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INFLUENCE OF INPUT DATA ON THE SOLUTION OF THE MULTIPHASE LIQUID FLOW MODEL

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ABSTRACT

In the paper, a filtration model for immiscible – compressible and incompressible – two-phase liquids is considered. The 1D case of the model is analyzed in detail and it is shown that approximation of the Buckley-Leverett function by a step-wise linear function leads to new fictitious jumps of the solution.

1. INTRODUCTION

Filtration of immiscible multiphase – especially two-phase – liquid has been much studied in the 2^{nd} half of our century (see, e.g. monographs [1], [3], [5], [7]). The reason is that by means of the models of two-phase liquid filtration a wide field of application can be described. At the same time solving of these models presents a real challenge - both in theory and calculation.

The two-phase filtration is described by partial differential equations of the hyperbolic type of the first order (or by a system of such equations). In the corresponding solutions jumps may arise even if the input data are highly smooth and consistent (there are initial and boundary conditions) [5,6]. As input data the so-called phase permeability curves should be considered, which are characterized by mobility of a phase depending on its permeability. The phase permeability curves determine the form of a function – the so-called Buckley-Leverett function – describing nonlinearity of the partial differential equation. In turn, the shape of the mentioned curves is determined by the properties of a porous medium and liquid. This was the reason for many authors to investigate how the solution of the Buckley-Leverett function behaves in a stochastic porous medium. A. N. Konovalov [5] has shown that minor variations in the phase permeability curves can essentially change the solution (the so-called instability of coefficients).

The main goal of the present work is to show that, if we refuse the smoothness of the Buckley-Leverett function (approximating it by a piecewise-linear function) we can create many new imaginary jumps of solution. Here we consider filtration of two-phase immiscible – compressible and incompressible – liquid, with the model of incompressible liquid serving as the limiting case for that of compressible liquid.

2. THE BASE RELATIONSHIPS DESCRIBING THE FILTRATION PROCESS OF TWO-PHASE IMMISCIBLE LIQUID

Filtration of two-phase immiscible liquid in a porous medium is described by saturation function for each phase, $s_i,\ i = 1, 2$. The conservation law for every phase can be written as [3], [5]-[7]

$$\frac{\partial (m s_i \rho_i)}{\partial t} + \operatorname{div}(\rho_i \vec{v_i}) = 0,\ i = 1, 2 \tag{2.1}$$

while the Darcy law for the i-th phase is described as

$$\vec{v_i} = -k_i(s_i)\,\mathrm{grad}\,\mathrm{p},\ \mathrm{i} = 1, 2. \tag{2.2}$$

Here m is the medium porosity, p is the pressure (common (equal) for both phases), $\rho_i,\ \vec{v_i}$ are the density of the i-th phase and velocity of filtration through it, respectively. The values

$$k_i(s_i) = k\,\frac{f_i(s_i)}{\mu_i}$$

are called the phase mobility functions, where k is the medium permeability and $f_i(s_i),\ \mu_i$ are the relative permeability of the i-th phase and its viscosity, respectively.

The equation system (2.1), (2.2) has to be supplemented by equations of state for the liquid and porous medium. These can be taken in the form [7]:

$$\frac{1}{\rho_i}\frac{d\rho_i}{dp} = \beta_{0i},\ i = 1, 2\ , \tag{2.3}$$

and, correspondingly,

$$\frac{dm}{dp} = \beta_1, \tag{2.4}$$

where β_{01}, β_{02}, β_1 denote compressibility. Then the model of incompressible liquid is obtained taking $\beta_{0i} = 0$, and, correspondingly, $\beta_1 = 0$.

The phase saturation functions describing the process of two-phase filtration meet the following obvious relationship:

$$s_1 + s_2 = 1. \tag{2.5}$$

Therefore, further we will only use the saturation of one phase – namely that of the second phase, $s = s_2$.

Finally, to describe the process it is useful (and convenient) to define the total velocity $\vec{v}$ of filtration:

$$\vec{v} = \vec{v_1} + \vec{v_2},$$

which by means of equation (2.2) can be expressed in the form

$$\vec{v} = -\eta(s)\,\mathrm{grad\,p}, \tag{2.6}$$

where

$$\eta(s) = k_1(s_1) + k_2(s_2).$$

3. A MATHEMATICAL MODEL OF FILTRATION PROCESS FOR INCOMPRESSIBLE TWO-PHASE LIQUID

Having computed both equations of system (2.1) for incompressible liquid filtration ($\rho_i \equiv \rho_{0i}$) in an incompressible porous medium ($m \neq m(t)$), we immediately arrive at the equation

$$\mathrm{div}\,(\vec{v}) = 0, \tag{3.1}$$

which with the help of (2.6) can be rewritten in the form

$$\mathrm{div}\,(\eta(s)\,\mathrm{grad\,p}) = 0. \tag{3.2}$$

In turn, having expressed through (2.2) the second phase filtration velocity $\vec{v_2}$ in a standard form:

$$\vec{v_2} = \frac{k_2(s)}{\eta(s)}\vec{v}$$

and substituting it in the second equation of system (2.1), we obtain:

$$m\frac{\partial s}{\partial t} + \mathrm{div}\,(\varphi\vec{v}) = 0, \tag{3.3}$$

where the function

$$\varphi(s) = \frac{k_2(s)}{k_1(s) + k_2(s)} = \frac{k_2(s)}{\eta(s)} \tag{3.4}$$

is called the Buckley-Leverett function.

We will analyze only qualitative properties of the solution of equation (3.3) for the 1D case and therefore transform the system of equations (3.2), (3.3) and (2.6) for this case. From (3.1) there immediately follows that $v = v(t)$, which gives for the 1D case the system of equations as below:

$$\frac{\partial}{\partial x}(\eta(s)\frac{\partial p}{\partial x}) = 0, \tag{3.5}$$

$$m\frac{\partial s}{\partial t} + v(t)\frac{\partial \varphi(s)}{\partial x} = 0, \tag{3.6}$$

$$v(t) = -\eta(s)\frac{\partial p}{\partial x}. \tag{3.7}$$

We will specify the initial and boundary conditions for finding saturation $s(x,t)$ in the form:

$$s\mid_{x=0}= s^1(t),\ s\mid_{t=0}= s^0(x). \tag{3.8}$$

In regard to the other two unknown functions, the simplest situation will take place if the function $v(t)$ is given, for example, $v(t) \equiv v \equiv const$ (according to (3.7), this is equivalent to giving the pressure gradient). Then from equation (3.6) with conditions (3.8) we find $s(x,t)$. This done, from (3.7) the pressure $p(x,t)$ can be derived if necessary (for this purpose one should know the pressure at some fixed point). In the case when the velocity $v(t)$ is not given, but the values of pressure are known, for example

$$p\mid_{x=0}= p^0(t),\ p\mid_{x=l}= p^1(t), \tag{3.9}$$

we can proceed as in [2]. Integration of equation (3.7) yields:

$$v(t) = \frac{p^0 - p^1}{\int\limits_0^l \frac{dx}{\eta(s)}}. \tag{3.10}$$

Having found $s(x,t)$ and $p(x,t)$ from equations (3.5) and (3.6) we can employ expression (3.7) to verify if at various points there is precise fulfillment of the condition that the total filtration velocity $v(t)$ is independent of x-coordinate.

4. A MATHEMATICAL MODEL FOR FILTRATION OF COMPRESSIBLE TWO-PHASE LIQUID

For compressible liquid using state equations (2.3), (2.4), we will transform the first term of equation (2.1). This yields:

$$m\frac{\partial s_i}{\partial t} + s_i(m\beta_{0i} + \beta_1)\frac{\partial p}{\partial t} + \frac{\mathrm{div}\,(\rho_i \vec{v_i})}{\rho_i} = 0,\ i = 1,2. \tag{4.1}$$

Next, we will use this equation with index $i = 2$ (then $s_i = s$) and transform the second term:

$$\frac{\mathrm{div}\,(\rho_2 \vec{v_2})}{\rho_2} = \mathrm{div}\,(\vec{v_2}) + \frac{\vec{v_2}}{\rho_2}\,\mathrm{grad}\,\rho_2 = \mathrm{div}\,(\varphi\vec{\mathrm{v}}) + \beta_{02}\,\vec{\mathrm{v}_2}\,\mathrm{grad}\,\mathrm{p}.$$

It remains now to express $\vec{v_2}$ from (2.2) so that equation (4.1) can be written in the form:

$$m\frac{\partial s}{\partial t} + \mathrm{div}\,(\varphi\vec{v}) + s(m\beta_{02} + \beta_1)\frac{\partial p}{\partial t} - \beta_{02}\,k_2(\mathrm{grad}\,\mathrm{p})^2 = 0. \tag{4.2}$$

To derive the second base equation, we have to solve both equations (4.1) using the identity (2.5):

$$\beta(s)\frac{\partial p}{\partial t} + \frac{\mathrm{div}\,(\rho_1\vec{v_1})}{\rho_1} + \frac{\mathrm{div}\,(\rho_2\vec{v_2})}{\rho_2} = 0, \tag{4.3}$$

where

$$\beta(s) = \beta_1 + m(s\beta_{02} + (1-s)\beta_{01})$$

is called the total compressibility [6].

Finally, we transform in this equation the sum of both last terms:

$$\frac{\mathrm{div}\,(\rho_1\vec{v_1})}{\rho_1} + \frac{\mathrm{div}\,(\rho_2\vec{v_2})}{\rho_2} = \mathrm{div}\,(\vec{v}) + (1-\varphi)\frac{\vec{v}}{\rho_1}\,\mathrm{grad}\,\rho_1 + \varphi\frac{\vec{\mathrm{v}}}{\rho_2}\,\mathrm{grad}\,\rho_2$$

$$= -\mathrm{div}\,(\eta\,\mathrm{grad}\,\mathrm{p}) + [(1-\varphi)\beta_{01} + \varphi\,\beta_{02}]\vec{\mathrm{v}}\,\mathrm{grad}\,\mathrm{p}.$$

This would allow to write equation (4.3) for finding the pressure, in the form:

$$\beta\frac{\partial p}{\partial t} + [(1-\varphi)\beta_{01} + \varphi\beta_{02}]\vec{v}\,\mathrm{grad}\,\mathrm{p} = \mathrm{div}\,(\eta\,\mathrm{grad}\,\mathrm{p}). \tag{4.4}$$

It is readily seen that the system of equations (4.2) and (4.4) for incompressible liquids ($\beta_{01} = \beta_{02} = 0$) and an incompressible porous medium ($\beta_1 = 0$) transforms into equations (3.3) and (3.2), respectively.

For the 1D case we have:

$$\beta\frac{\partial p}{\partial t} + [(1-\varphi)\beta_{01} + \varphi\beta_{02}]v\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}(\eta(s)\frac{\partial p}{\partial x}), \tag{4.5}$$

$$m\frac{\partial s}{\partial t} + \frac{\partial}{\partial x}(\varphi v) + s(m\beta_{02} + \beta_1)\frac{\partial p}{\partial t} - \beta_{02}\,k_2(\frac{\partial p}{\partial x})^2 = 0. \tag{4.6}$$

One can see that these equations present a general form of equations (3.5), (3.6), respectively. Equation (3.7) retains its form, however velocity v may now be dependent also on coordinate x:

$$v(x,t) = -\eta(s)\frac{\partial p}{\partial x}. \tag{4.7}$$

Additional conditions remain as above, with the only difference that the initial condition must be added to the pressure equation (4.5). As an alternative we can also preset the pressure gradient on one of the boundaries.

5. A NUMERICAL ALGORITHM FOR FILTRATION OF COMPRESSIBLE TWO-PHASE LIQUID

As mentioned above, the filtration model for incompressible liquid is obtainable from the general model by setting all the elasticity coefficients equal to zero. Then its numerical algorithm can be worked out directly for the general model. Since our intention is to analyze the qualitative properties of the solution (or appearance of new jumps of the solution), we can restrict ourselves to the 1D model.

As a result of numerical experiments, the best choice from various schemes has proved to be the following system of difference equations (corresponding to equations (4.5) – (4.7)):

$$\eta^j_{i-\frac{1}{2}} p^{j+1}_{i-1} - (\frac{h^2\beta^j_i}{\tau} + (\eta^j_{i-\frac{1}{2}} + \eta^j_{i+\frac{1}{2}}))p^{j+1}_i + \eta^j_{i+\frac{1}{2}} p^{j+1}_{i+1}$$

$$= h(\beta_{01}(1-\varphi^j_i) + \beta_{02}\varphi^j_i) v^j_{i-\frac{1}{2}}(p^j_i - p^j_{i-1}) - \frac{h^2\beta^j_i}{\tau} p^j_i, \tag{5.1}$$

$$j = \overline{0, J-1},\ i = \overline{1, n-1},$$

with additional conditions (3.9) (assuming that the initial time moment is given as $p\mid_{t=0} = p^0_0(x)$):

$$p^{j+1}_0 = p^0(t_{j+1}),\ p^{j+1}_n = p^1(x_n),\ p^0_i = p^0_0(x_i),\ j = \overline{0, J-1},\ i = \overline{0, n}, \tag{5.1$_1$}$$

$$s^{j+1}_i = s^j_i + \frac{\tau}{m}(-s^j_i(\beta_1 + m\,\beta_{02})\frac{p^{j+1}_i - p^j_i}{\tau} -$$

$$-v^j_{i-\frac{1}{2}}\frac{\varphi^j_i - \varphi^j_{i-1}}{h} + \beta_{02}\, k^j_{2,i}(\frac{p^j_i - p^j_{i-1}}{h})^2), \tag{5.2}$$

$$j = \overline{0, J-1},\ i = \overline{1, n},$$

with additional conditions

$$s_0^{j+1} = s^1(t_{j+1}),\ s_i^0 = s^0(x_i),\ j = \overline{0, J-1},\ i = \overline{1, n} \tag{5.2$_1$}$$

and

$$v_{i-\frac{1}{2}}^{j+1} = -\eta_{i-\frac{1}{2}}^{j+1} \frac{p_i^{j+1} - p_{i-1}^{j+1}}{h},\ j = \overline{0, J-1},\ i = \overline{1, n}. \tag{5.3}$$

Here for equations (5.1) and (5.3) we have

$$\eta_{i\mp\frac{1}{2}}^{j} = \frac{1}{2}(\eta_i^j + \eta_{i\mp 1}^j).$$

The standard notations are used in representation of the difference scheme. For example, τ is a step along the time axis with index j, h is a step along x-axis with index i, and so on.

The algorithm for solving the difference scheme is simple: first, by the factorization method for a three-diagonal system of equations, system (5.1) is solved in regard to p_i^{j+1}; then from equations (5.2) and (5.3) we find s_i^{j+1} and $v_{i-\frac{1}{2}}^{j+1}$ in an open form. Here the following details should only be mentioned: 1) at the initial time moment the total filtration velocity is found from equation (3.10); that is, it is assumed that to derive the filtration velocity at the initial moment one can use the model of incompressible liquid. 2) at the point of saturation jump the differential equation (5.1) yields for the pressure distribution one small peak (1-2%) on its either side, with a total width of 2-3 grid points. Such a defect can be remedied with the help of difference schemes of a higher order of precision, for example, those of ENO type [6] (it is easy to see that the scheme proposed by us possesses only the first order of approximation, with respect both to h and τ). Yet, since we analyze here only qualitative properties of the solution in places where saturation field is smooth, we will reconcile ourselves to this defect of the difference scheme as well as to the well-known numerical diffusion in the explicit up-wind scheme for saturation. To control this diffusion, we have chosen time step τ maximally close to the upper limit of the scheme stability, that is CFL≈1, and compared our numerical results with the exact solution, which for the given Buckley-Leverett function (3.4) is found by the characteristic method.

6. RESULTS AND DISCUSSION

The Buckley-Leverett function can be given, with the help of the phase mobility function, in the form

$$k_1(s) = \frac{k}{\mu_1}(1-s)^2,\ k_2(s) = \frac{k}{\mu_2}s^2, \tag{6.1}$$

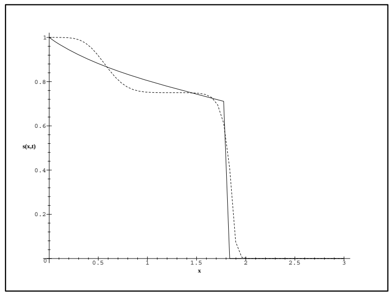

Figure 1. Saturation field at the time $t = 1.5$. The solid line corresponds to the solution by function $\varphi(s)$ and the broken line - to that by function $\overline{\varphi}(s)$. ($m = 4$, $h = 0.06122$, $\tau = 0.01515$).

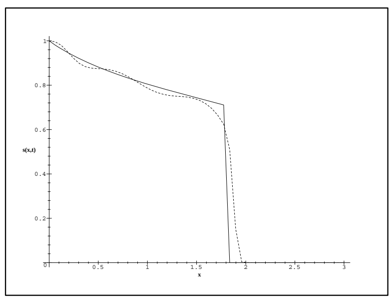

Figure 2. The same as in figure 1, but with a greater number of interpolation points ($m = 8$) for function $\varphi(s)$.

where $k = 1$, $\mu_1 = 1$, $\mu_2 = 0.1$. In turn, a perturbed Buckley-Leverett function $\tilde{\varphi}(s)$ is constructed as follows: we set the points s_i, $i = \overline{0,m}$, ($s_0 = 0$, $s_m = 1$), then according to (6.1) and (3.4) find $\varphi_i = \varphi(s_i)$ and based on this latter we construct a new piecewise-linear Buckley-Leverett function:

$$\tilde{\varphi}(s) = \varphi_i + \frac{\varphi_{i+1} - \varphi_i}{s_{i+1} - s_i}(s - s_i),\ s \in [s_i, s_{i+1}],\ i = \overline{0, m-1}. \tag{6.2}$$

Figure 1 shows the distribution of saturation for smooth function $\varphi(s)$ (a solid line) and for piecewise-linear function $\tilde{\varphi}(s)$ (a broken line). This calculation has been performed for incompressible liquid ($\beta = 0$) with $v(t) = const = m$ and uniform distribution of interpolation points s_i: $s_{i+1} - s_i = \Delta s = 0.25$ ($m = 4$). The second figure gives the same comparison, only for smaller interpolation steps $\Delta s = 0.125$ ($m = 8$).

It is readily seen that in place of a smooth saturation distribution we obtain a stepwise solution. In turn, in place of (6.2) one should employ function $\overline{\varphi}(s)$, which interpolates at the same points by parabolic spline (this ensures smoothness of the Buckley-Leverett function: $\overline{\varphi}(s) \in C^1[0,1]$), then in the saturation field the stepwise zones disappears. It means that, when specifying the phase mobility function $k_i(s)$ based on experimental data, we must ensure the smoothness of function $\varphi(s)$. This however is not always observed, that is why even in well-known programs when a 1D solution is sought for a homogeneous layer with phase permeability curves constructed on separate points, a solution of step-wise structure is obtained as is seen in figures 1 and 2.

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SKYSČIO TEKĖJIMO MODELIO SPRENDINIO PRIKLAUSOMYBĖ NUO PRADINIŲ DUOMENŲ

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Darbe nagrinėjamas dvisluoksnis filtracijos modelis spūdžiųir nespūdžiųskysčiųatvejais. Detaliai išnagrinėtas vienmatis modelis ir parodyta, kad Buckley-Leverett funkcijąaproksimuojant dalimis tiesine funkcija, sprendinyje atsiranda fiktyvus šuolis.