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A New Second-Order Difference Approximation for Nonlocal Boundary Value Problem with Boundary Layers

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Abstract. The aim of this paper is to present finite difference method for numerical solution of singularly perturbed linear differential equation with nonlocal boundary condition. Initially, the nature of the solution of the presented problem for the numerical solution is discussed. Subsequently, the difference scheme is established on Bakhvalov-Shishkin mesh. Uniform convergence in the second-order is proven with respect to the $\varepsilon-$ perturbation parameter in the discrete maximum norm. Finally, an example is provided to demonstrate the success of the presented numerical method. Thus, it is shown that indicated numerical results support theoretical results.

Keywords: singular perturbation, finite difference method, Bakhvalov-Shishkin mesh, uniformly convergence, nonlocal condition.

AMS Subject Classification: 65L10; 65L11; 65L12; 65L15; 65L20; 65L70; 34B10.

1 Introduction

In the present study, linear singularly perturbed problem with nonlocal boundary condition is discussed as follows:

$$\varepsilon^2 u''(x) + \varepsilon a\,(x)\,u'\,(x) - b(x)u(x) = f\,(x)\,, \quad 0 < x < 1, \tag{1.1}$$

$$u\,(0) = A, \tag{1.2}$$

$$u(1) - \gamma u(l_1) = B, \quad 0 < l_1 < 1, \tag{1.3}$$

where $0 < \varepsilon << 1$ is a small perturbation parameter; A, B and $|\gamma| < 1$ are given constants; $a(x) \geq 0$, $b\,(x) \geq \beta > 0$ and $f\,(x)$ are assumed to be sufficiently smooth functions in $[0,1]$. Furthermore, the solution of the problem (1.1)–(1.3) is within general boundary layers at $x=0$ and $x=1$ points.

Problems such as the nonlocal singular perturbation problem (1.1)–(1.3), are problems where the coefficients of the highest order derivative are a very small positive parameters such as $0<\varepsilon<<1$. Solving this problems with classical numerical methods may not be the right choice due to the difficulties that may arise from the small perturbation parameter [18, 23, 24, 25, 26, 27]. These difficulties are quick and fairly irregular variations within thin transition layers. These lead to unlimited number of derivatives in the solution of singular perturbation problems. Therefore, it is important to choose the most suitable numerical methods for singularly perturbed problems. These include finite difference and finite element methods. Thus, in the present study, we wanted to demonstrate that these difficulties can be overcome with the finite difference method.

Studies conducted on singular perturbation problems commenced in the 1900s. These problems were known to be common in the fields of natural sciences, engineering, medical sciences, fluid mechanics, aerodynamics, magnetic dynamics, diffusion theory, reaction diffusion, light emitting waves, electron plasma waves, communication networks, plasma dynamics, refined gas dynamics, mass transport, plastics, chemical reactor theory, oceanography, meteorology, electricity current, ion acoustic waves plasma and several physical modelling techniques (see, [2, 4, 9, 14, 15, 18, 24, 25, 26, 27]). Lately, singularly perturbed problems, particularly with the nonlocal boundary condition and boundary layers have been studied by several researchers (e.g., [1,7,8,10,11,12, 16, 17, 19, 20, 23] and the references therein). Bakhvalov used a special transformation in numerical solution of boundary solid problems [5]. Bitsadze and Samarskii obtained several generalizations for linear elliptic boundary value problems [6]. Čiegis, studied numerical solution of the singular perturbation problem with nonlinear boundary condition [13]. Different from the previous studies in the literature and for the first time, this problem is solved with the presented finite difference method on Bakhvalov-Shishkin mesh in order to demonstrate that the difference scheme has second-order convergence and a better result could be obtained. Especially, this method shows uniformly convergent provided only that $\varepsilon \leq CN^{-1}$. Namely, Bakhvalov-Shishkin mesh gives a stronger error bound for $\varepsilon \leq CN^{-1}$. Bakhvalov-Shishkin mesh is a modification of the Shishkin mesh described that incorporates idea by Bakhvalov. But the original Bakhvalov mesh requires the solution of a nonlinear equation to determine the transition point where the mesh switches from coarse to fine. Instead, the transition points are as in the Shishkin mesh [21]. There are many studies on the B-S (Bakhvalov-Shishkin) mesh: T. Linss has studied simple upwind difference scheme on a B-S mesh [21]. Analysis of a Galerkin finite element method on Bakhvalov-Shishkin mesh for a linear convection-diffusion problem is investiagated by Linss [22]. Uniform second-order hybrid schemes on Bakhvalov-Shishkin mesh are analyzed in [29]. Hybrid difference schemes with variable weights on Bakhvalov-Shishkin mesh are examined to the derivative for quasi-linear singularly perturbed convection-diffusion boundary value problems in [28]. Linear Galerkin finite element method on Bakhvalov-Shishkin mesh for singularly perturbed convection-diffusion problem is worked in [30].

The present study is structured as follows: Section 2 focused on the exact solution of the problem provided in (1.1)–(1.3) and several asymptotic evaluations on the fourth-order derivatives of the exact solution. In Section 3, the difference scheme is constructed as hybrid scheme. Subsequently, the structure of the Bakhvalov-Shishkin mesh is introduced. In Section 4, the second-order uniform convergence of the difference scheme is obtained according to ε. The present study is finalized with the conclusion section. Henceforth, C and C_0 are positive constants independent of ε and the mesh parameter in the following sections.

2 Certain properties of the continuous problem

This section focuses on some properties of the solution $u(x)$ of the problem (1.1)–(1.3), which will be essential in the further sections of the study.

Lemma 1. *Given that $a(x)$, $b(x)$ and $f(x) \in C^3[0,1]$. Then, the solution of the problem (1.1)–(1.3) fulfills the following inequalities:*

$$|u(x)| \le C_0, \tag{2.1}$$

$$\left|u^{(k)}(x)\right| \le C\left\{1+\frac{1}{\varepsilon^k}\left(e^{-\frac{\mu_1 x}{\varepsilon}}+e^{-\frac{\mu_2(1-x)}{\varepsilon}}\right)\right\},\quad 0<x<1,\ \ k=1,2,3, \tag{2.2}$$

where

$$C_0 = |A| + (1-|\gamma|)^{-1}\left[|B| + |\gamma|(|A| + \beta^{-1}\|f\|_\infty)\right] + \beta^{-1}\|f\|_\infty,\ \ |\gamma|<1,$$
$$\mu_1 = \frac{1}{2}\left(\sqrt{a^2(0)+4b(0)}+a(0)\right),\quad \mu_2 = \frac{1}{2}\left(\sqrt{a^2(1)+4b(0)}-a(1)\right).$$

Proof. Once maximum principle for (1.1) is used, we obtain that

$$|u(x)| \le |A| + |u(1)| + \beta^{-1}\|f\|_\infty. \tag{2.3}$$

Next, from boundary condition (1.3), we attain

$$|u(1)| \le |B| + |\gamma||u(l_1)|. \tag{2.4}$$

If $x = l_1$ is written in inequality (2.3), the following inequality is found

$$|u(l_1)| \le |A| + |u(1)| + \beta^{-1}\|f\|_\infty. \tag{2.5}$$

By setting (2.5) in inequality (2.4), we get

$$|u(1)| \le (1-|\gamma|)^{-1}\left[|B| + |\gamma|(|A| + \beta^{-1}\|f\|_\infty)\right]. \tag{2.6}$$

Then by setting (2.6) in inequality (2.3), we have

$$|u(x)| \le |A| + (1-|\gamma|)^{-1}\left[|B| + |\gamma|(|A| + \beta^{-1}\|f\|_\infty)\right] + \beta^{-1}\|f\|_\infty,$$

where

$$C_0 = |A| + (1-|\gamma|)^{-1}\left[|B| + |\gamma|(|A| + \beta^{-1}\|f\|_\infty)\right] + \beta^{-1}\|f\|_\infty,$$

and this prove the inequality (2.1).
The proof of inequality (2.2) is almost identical to that of [1,9] for $k=1$ as

$$|u'(x)| \le C\left\{1+\frac{1}{\varepsilon}\left(e^{-\frac{\mu_1 x}{\varepsilon}}+e^{-\frac{\mu_2(1-x)}{\varepsilon}}\right)\right\}, \quad 0<x<1, \quad k=1.$$

Now, we obtain inequality (2.2) for $k=2$. The proof of (2.2) for $k=3$ is obtained in the same way. Let us begin by taking the derivative of equation (1.1) two times,

$$\begin{aligned}&\varepsilon^2 v''(x)+\varepsilon a(x)v'(x)-b(x)v(x)=G(x),\\ &v(0)=u''(0), \quad v(1)=u''(1),\end{aligned}$$

where

$$u''(x)=v(x), \tag{2.7}$$

$$G(x)=f''(x)-2\varepsilon a'(x)u''(x)-(\varepsilon a''(x)-2b')\,u'+b''(x)u(x), \tag{2.8}$$

and also, from (1.1) we get

$$u''(0)\le \frac{C}{\varepsilon^2}, \quad u''(1)\le \frac{C}{\varepsilon^2}.$$

Now, let us take $v(x)$ as follows:

$$v(x)=v_1(x)+v_2(x), \tag{2.9}$$

where $v_1(x)$ and $v_2(x)$ are the solutions of the following problems:

$$Lv_1(x)=G(x), \quad v_1(0)=0, \quad v_1(1)=0, \tag{2.10}$$

$$Lv_2(x)=0, \quad v_1(0)=u''(0), \quad v_1(1)=u''(1), \tag{2.11}$$

From (2.8), (2.10) and [1], we have

$$|v_1(x)|\le C. \tag{2.12}$$

We can give the solution of (2.11) in the form

$$v_2(x)=p_0(x)+q_0(x)+R_\varepsilon(x), \tag{2.13}$$

where the functions $p_0(x)$, $q_0(x)$ and $R_\varepsilon(x)$ are, respectively, the solutions of (2.21), (2.22) and (2.23) from [1]. Also, we see that these solutions have the following estimations:

$$\left|p_0^{(m)}(x)\right| \le \frac{C}{\varepsilon^{m+1}}e^{-\frac{\mu_1 x}{\varepsilon}}, \quad \left|q_0^{(m)}(x)\right| \le \frac{C}{\varepsilon^{m+1}}e^{-\frac{\mu_2(1-x)}{\varepsilon}}, \quad m=0,1,2,$$

and $|R_\varepsilon(x)|\le C$. From (2.7), (2.9), (2.12) and (2.13) the following inequality clearly leads to (2.2) for $k=2$.

$$\begin{aligned}|u''(x)| &= |v(x)| \le |v_1(x)|+|v_2(x)| \le |p_0(x)|+|q_0(x)|+|R_\varepsilon(x)|\\ &\le \frac{C}{\varepsilon}e^{-\frac{\mu_1 x}{\varepsilon}}+\frac{C}{\varepsilon}e^{-\frac{\mu_2(1-x)}{\varepsilon}}+C \le \frac{C}{\varepsilon^2}e^{-\frac{\mu_1 x}{\varepsilon}}+\frac{C}{\varepsilon^2}e^{-\frac{\mu_2(1-x)}{\varepsilon}}+C\\ &\le C\left\{1+\frac{1}{\varepsilon^2}\left(e^{-\frac{\mu_1 x}{\varepsilon}}+e^{-\frac{\mu_2(1-x)}{\varepsilon}}\right)\right\}, \quad 0<x<1, \quad k=2.\end{aligned}$$

All these estimations conclude our proof. □

3 The construction of difference scheme and mesh

In this section, the discretization of the problem (1.1)–(1.3) using finite difference method on Bakhvalov-Shishkin mesh is presented.

3.1 Bakhvalov-Shishkin mesh

The interval $[0,1]$ is divided into the three subintervals $[0,\sigma_1]$, $[\sigma_1, 1-\sigma_2]$ and $[1-\sigma_2, 1]$. Here σ_1 and σ_2 are referred as the transition points and are written as follows:

$$\sigma_1 = \min\left\{\frac{1}{4}, \mu_1^{-1}\varepsilon \ln N\right\}, \quad \sigma_2 = \min\left\{\frac{1}{4}, \mu_2^{-1}\varepsilon \ln N\right\}.$$

Assumption 1: We shall assume throughout the paper that $\varepsilon \le CN^{-1}$ as is generally the case in practice, where, N is a positive even integer.
The mesh points $\bar{\omega}_N = \{x_i\}_{i=0}^{N}$ are introduced through a set of the equalities:

$$x_i = \begin{cases} -\mu_1^{-1}\varepsilon \ln\left[1 - 4(1-N^{-1})\frac{i}{N}\right], & i = 0, ..., \frac{N}{4}; \\ \sigma_1 + (i - \frac{N}{4})h, \quad h = \frac{2(1-\sigma_2-\sigma_1)}{N}, & i = \frac{N}{4}+1, ..., \frac{3N}{4}; \\ 1 + \mu_2^{-1}\varepsilon \ln\left[1 - 4(1-N^{-1})(1-\frac{i}{N})\right], & i = \frac{3N}{4}, ..., N; \end{cases}$$

3.2 The construction of the difference scheme

Here the following finite differences for any mesh function $g_i = g(x_i)$ are presented on $\bar{\omega}_N$ as:

$$g_{\bar{x},i} = \frac{g_i - g_{i-1}}{h_i}, \quad g_{x,i} = \frac{g_{i+1} - g_i}{h_{i+1}}, \quad g_{\overset{0}{x},i} = \frac{g_{x,i} + g_{\bar{x},i}}{2},$$
$$g_{\widehat{x},i} = \frac{g_{i+1} - g_i}{\hbar_i}, \quad g_{\bar{x}\widehat{x},i} = \frac{g_{x,i} - g_{\bar{x},i}}{\hbar_i}, \quad \hbar_i = \frac{h_i + h_{i+1}}{2}, \quad h_i = x_i - x_{i-1},$$
$$\|g\|_\infty \equiv \|g\|_{\infty,\bar{\omega}_N} := \max_{0\leqslant i\leqslant N} |g_i|.$$

Now, the difference scheme for the problem (1.1) should be constructed.

The following exact relation is obtained through the use of the interpolating quadrature formulas on subintervals $[x_{i-1}, x_{i+1}]$ [2]. Initially, the equation (1.1) is integrated over (x_{i-1}, x_{i+1}) as

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} Lu(x)\varphi_i(x)dx = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)dx, \; i=1,\dots,\frac{N}{4}, \; i=\frac{3N}{4},\dots,N-1.$$

If the above equality is arranged, it gives

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \left[\varepsilon^2 u''(x) + \varepsilon a(x)\, u'(x) - b(x)u(x)\right]\varphi_i(x)dx = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)dx.$$

From here, the following equality is obtained by implementing partial integration and then by using the formula (2.2) of [3]:

$$\hbar_i^{-1}\varepsilon^2\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i'(x)dx + \hbar_i^{-1}\varepsilon a_i\int_{x_{i-1}}^{x_{i+1}} u'(x)\,\varphi_i(x)dx - b_i u_i = f_i - R_i^1.$$

Finally, we propose the following difference scheme for approximating (1.1)–(1.3):

$$\varepsilon^2 u_{\bar{x}\widehat{x},i} + \varepsilon a_i u_{x,i} - b_i u_i = f_i - R_i^1,\ i = 1,...,\frac{N}{4}, i = \frac{3N}{4},...,N-1,$$

and the reminder term

$$R_i^1 = -\frac{\varepsilon^2}{2}\int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)u'''(x)dx - \frac{\varepsilon a_i}{h_{i+1}}\int_{x_i}^{x_{i+1}} (x_{i+1}-x)u''(x)dx, \qquad (3.1)$$

where the functions $\varphi_i(x)$ are in the form:

$$\varphi_i(x) = \begin{cases} \dfrac{(x-x_{i-1})^2}{\hbar_i h_i}, & x_{i-1} < x < x_i, \\ \dfrac{(x_{i+1}-x)^2}{\hbar_i h_{i+1}}, & x_i < x < x_{i+1}. \end{cases}$$

Secondly, Equation (1.1) is integrated over (x_{i-1}, x_{i+1}) as

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}} Lu(x)\psi_i(x)dx = h^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x)\psi_i(x)dx,\ \ i = \frac{N}{4}+1,\dots,\frac{3N}{4}-1,$$

$$h^{-1}\int\limits_{x_{i-1}}^{x_{i+1}} \left[\varepsilon^2 u''(x) + \varepsilon a\left(x\right)u'\left(x\right) - b(x)u(x)\right]\psi_i(x)dx = h^{-1}\int\limits_{x_{i-1}}^{x_{i+1}} f(x)\psi_i(x)dx.$$

Here, applying partial integration in the first expression of the left integral, we get

$$h^{-1}\varepsilon^2\int_{x_{i-1}}^{x_{i+1}} u'(x)\psi_i'(x)dx + h^{-1}\varepsilon a_i\int_{x_{i-1}}^{x_{i+1}} u'\left(x\right)\psi_i(x)dx - b_i u_i = f_i - R_i^2,$$

and from here it follows that

$$\begin{aligned} h^{-1}&\int_{x_{i-1}}^{x_i} \left(\varepsilon^2 u'(x)\psi_i'(x) + \varepsilon a_i u'\left(x\right)\psi_i(x)\right)dx \\ &+ \int_{x_i}^{x_{i+1}} \left(\varepsilon^2 u'(x)\psi_i'(x) + \varepsilon a_i u'\left(x\right)\psi_i(x)\right)dx = b_i u_i + f_i - R_i^2, \end{aligned}$$

we use the formula (2.2) of [3] in (3.2) and propose the following difference scheme:

$$\varepsilon^2 u_{\bar{x}x,i} + \varepsilon a_i u_{x,i} - b_i u_i = f_i - R_i^2, \quad i = \frac{N}{4}+1,\dots,\frac{3N}{4}-1, \qquad (3.2)$$

where reminder term R_i^2

$$R_i^2 = -\frac{\varepsilon^2}{2}\int_{x_{i-1}}^{x_{i+1}} \psi_i(x)u'''(x)dx - \frac{\varepsilon a_i}{h}\int_{x_i}^{x_{i+1}} (x_{i+1}-x)u''(x)dx, \qquad (3.3)$$

and the functions $\psi_i(x)$ take the form

$$\psi_i(x) = \begin{cases} -\big(\dfrac{x - x_{i-1}}{h}\big)^2, & x_{i-1} < x < x_i, \\ \big(\dfrac{x_{i+1} - x}{h}\big)^2, & x_i < x < x_{i+1}. \end{cases}$$

Here, it is necessary to define an approximation for the second boundary condition (1.3). The following equation is written using the interpolation quadrature formula with respect to x_{N_0} and x_{N_0+1}:

$$u(x) = \frac{x - x_{N_0+1}}{x_{N_0} - x_{N_0+1}} u(x_{N_0}) + \frac{x - x_{N_0}}{x_{N_0+1} - x_{N_0}} u(x_{N_0+1}) + r_0,$$

where, reminder term r_0

$$r_0 = \frac{1}{2} f''(\xi)(x - x_{N_0})(x - x_{N_0+1}), \quad \xi \in (x_{N_0}, l_1). \tag{3.4}$$

Once reminder terms R_i^1 and R_i^2 and r_0 are neglected from (3.1), (3.3) and (3.4), it is possible to propose the following difference schemes for the problem (1.1)–(1.3):

$$\varepsilon^2 y_{\bar{x}\widehat{x},i} + \varepsilon a_i y_{x,i} - b_i y_i = f_i, \; i = 1, \ldots, \frac{N}{4}, \; i = \frac{3N}{4}, ...N, \tag{3.5}$$

$$\varepsilon^2 y_{\bar{x}x,i} + \varepsilon a_i y_{x,i} - b_i y_i = f_i, \; i = \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1,$$

$$y_0 = A,$$

$$y_N - \gamma \left[\frac{l_1 - x_{N_0+1}}{x_{N_0} - x_{N_0+1}} y(x_{N_0}) + \frac{l_1 - x_{N_0}}{x_{N_0+1} - x_{N_0}} y(x_{N_0+1}) \right] = B, \tag{3.6}$$

where x_{N_0} is the mesh point nearest to l_1.

4 Uniform error estimates

With respect to the examination of the presented method for the problem (1.1)–(1.3), this section provides the following discrete problem and its solution:

$$\varepsilon^2 z_{\bar{x}\widehat{x},i} + \varepsilon a_i z_{x,i} - b_i z_i = R_i^1, \quad i = 0, \ldots, \frac{N}{4}, i = \frac{3N}{4}, \ldots, N, \tag{4.1}$$

$$\varepsilon^2 z_{\bar{x}x,i} + \varepsilon a_i z_{x,i} - b_i z_i = R_i^2, \; i = \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1,$$

$$z_0 = 0, \; z_N - \gamma \left[\frac{l_1 - x_{N_0+1}}{x_{N_0} - x_{N_0+1}} z(x_{N_0}) + \frac{l_1 - x_{N_0}}{x_{N_0+1} - x_{N_0}} z(x_{N_0+1}) \right] = r_0, \tag{4.2}$$

where, R_i^1, R_i^2 and r_0 are given by (3.1), (3.3) and (3.4) respectively.

Lemma 2. *If z_i is the solution to (4.1)–(4.2), then the estimate becomes:*

$$\|z\|_{\infty,\bar{\omega}_N} \le C \left(\|R^1\|_{\infty,\omega_N} + \|R^2\|_{\infty,\omega_N} + |r_0| \right),$$

holds.

Proof. Once the maximum principle is used for (4.1)–(4.2), we deduce

$$\begin{aligned}|z_i| &\leq |z_N| + \beta^{-1}\left[\|R^1\|_{\infty,\omega_N} + \|R^2\|_{\infty,\omega_N}\right], \\ |z_N| &\leq |r_0| + |\gamma||z_{N_0}|,\end{aligned} \tag{4.3}$$

where $|\gamma| \leq k < 1$. For $i = N_0$ in the Equation (4.3), it yield:

$$|z_{N_0}| \leq |z_N| + \beta^{-1}\left[\|R^1\|_{\infty,\omega_N} + \|R^2\|_{\infty,\omega_N}\right]. \tag{4.4}$$

As a result, from (4.3)–(4.4), we obtain

$$\begin{aligned}|z_i| \quad \leq \quad & (1-k)^{-1}\left\{|r_0| + k\beta^{-1}\left[\|R^1\|_{\infty,\omega_N} + \|R^2\|_{\infty,\omega_N}\right]\right\} \\ & +\beta^{-1}\left[\|R^1\|_{\infty,\omega_N} + \|R^2\|_{\infty,\omega_N}\right].\end{aligned}$$

And conclusively, from here, we obtain Lemma 2. □

Lemma 3. *Based on the assumptions of Lemma 1 and Lemma 2, the solution of the problem (1.1)–(1.3) fulfills the following estimates for the reminder terms* R_i^1, R_i^2 *and* r_0 :

$$\|R^1\|_{\infty,\omega_N} \leq CN^{-2}, \quad \|R^2\|_{\infty,\omega_N} \leq CN^{-2}, \quad |r_0| \leq CN^{-2}.$$

Proof. The remainder terms R_i^1, R_i^2 and r_0 are evaluated for the subintervals $[0, \sigma_1]$, $[\sigma_1, 1-\sigma_2]$ and $[1-\sigma_2, 1]$ on Bakhvalov-Shishkin mesh.

1) The remainder term R_i^1 is evaluated for $x_i \in [0, \sigma_1]$, $\sigma_1 \leq 1/4$:

$$x_{i-1} = -\mu_1^{-1}\varepsilon \ln\left[1 - 4(1-N^{-1})\frac{(i-1)}{N}\right], \quad i = 1, \ldots, \frac{N}{4}, \tag{4.5}$$

$$h_i = -\mu_1^{-1}\varepsilon \ln\left[1 - 4(1-N^{-1})\frac{i}{N}\right] + \mu_1^{-1}\varepsilon \ln\left[1 - 4(1-N^{-1})\frac{(i-1)}{N}\right]. \tag{4.6}$$

Applying the mean value theorem in (4.6), we obtain that

$$h_i = \mu_1^{-1}\varepsilon \frac{4(1-N^{-1})N^{-1}}{1 - 4i_1(1-N^{-1})N^{-1}} \leq CN^{-1}. \tag{4.7}$$

Thus, from (3.1) and (4.5)–(4.7), we can write

$$\begin{aligned}\left|R_i^1\right| \leq & C\left\{\varepsilon^2 \int_{x_{i-1}}^{x_i} |u'''(x)| \frac{(x_{i-1}-x)^2}{\hbar_i h_i} dx\right\} \\ & + C\left\{\int_{x_i}^{x_{i+1}} \left[\varepsilon^2 |u'''(x)| \frac{(x_{i+1}-x)^2}{\hbar_i h_{i+1}} + \varepsilon h_{i+1}^{-1} |u''(x)|(x_{i+1}-x)\right] dx\right\} \\ \leq & C\left\{\varepsilon^2 \int_{x_{i-1}}^{x_{i+1}} |1 + \frac{1}{\varepsilon^3}(e^{-\frac{\mu_1(x)}{\varepsilon}} + e^{-\frac{\mu_2(1-x)}{\varepsilon}})| dx\right\} \\ & + C\left\{\varepsilon \int_{x_i}^{x_{i+1}} |1 + \frac{1}{\varepsilon^2}(e^{-\frac{\mu_1(x)}{\varepsilon}} + e^{-\frac{\mu_2(1-x)}{\varepsilon}})| dx\right\}.\end{aligned} \tag{4.8}$$

Since

$$e^{-\frac{\mu_1(x_{i-1})}{\varepsilon}}-e^{-\frac{\mu_1(x_{i+1})}{\varepsilon}} = e^{\ln\left[1-(1-N^{-1})\frac{4(i-1)}{N}\right]}-e^{\ln\left[1-(1-N^{-1})\frac{4(i+1)}{N}\right]}$$
$$\le 8\left(1-N^{-1}\right)N^{-1}\le CN^{-2},$$

the following estimations are obtained in a similar manner:

$$e^{-\frac{\mu_2(1-x_{i+1})}{\varepsilon}}-e^{-\frac{\mu_2(1-x_{i-1})}{\varepsilon}}\le CN^{-2},$$
$$e^{-\frac{\mu_1(x_i)}{\varepsilon}}-e^{-\frac{\mu_1(x_{i+1})}{\varepsilon}}\le CN^{-2},\quad e^{-\frac{\mu_2(1-x_{i+1})}{\varepsilon}}-e^{-\frac{\mu_2(1-x_i)}{\varepsilon}}\le CN^{-2}.$$

It then follows from (4.8), we come to conclusion as

$$\left|R_i^1\right|\le CN^{-2},\quad i=1,\ldots,N/4.$$

2) The remainder term R_i^2 is evaluated for $x_i\in[\sigma_1,1-\sigma_2]$:

$$x_i=\sigma_1+(i-N/4)h,\quad i=N/4+1,\ldots,3N/4, \tag{4.9}$$

where

$$h=2(1-\sigma_2-\sigma_1)/N. \tag{4.10}$$

It then follows from (3.3), (4.9) and (4.10), we have

$$\left|R_i^2\right|\le C\left\{\varepsilon^2\int_{x_{i-1}}^{x_{i+1}}|u'''(x)|dx+\varepsilon\int_{x_i}^{x_{i+1}}|u''(x)|dx\right\}$$
$$\le C\left\{\mu_1^{-1}\left[e^{\frac{-\mu_1x_{i-1}}{\varepsilon}}-e^{\frac{-\mu_1x_{i+1}}{\varepsilon}}\right]-\mu_2^{-1}\left[e^{\frac{-\mu_2(1-x_{i+1})}{\varepsilon}}-e^{\frac{-\mu_2(1-x_{i-1})}{\varepsilon}}\right]\right\}$$
$$\le CN^{-2},\quad i=N/4+1,\ldots,3N/4-1,$$

where

$$e^{-\frac{\mu_1(x_{i-1})}{\varepsilon}}-e^{-\frac{\mu_1(x_{i+1})}{\varepsilon}}\quad\le\quad\frac{1}{N^2}e^{\frac{-\mu_1(i-1-\frac{N}{4})h}{\varepsilon}}\left(1-e^{\frac{-2\mu_1h}{\varepsilon}}\right)\le CN^{-2}.$$

and similarly

$$e^{-\frac{\mu_2(1-x_{i+1})}{\varepsilon}}-e^{-\frac{\mu_2(1-x_{i-1})}{\varepsilon}}\le CN^{-2}.$$

3) The remainder term R_i^1 is evaluated for $x_i\in[1-\sigma_2,1]$:

$$x_{i-1}=1+\mu_2^{-1}\varepsilon\ln\left[1-4(1-N^{-1})(1-\frac{i-1}{N})\right],\quad i=\frac{3N}{4},\ldots,N, \tag{4.11}$$
$$h_i=\mu_2^{-1}\varepsilon\left\{\ln[1-4(1-N^{-1})(1-\frac{i}{N})]-\ln[1-4(1-N^{-1})(1-\frac{i-1}{N})]\right\}. \tag{4.12}$$

By applying the mean value theorem in (4.12), we obtain

$$h_i\le CN^{-1}. \tag{4.13}$$

Using the inequality (4.13), we have

$$\hbar_i \le CN^{-1}. \tag{4.14}$$

Thus, from (3.1) and (4.11)–(4.14), we can write

$$\left|R_i^1\right| \ \le \ CN^{-2}, \ \ i = 3N/4, \ldots, N,$$

where

$$e^{-\frac{\mu_1(x_{i-1})}{\varepsilon}} - e^{-\frac{\mu_1(x_{i+1})}{\varepsilon}} = e^{-\mu_1(1+\mu_2^{-1}\varepsilon\ln\left[1-4(1-N^{-1})(1-\frac{i-1}{N})\right])} - e^{-\mu_1(1+\mu_2^{-1}\varepsilon\ln\left[1-4(1-N^{-1})(1-\frac{i+1}{N}\right])} \le CN^{-2}.$$

4) Now, we estimate the remainder term r_0. In the following estimation, x_{N_0} is the mesh point nearest to l_1. Also, we assume that $l_1 \in [2\alpha^{-1}\varepsilon|\ln\varepsilon|, 1-2\alpha^{-1}\varepsilon|\ln\varepsilon|], \alpha \ge 0$, and the second derivative of $f(x)$ is bounded. So, we obtain from (3.4),

$$\begin{aligned} |r_0| \ &\le \ C\left\{|f''(\xi)(x-x_{N_0})(x-x_{N_0+1})|\right\} \\ &\le \ C\left\{(x-x_{N_0})(x-x_{N_0+1})\right\} \le C\left\{h^2\right\} \le CN^{-2}, \ \ \xi \in (x_{N_0}, l_1). \end{aligned}$$

These estimations complete the proof of Lemma 3. □

We can state the convergence result of this study the following Theorem 1.

Theorem 1. *Let $u(x)$ be the solution of the problem (1.1)–(1.3) and y be the solution of (3.5)–(3.6). Then, the following uniform error estimate satisfies*

$$\|y-u\|_{\infty,\bar{\omega}_N} \le CN^{-2}.$$

Proof. This follows immediately by mixing previous lemmas. □

5 Algorithm and numerical results

This section focuses on the demonstration of the following procedure for the difference scheme (3.5)–(3.6). Moreover, the effectiveness of the presented method is confirmed by applying it to a linear problem (1.1)–(1.3). Initially, the algorithm for the solution of the difference scheme (3.5)–(3.6) is provided:

$$\left(\frac{\varepsilon^2}{\hbar h_i}\right) y_{i-1} - \left(\frac{\varepsilon^2}{\hbar h_{i+1}} + \frac{\varepsilon^2}{\hbar h_i} + \frac{\varepsilon a_i}{h_{i+1}} + b_i\right) y_i + \left(\frac{\varepsilon^2}{\hbar h_{i+1}} + \frac{\varepsilon a_i}{h_{i+1}}\right) y_{i+1} = -f_i,$$
$$i = 1, \ldots, N/4, \ \ i = 3N/4, \ldots, N-1;$$

$$\left(\frac{\varepsilon^2}{h^2}\right) y_{i-1} - \left(\frac{2\varepsilon^2}{h^2} + \frac{\varepsilon a_i}{h} + b_i\right) y_i + \left(\frac{\varepsilon^2}{h^2} + \frac{\varepsilon a_i}{h}\right) y_{i+1} = -f_i, \ i = \frac{N}{4}+1, \ldots, \frac{3N}{4}-1;$$

$$\alpha_1 = 0, \ \ \beta_1 = 0,$$

$$\alpha_{i+1} = \frac{B_i}{C_i - A_i\alpha_i}, \ \beta_{i+1} = \frac{F_i + A_i\beta_i}{C_i - A_i\alpha_i}, \ \ i = 1, \ldots, N-1,$$

$$y_i = \alpha_{i+1}y_{i+1} + \beta_{i+1}, \ \ i = N-1, \ldots, 1.$$

This algorithm is stable due to $A_i > 0$, $B_i > 0$, $C_i > A_i + B_i$, $i = 1, 2, \ldots, N$.

Subsequently, the following problem is taken into consideration in order to prove that the presented method is working:

$$\varepsilon^2 u''(x) + \varepsilon(1+\cos(\pi x))u'(x) - (1+\sin(\pi x/2))u(x) = f(x), \quad 0 < x < 1,$$
$$u'(0) = 0, \qquad u(1) - 0.5u(0.5) = 0.5.$$

The exact solution of the problem is

$$u(x) = \frac{\frac{(1-e^{1-x})(1+\cos(\pi x)+d)}{2\varepsilon}(1 - e^{\frac{xd}{\varepsilon}})}{(-1+e^{\frac{d}{2\varepsilon}})(-2-2e^{\frac{d}{2\varepsilon}}) + \frac{e^{1+\cos(\pi x)+d}}{4\varepsilon}} + \sin(\pi x)^2,$$

where $d = \sqrt{5 + 2\cos(\pi x) + \cos(\pi x)^2 + 4\sin(\pi x/2)}$. The ε-uniform convergence rates are calculated using the following expression:

$$P^N = \ln\left(e^N/e^{2N}\right)/\ln 2.$$

The error estimates are also denoted by

$$e^N_\varepsilon = \left\| y^{\varepsilon,N} - u^{\varepsilon,N} \right\|_{\infty,\bar{\omega}_N}, \quad e^N = \max_\varepsilon e^N_\varepsilon.$$

As presented in Table 1, when the ε is small, the solution changes fastly in the boundary layer regions.

Table 1. Errors e^N and rates of convergence p^N for test problem.

$\varepsilon \vert N$	16	32	64	128	256	512	1024
2^{-15}	0.1303665	0.0379424	0.0101458	0.0026100	0.0006577	0.0001630	0.0000395
	1.78	1.90	1.95	1.98	2.01	2.04	
2^{-16}	0.1303729	0.0379483	0.0101519	0.0026147	0.0006609	0.0001650	0.0000406
	1.78	1.90	1.95	1.98	2.00	2.02	
2^{-17}	0.1303761	0.0379513	0.0101549	0.0026171	0.0006625	0.0001660	0.0000412
	1.78	1.90	1.95	1.98	1.99	2.01	
2^{-18}	0.1303777	0.0379528	0.0101564	0.0026183	0.0006633	0.0001666	0.0000415
	1.78	1.90	1.95	1.98	1.99	2.00	
2^{-19}	0.1303785	0.0379535	0.0101572	0.0026189	0.0006637	0.0001668	0.0000417
	1.78	1.90	1.95	1.98	1.99	2.00	
2^{-20}	0.1303789	0.0379539	0.0101575	0.0026192	0.0006639	0.0001669	0.0000418
	1.78	1.90	1.95	1.98	1.99	1.99	
2^{-21}	0.1303791	0.0379541	0.0101577	0.0026193	0.0006640	0.0001670	0.0000418
	1.78	1.90	1.95	1.98	1.99	1.99	
e^N	0.1303791	0.0379541	0.0101577	0.0026193	0.0006640	0.0001670	0.0000418
p^N	1.78	1.90	1.95	1.98	1.99	1.99	

When $N = 32, 64, \ldots, 1024$ takes increasing values, it is observed in table that the convergence rate p^N is of the second-order. The exact solution and approximate solution curves are determined to be almost same, as presented in Figure 1. Therefore, it is possible to conclude that convergence is achieved. As indicated in Figure 2, errors in boundary layer regions with respect to the examination of the presented method for the problem (1.1)–(1.3), are maximum due to the irregularity caused by the sudden and rapid change of the solution in these regions around $x = 0$ and $x = 1$ for the different values of ε. Therefore, the numerical results indicated that the proposed scheme is working effectively.

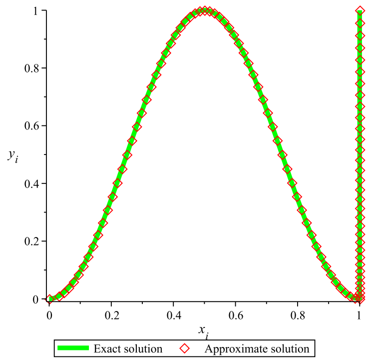

Figure 1. Comparison of the exact and approximate solutions of test problem for $N = 128$, $\varepsilon = 2^{-15}$.

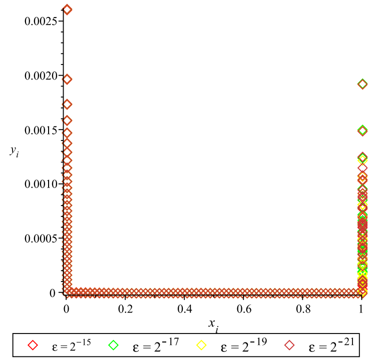

Figure 2. Error distribution of test problem for $N = 128$, $\varepsilon = 2^{-15}$, $\varepsilon = 2^{-17}$, $\varepsilon = 2^{-19}$, $\varepsilon = 2^{-21}$.

6 Conclusions

In this study, we offered an effective finite difference method for solving second-order linear singularly perturbed nonlocal boundary value problem. Uniform convergence in the second-order was proven with respect to the $\varepsilon-$ perturbation parameter in the discrete maximum norm of the difference scheme. As a result, it was possible to conclude that the finite difference method, taken into consideration for the solution of problems that are not easy to solve with every numerical method and that have both nonlocal and singular perturbation properties, was very effective and convenient on nonuniform meshes (Shishkin, Bakhvalov, Bakhvalov-Shishkin etc.). The present study findings demonstrate that it would be possible to conduct further studies on delayed and partial differential equations, which contain more complex nonlocal conditions. Furthermore, it could be suggested that a study on the increase in the convergence rate to three or higher orders would be possible.

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