0.71995	0.66417	0.68289	0.7196	0.6647	0.6830
1.0	0.79374	0.74749	0.76305	0.7934	0.7481	0.7632
1.2	0.85178	0.81528	0.82758	0.8516	0.8160	0.8278
1.4	0.89553	0.86791	0.87723	0.8954	0.8686	0.8776
1.6	0.92742	0.90721	0.91404	0.9274	0.9079	0.9144
1.8	0.95010	0.93569	0.94056	0.9501	0.9363	0.9409
2.0	0.96594	0.95585	0.95926	0.9660	0.9563	0.9596

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PAPRASTAS ALGORITMAS ŠILUMOS LAIDUMO UŽDAVINIUI PLOKŠTELĖJE SPRĘSTI

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Šilumos laidumo uždavinio plonoje plokštelėje aproksimacija pagrįsta baigtinių tūrių ir konservatyviuoju vidurkinimo metodu. Šie metodai leidžia dvimatį šilumos laidumo uždavinį, aprašomą dalinėmis išvestinėmis, suvesti į dviejų pirmos eilės paprastųjų diferencialinių lygčių sistemą arba į vieną pirmos eilės paprastąją diferencialinę lygtį su papildoma algebrine lygtimi. Atitinkami parastųjų diferencialinių lygčių sprendiniai randami naudojant MAPLE paketą.