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COLLOCATION APPROXIMATIONS FOR WEAKLY SINGULAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS [1]

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ABSTRACT

A piecewise polynomial collocation method for solving linear weakly singular integro-differential equations of Volterra type is constructed. The attainable order of convergence of collocation approximations on arbitrary and quasi-uniform grids is studied theoretically and numerically.

Key words: Weakly singular Volterra integro-differential equation, piecewise polynomial collocation method, order of convergence

1. INTRODUCTION

We present an analysis of collocation approximations for Volterra integro-differential equations (VIDEs)

$$y'(t) = p(t)y(t) + q(t) + \int_0^t K_1(t,s)y(s)ds + \int_0^t K_2(t,s)y'(s)ds, \tag{1.1}$$

with $0 \le t \le T,\ 0 < T < \infty$, and with a given initial condition

$$y(0) = y_0,\ y_0 \in \mathbb{R} = (-\infty, \infty). \tag{1.2}$$

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We assume that

$$p, q \in C^{m,\nu}[0,T],\ K_1, K_2 \in \mathcal{W}^{m,\nu}(\Delta_T), m \in \mathbb{N} = \{1,2,\ldots\},\ \nu \in \mathbb{R},\ \nu < 1. \tag{1.3}$$

Here $C^{m,\nu}[0,T]$, $m \in \mathbb{N}$, $\nu < 1$, is defined as the collection of continuous functions $x : [0,T] \to \mathbb{R}$ which are m times continuously differentiable in $(0,T]$ and such that the estimate

$$\left|x^{(k)}(t)\right| \le c \begin{cases} 1 & \text{if} \quad k < 1-\nu, \\ 1 + |\log t| & \text{if} \quad k = 1-\nu, \\ t^{1-\nu-k} & \text{if} \quad k > 1-\nu \end{cases} \tag{1.4}$$

holds with a constant $c = c(x)$ for all $t \in (0,T]$ and $k = 1,\ldots,m$. Note that $C^m[0,T]$, the set of m times continuously differentiable functions $x : [0,T] \to \mathbb{R}$, is contained in $C^{m,\nu}[0,T]$ with arbitrary $\nu < 1$. Some other examples are $x_1(t) = t^{3/2}$, $x_2(t) = t^{3/4}$ and $x_3(t) = t \log t$ with $x_3(0) = 0$. Clearly, $x_1 \in C^{m,-1/2}[0,T]$, $x_2 \in C^{m,1/4}[0,T]$ and $x_3 \in C^{m,0}[0,T]$, $m \in \mathbb{N}$.

The set

$$\mathcal{W}^{m,\nu}(\Delta_T), \quad m \in \mathbb{N}, \quad \nu < 1, \quad \Delta_T = \{(t,s) \in \mathbb{R}^2 : 0 \le t \le T, 0 \le s < t\}$$

consists of m times continuously differentiable functions $K : \Delta_T \to \mathbb{R}$ satisfying

$$\left|\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j K(t,s)\right| \le c \begin{cases} 1 & \text{if} \quad \nu + i < 0, \\ 1 + |\log(t-s)| & \text{if} \quad \nu + i = 0, \\ (t-s)^{-\nu-i} & \text{if} \quad \nu + i > 0, \end{cases} \tag{1.5}$$

with a constant $c = c(K)$ for all $(t,s) \in \Delta_T$ and all non-negative integers i and j such that $i + j \le m$. Notice that, if $0 \le \nu < 1$, then $K(t,s)$ may possess a weak singularity as $s \to t$ (see (1.5) with $i = j = 0$). If $\nu < 0$, then $K(t,s)$ itself is bounded on Δ_T, but its derivatives may be singular as $s \to t$.

Observe that, in contrast to "standard" VIDEs, the integrand $K_2(t,s)y'(s)$ in (1.1) depends on the derivative y' instead of the solution y itself. Moreover, the derivatives of $p(t)$ and $q(t)$ in (1.1) may have singularities at $t = 0$ and the kernel functions $K_1(t,s)$ and $K_2(t,s)$ may be weakly singular at $t = s$. In particular, K_1 and K_2 may have the form

$$K_\alpha(t,s) = \kappa(t,s)(t-s)^{-\alpha}, \quad 0 < \alpha < 1,$$

where κ is a m times continuously differentiable function on

$$\bar{\Delta}_T = \{(t,s) : 0 \le s \le t \le T\}.$$

Clearly, $K_\alpha \in \mathcal{W}^{m,\alpha}(\Delta_T)$. Especially, if $K_1 = 0$ and $K_2 = K_\alpha$, $0 < \alpha < 1$, then equation (1.1) is of type which is often referred to as the Basset equation, which is playing important role in the mathematical modelling of the diffusion of discrete particle in a turbulent fluid (see, for example, [3]).

Introducing a new unknown function

$$z = y', \tag{1.6}$$

and using (1.2), equation (1.1) may be rewritten as a linear Volterra integral equation of the second kind with respect to z,

$$z(t) = f(t) + \int_0^t K_1(t,s)\int_0^s z(\tau)d\tau ds + \int_0^t [p(t) + K_2(t,s)]\, z(s)ds,\ t \in [0,T], \tag{1.7}$$

which may also be expressed in the form

$$z(t) = f(t) + \int_0^t K(t,s)z(s)ds, \quad t \in [0,T], \tag{1.8}$$

with

$$f(t) = q(t) + y_0 p(t) + y_0 \int_0^t K_1(t,s)ds, \quad t \in [0,T], \tag{1.9}$$

$$K(t,s) = p(t) + \int_s^t K_1(t,\tau)d\tau + K_2(t,s), \quad (t,s) \in \Delta_T. \tag{1.10}$$

We will employ (1.7) in the construction of numerical solutions for problem (1.1) – (1.2) (see Section 2). For the smoothness analysis of the solution of (1.1) – (1.2) is more convenient to use (1.8). Actually, the regularity of the solution of problem (1.1) – (1.2) is described in the following lemma.

Lemma 1.1. *Let* $y_0 \in \mathbb{R}$ *and assume* (1.3). *Then equation* (1.8) *has a unique solution* $z \in C^{m,\nu}[0,T]$, *implying that problem* (1.1) – (1.2) *has a unique solution* $y \in C^{m+1,\nu-1}[0,T]$.

Proof. The statement of this lemma can be established by using arguments similar to those used in [1] for the proof of Theorem 2.5. ■

Thus, under the conditions of Lemma 1.1, the solution y of (1.1) – (1.2) belongs to $C^1[0,T]$. On the other hand, an analysis shows that y may not belong to $C^2[0,T]$ even if p and q belong to $C^\infty[0,T]$. This complicates the construction of effective numerical methods with high accuracy for (1.1) – (1.2). In this paper a discussion of the attainable order of convergence of a piecewise polynomial collocation method on arbitrary and quasi-uniform grids is given, with numerical illustrations. Our analysis is based on two equivalent integral equation reformulations (1.7) – (1.8) of the Cauchy problem (1.1) – (1.2) and on the smoothness properties of the solution of (1.1) – (1.2) (given by Lemma 1.1). The main results of the paper extend the corresponding results of [1; 2; 4; 6] and are formulated in Theorems 2.1 – 2.2. Notice also that our method (2.4) – (2.7) below, where we have discretized the integral equation (1.7), is equivalent to the collocation method applied directly to the initial-value problem (1.1) – (1.2). In the latter form the collocation method in more particular case has been examined in [4; 5; 7].

2. COLLOCATION METHOD

For given $N \in \mathbb{N}$, let

$$\Pi_N = \{t_0, t_1, \ldots, t_N : 0 = t_0 < t_1 < \ldots < t_N = T\}$$

be a partition (a grid) of the interval $[0,T]$ (for ease of notation we suppress the index N in $t_j = t_j^{(N)}$ indicating the dependence of the grid points on N). A sequence of partitions for $[0,T]$ is called *quasi-uniform* if there exists a constant $\theta \geq 1$ independent of N such that

$$\max_{j=1,\ldots,N}(t_j - t_{j-1}) / \min_{j=1,\ldots,N}(t_j - t_{j-1}) \leq \theta, \quad N \in \mathbb{N}. \tag{2.1}$$

It follows from (2.1) that

$$t_j - t_{j-1} \leq cN^{-1}, \quad j = 1, \ldots, N, \tag{2.2}$$

where c is a positive constant which does not depend on N. Therefore, for the quasi-uniform grid $\Pi_N = \Pi_{N,\Theta}$ we have

$$h_N = \max_{j=1,\ldots,N}(t_j - t_{j-1}) \to 0 \quad \text{as} \quad N \to \infty\,. \tag{2.3}$$

If $\Theta = 1$, then $\Pi_{N,1}$ is uniform, with $h_N = TN^{-1}$. Denote

$$\sigma_1 = [t_0, t_1], \ \sigma_j = (t_{j-1}, t_j], \ j = 2, \ldots, N.$$

For given integers $m \geq 0$ and $-1 \leq d \leq m-1$, let $S_m^{(d)}(\Pi_N)$ be the spline space of piecewise polynomial functions on the grid Π_N:

$$S_m^{(d)}(\Pi_N) = \Big\{ \quad u : u\big|_{\sigma_j} = u_j \in \pi_m, \quad j = 1, \ldots, N; \\ u_j^{(k)}(t_j) = u_{j+1}^{(k)}(t_j), \qquad 0 \leq k \leq d, \ j = 1, \ldots, N-1 \Big\}.$$

Here π_m denotes the set of all polynomials of degree not exceeding m, $u\big|_{\sigma_j}$ is the restriction of u to the subinterval σ_j and

$$u_{j+1}^{(k)}(t_j) = \lim_{t\to t_j, t>t_j} u_{j+1}^{(k)}(t).$$

Note that the elements of

$$S_m^{(-1)}(\Pi_N) = \{u : u\big|_{\sigma_j} \in \pi_m,\ j = 1,\ldots,N\}$$

may have jump discontinuities at the interior points $t_1,\ldots,t_{N-1}$ of Π_N.

For given $m, N \in \mathbb{N}$ we find an approximation v to the solution z of equation (1.7) in the space $S_{m-1}^{(-1)}(\Pi_N)$. Function $v = v^{(N,m)} \in S_{m-1}^{(-1)}(\Pi_N)$ is determined from the following conditions:

$$\begin{aligned} v_j(t_{jk}) &= f(t_{jk}) + \int_0^{t_{jk}} K_1(t_{jk},s)\left(\int_0^s v(\tau)d\tau\right)ds \\ &+ \int_0^{t_{jk}} [p(t_{jk}) + K_2(t_{jk},s)]\, v(s)ds,\ \ k = 1,\ldots,m;\ j = 1,\ldots,N\,. \end{aligned} \tag{2.4}$$

Here $v_j = v\big|_{\sigma_j}$ is the restriction of v to σ_j, $j = 1,\ldots,N$, and the points

$$t_{jk} = t_{j-1} + \eta_k(t_j - t_{j-1}), \quad k = 1,\ldots,m;\ j = 1,\ldots,N \tag{2.5}$$

are completely characterized by the points $t_0, t_1, \ldots, t_N$ of the grid Π_N and parameters $\eta_1,\ldots,\eta_m$ which do not depend on j and N and satisfy

$$0 \le \eta_1 < \ldots < \eta_m \le 1\,. \tag{2.6}$$

Having determined the approximation v for z, we can also determine the approximation u for y, the solution of the initial-value problem (1.1) – (1.2), setting (see (1.6))

$$u(t) = y_0 + \int_0^t v(s)ds, \qquad t \in [0,T]\,. \tag{2.7}$$

Remark 2.1. The choice of collocation points (2.5) with $\eta_1 = 0$, $\eta_m = 1$ in (2.6) actually implies that the resulting collocation approximation v belongs to the smoother polynomial spline space $S_{m-1}^{(0)}(\Pi_N)$. Note also that $v \in S_{m-1}^{(-1)}(\Pi_N)$ implies that $u \in S_m^{(0)}(\Pi_N)$, and $v \in S_{m-1}^{(0)}(\Pi_N)$ implies that $u \in S_m^{(1)}(\Pi_N)$.

Remark 2.2. Conditions (2.4) form a system of equations whose exact form is determined by the choice of a basis in $S_{m-1}^{(-1)}(\Pi_N)$ (or in $S_{m-1}^{(0)}(\Pi_N)$ if $\eta_1 = 0$, $\eta_m = 1$). For instance, in each subinterval $[t_{j-1}, t_j]$ $(j = 1, \ldots, N)$ we may use the representation

$$v_j(t_{j-1} + \tau(t_j - t_{j-1})) = \sum_{k=1}^{m} c_{jk} L_k^{(m-1)}(\tau), \tau \in [0,1],$$

where $L_k^{(m-1)}(\tau)$ denotes the kth Lagrange fundamental polynomial of degree $m-1$ associated with the parameters $0 \leq \eta_1 < \ldots < \eta_m \leq 1$, that is $L_k^{(m-1)}(\tau) = \prod_{i \neq k}^{m} (\tau - \eta_i)/(\eta_k - \eta_i)$, $\tau \in [0,1]$. The collocation conditions (2.4) then lead to a linear system of equations for the coefficients

$$c_{jk} = c_{jk}^{(N)} = v_j(t_{jk}), \quad k = 1, \ldots, m; \ j = 1, \ldots, N.$$

Our main results are contained in the following theorems. The proofs of these results are given in Section 3.

Theorem 2.1. *Let conditions* (1.3) *be fulfilled and assume that the underlying grid sequence* (Π_N) *satisfies* (2.3). *Then, for all sufficiently large* $N \in \mathbb{N}$, *say* $N \geq N_0$, *and for every choice of parameters* $0 \leq \eta_1 < \ldots < \eta_m \leq 1$ *with* $\eta_1 > 0$ *or* $\eta_m < 1$, *the equations* (2.7) *and* (2.4) *determine unique approximations* $u \in S_m^{(0)}(\Pi_N)$ *and* $v \in S_{m-1}^{(-1)}(\Pi_N)$ (*with* $v\big|_{\sigma_j} = \big(u\big|_{\sigma_j}\big)'$, $j = 1, \ldots, N$) *to the solution* y *of problem* (1.1) – (1.2) *and its derivative* y', *respectively. If* $\eta_1 = 0$, $\eta_m = 1$, *then* $u \in S_m^{(1)}(\Pi_N)$ *and* $v = u' \in S_{m-1}^{(0)}(\Pi_N)$. *For all* $N \geq N_0$ *the collocation error* $e^{(k)}$ *with* $k = 0$ *and* $k = 1$ *satisfies*

$$\|e^{(k)}\|_\infty \leq c\,\mu_N. \tag{2.8}$$

Here

$$\mu_N = \begin{cases} h_N^m & \text{for} \quad m < 1 - \nu, \\ h_N^m(1 + |\log h_N|) & \text{for} \quad m = 1 - \nu, \\ h_N^{1-\nu} & \text{for} \quad m > 1 - \nu, \end{cases} \tag{2.9}$$

(h_N *is given by* (2.3)), c *is a constant not depending on* N *and*

$$\|e^{(k)}\|_\infty = \max_{j=1,\ldots,N} \big(\sup_{t \in \sigma_j} \big|u_j^{(k)}(t) - y^{(k)}(t)\big| \big), \ u_j = u\big|_{\sigma_j}, \ k = 0, 1. \tag{2.10}$$

Theorem 2.2. *Let* $y_0 \in \mathbb{R}$, p, $q \in C^{m,\nu}[0,T]$, $K_1 \in \mathcal{W}^{m,\nu}(\Delta_T)$, $K_2 \in \mathcal{W}^{m,\nu-1}(\Delta_T)$, $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$, *and assume that the underlying grid sequence* (Π_N) *is quasi-uniform* (*i.e. satisfies* (2.1)). *Then, for all* *sufficiently large $N \in \mathbb{N}$, in the notation of Theorem 2.1, the collocation error $e^{(k)}$ for $k = 0$ and $k = 1$ satisfies the following estimates:*

$$\|e^{(k)}\|_\infty \le c \begin{cases} N^{-m} & \text{for} \quad m < 2 - \nu - k, \\ N^{-m}(1 + \log N) & \text{for} \quad m = 2 - \nu - k, \\ N^{-(2-\nu-k)} & \text{for} \quad m > 2 - \nu - k. \end{cases} \tag{2.11}$$

Here c is a constant not depending on N and $\|e^{(k)}\|_\infty$ is given by (2.10).

3. PROOFS

First we present some auxiliary results which we need for the proof of Theorems 2.1 and 2.2.

To a continuous function $x : [0, T] \to \mathbb{R}$ we assign a piecewise interpolation function $P_N x = P_N^{(m)} x \in S_{m-1}^{(-1)}(\Pi_N)$ which interpolates x at the nodes (2.5): $(P_N x)(t_{jk}) = x(t_{jk})$, $k = 1, \ldots, m$; $j = 1, \ldots, N$. Thus, $(P_N x)(t)$ is independently defined in every subinterval $[t_{j-1}, t_j]$, $j = 1, \ldots, N$, and may be discontinuous at the interior grid points t_j, $j = 1, \ldots, N-1$. Note that, in case $\eta_1 = 0$, $\eta_m = 1$, $P_N x$ is a continuous function on $[0, T]$. We also introduce an interpolation operator $P_N = P_N^{(m)}$ which assigns to every continuous function $x : [0, T] \to \mathbb{R}$ its piecewise interpolation function $P_N x$. In the sequel, for Banach spaces E and F, we denote by $\mathcal{L}(E, F)$ the Banach space of linear bounded operators $A : E \to F$ with the norm

$$\|A\| = \sup\{\|Ax\| : x \in E, \|x\| \le 1\}.$$

By $C[a, b]$ we denote the Banach space of continuous functions $x : [a, b] \to \mathbb{R}$ with the norm $\|x\| = \max\{|x(t)| : a \le t \le b\}$. By $c, c_1, c_2, \ldots$ we will denote positive constants, which may be different in different inequalities.

Lemma 3.1. *Let $S : L^\infty(0, T) \to C[0, T]$ be a linear compact operator and assume that the underlying grid sequence (Π_N) satisfies (2.3). Then*

$$\|S - P_N S\|_{\mathcal{L}(L^\infty(0,T), L^\infty(0,T))} \to 0 \quad \text{as} \quad N \to \infty.$$

Proof. We observe that

$$\max_{j=1,\ldots,N} \|P_N\|_{\mathcal{L}(C[t_{j-1},t_j], C[t_{j-1},t_j])} \le c,$$
$$\|P_N\|_{\mathcal{L}(C[0,T], L^\infty(0,T))} \le c \tag{3.1}$$

with a constant c which is independent of N. It follows from (2.3) and (3.1) that $\|x - P_N x\|_{L^\infty(0,T)} \to 0$ as $N \to \infty$ for every $x \in C[0, T]$. Together with the compactness of $S : L^\infty(0,T) \to C[0,T]$ we obtain the assertion of Lemma 3.1. ■

Lemma 3.2. *Let $x \in C^{m,\nu}[0,T]$, $m \in \mathbb{N}$, $\nu < 1$. Then*

$$\|x - P_N x\|_{L^\infty(0,T)} \le c \max_{j=1,\dots,N} \max_{t_{j-1}\le t\le t_j} |\gamma_j(t)|, \tag{3.2}$$

where

$$\gamma_j(t) = \int_t^{t_j} (s-t)^{m-1} \left\{ \begin{array}{ll} 1, & m < 1-\nu, \\ 1+|\log s|, & m = 1-\nu, \\ s^{1-\nu-m}, & m > 1-\nu \end{array} \right\} ds, \quad t_{j-1} \le t \le t_j, \tag{3.3}$$

$j = 1, \dots, N$, and c is a constant which does not depend on j and N.

Proof. Taking $v \in S_{m-1}^{(-1)}(\Pi_N)$, on the base of (3.1), we obtain that

$$\begin{aligned} \|x - P_N x\|_{L^\infty(0,T)} &= \max_{j=1,\dots,N} \max_{t_{j-1}\le t\le t_j} |x(t) - v(t) - (P_N(x-v))(t)| \\ &\le c \max_{j=1,\dots,N} \max_{t_{j-1}\le t\le t_j} |x(t) - v(t)|, \end{aligned} \tag{3.4}$$

with a constant c which is independent of j and N. We fix $v(t)$, $t_{j-1} \le t \le t_j$, as a Taylor polynomial for x at $t = t_j$:

$$v(t) = \sum_{k=0}^{m-1} \frac{x^{(k)}(t_j)}{k!}(t-t_j)^k, \quad t_{j-1} \le t \le t_j \quad (j = 1, \dots, N). \tag{3.5}$$

Then, due to $x \in C^{m,\nu}[0,T]$, the statement of lemma follows from (3.4) and (3.5). ■

For given $\lambda \in \mathbb{R}$ we introduce the weight function $w_\lambda(t)$ by

$$w_\lambda(t) = \left\{ \begin{array}{ll} 1 & \text{if } \lambda < 0, \\ (1+|\log t|)^{-1} & \text{if } \lambda = 0, \\ t^\lambda & \text{if } \lambda > 0, \end{array} \right. \qquad t > 0. \tag{3.6}$$

Using (3.6), we redefine the space $C^{m,\nu}[0,T]$, $m \in \mathbb{N}$, $\nu < 1$, as the set of functions $x \in C[0,T]$ which are m times continuously differentiable in $(0,T]$ and such that $\sum_{k=1}^m \sup_{0<t\le T} \left(w_{k-(1-\nu)}(t)|x^{(k)}(t)|\right) \le c$. In other words, a function $x \in C[0,T]$ belongs to $C^{m,\nu}[0,T]$ if it is m times continuously differentiable in $(0,T]$ and its derivatives can be estimated by (1.4). Equipped with the norm

$$\|x\|_{m,\nu} = \max_{0\le t\le T} |x(t)| + \sum_{k=1}^m \sup_{0<t\le T} \left(w_{k-(1-\nu)}(t)|x^{(k)}(t)|\right),$$

$C^{m,\nu}[0,T]$ is a Banach space.

Lemma 3.3. [1]. *If* $x_1, x_2 \in C^{m,\nu}(0,T]$, $m \in \mathbb{N}$, $\nu < 1$, *then* $x_1 x_2 \in C^{m,\nu}(0,T]$ *and* $\|x_1 x_2\|_{m,\nu} \le c\,\|x_1\|_{m,\nu}\|x_2\|_{m,\nu}$, *where* c *is a constant which does not depend on* x_1 *and* x_2.

Lemma 3.4. [1]. *Let* $L \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$. *Then the operator* M, *defined by* $(Mx)(t) = \int_0^t L(t,s)x(s)ds$, $t \in [0,T]$, *is compact as an operator from* $C^{m,\nu}[0,T]$ *to* $C^{m,\nu}[0,T]$.

Lemma 3.5. [1]. *For* $L \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$, *let* L_1 *be defined by* $L_1(t,s) = \int_s^t L(t,\tau)d\tau$, $(t,s) \in \Delta_T$. *Then* $L_1 \in \mathcal{W}^{m,\nu-1}(\Delta_T)$.

Further, we consider an integral operator S :

$$(Sx)(t) = \int_0^t K(t,s)x(s)ds, \quad t \in [0,T], \tag{3.7}$$

with K given by (1.10).

Lemma 3.6. *Let* $p \in C^{m,\nu}[0,T]$, $K_1, K_2 \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$. *Then* S *is compact as an operator from* $L^\infty(0,T)$ *to* $C[0,T]$. *Moreover,* S *is compact as an operator from* $C^{m,\nu}[0,T]$ *to* $C^{m,\nu}[0,T]$.

Proof. We observe by (1.10) and (3.7) that S can be presented in the form $S = PJ + S_1 + S_2$, where the operators P, J, S_2 and S_1 are defined by settings

$$(Px)(t) = p(t)x(t), \quad (Jx)(t) = \int_0^t x(s)ds, \quad (S_2x)(t) = \int_0^t K_2(t,s)x(s)ds,$$

$$(S_1x)(t) = \int_0^t L_1(t,s)x(s)ds, \quad L_1(t,s) = \int_s^t K_1(t,\tau)d\tau.$$

Due to Lemma 3.5, $L_1 \in \mathcal{W}^{m,\nu-1}(\Delta_T)$. Therefore L_1 is bounded and continuous on Δ_T and S_1 is compact as an operator from $L^\infty(0,T)$ to $C[0,T]$. Since $K_2 \in \mathcal{W}^{m,\nu}(\Delta_T)$ is at most weakly singular, $S_2 : L^\infty(0,T) \to C[0,T]$ is compact. Clearly, $P \in \mathcal{L}(C[0,T], C[0,T])$ and $J \in \mathcal{L}(L^\infty(0,T), C[0,T])$ is compact. This implies $PJ : L^\infty(0,T) \to C[0,T]$ is linear and compact. In summary, $PJ + S_1 + S_2 = S \in \mathcal{L}(L^\infty(0,T), C[0,T])$ is compact.

Further, $L_1, K_2 \in \mathcal{W}^{m,\nu}(\Delta_T)$. Due to Lemma 3.4, $S_1 + S_2$ is compact as an operator from $C^{m,\nu}[0,T]$ to $C^{m,\nu}[0,T]$. Since $1 \in \mathcal{W}^{m,\nu}(\Delta_T)$, we also deduce that $J : C^{m,\nu}[0,T] \to C^{m,\nu}[0,T]$ is compact. It follows from Lemma 3.3 that $P \in \mathcal{L}(C^{m,\nu}[0,T], C^{m,\nu}[0,T])$. Therefore PJ is compact as an operator from $C^{m,\nu}[0,T]$ to $C^{m,\nu}[0,T]$. In summary, $PJ + S_1 + S_2 = S \in \mathcal{L}(C^{m,\nu}[0,T], C^{m,\nu}[0,T])$ is compact. ∎

Proof of Theorem 2.1 Let the conditions of Theorem 2.1 be fulfilled. As we know from Section 1, problem (1.1) – (1.2) is equivalent to the integral equation (1.8) where $z = y'$ and the forcing function f and the kernel K are given by (1.9) and (1.10), respectively. We rewrite (1.8) in the form $z = f + Sz$, with S defined by (3.7). We find that $f \in C^{m,\nu}[0,T] \subset L^\infty(0,T)$. It follows from Lemma 3.6 that S is linear and compact as an operator from $L^\infty(0,T)$ to $L^\infty(0,T)$. Therefore, $z = f + Sz$ has a unique solution $z \in L^\infty(0,T)$. Moreover, on the base of Lemma 1.1 we obtain that $z \in C^{m,\nu}[0,T]$.

Further, conditions (2.4) are equivalent to the operator equation representation $v = P_N f + P_N Sv$, with P_N defined at the beginning of Section 3. From Lemma 3.1 and from the boundedness of $(I - S)^{-1}$ in $L^\infty(0,T)$, with I, the identity transformation, we see that $I - P_N S$ is invertible in $L^\infty(0,T)$ for all sufficiently large N, say $N \geq N_0$. Moreover, the norms of $(I - P_N S)^{-1}$ are uniformly bounded in N :

$$\|(I - P_N S)^{-1}\|_{\mathcal{L}(L^\infty(0,T),L^\infty(0,T))} \leq c, \quad N \geq N_0. \tag{3.8}$$

Thus, for $N \geq N_0$ the equation $v = P_N f + P_N Sv$ provides a unique solution $v \in S_{m-1}^{(-1)}(\Pi_N)$ ($v \in S_{m-1}^{(0)}$ if $\eta_1 = 0$, $\eta_m = 1$). For v and z, the solutions of equations $v = P_N f + P_N Sv$ and $z = f + Sz$ respectively, we have

$$v - z = (I - P_N S)^{-1}(P_N z - z), \ N \geq N_0. \tag{3.9}$$

Now (3.8) yields $\|v - z\|_{L^\infty(0,T)} \leq c\,\|P_N z - z\|_{L^\infty(0,T)}$, $N \geq N_0$, with a constant c which is independent of N. Applying Lemma 3.2 we obtain that

$$\begin{aligned}
&\|v - z\|_{L^\infty(0,T)} \leq c_1 \max_{j=1,\dots,N} \max_{t_{j-1} \leq t \leq t_j} |\gamma_j(t)| \\
&\leq c_1 \max_{j=1,\dots,N} \max_{t_{j-1} \leq t \leq t_j} \int_t^{t_j} (s-t)^{m-1} \left\{ \begin{array}{ll} 1, & m < 1-\nu, \\ 1 + |\log(s-t)|, & m = 1-\nu, \\ (s-t)^{1-\nu-m}, & m > 1-\nu \end{array} \right\} ds \\
&\leq c_2 \mu_N,
\end{aligned}$$

where γ_j and μ_N are defined by (3.3) and (2.9), respectively. In fact, this is the estimate (2.8) with $k = 1$, $z = y'$ and $v\big|_{\sigma_j} = (u\big|_{\sigma_j})'$, $j = 1, \dots, n$.

Further, due to (2.7),

$$u_j(t) - y(t) = u_j(t_{j-1}) - y(t_{j-1}) + \int_{t_{j-1}}^t [u_j'(s) - y'(s)]ds, \tag{3.10}$$

where $u_j = u\big|_{\sigma_j}$ and $t \in [t_{j-1}, t_j]$, $j = 1, \dots, N$. Moreover, it follows from (2.7) and (1.2) that $u_1(0) - y(0) = 0$. Applying (2.8) with $k = 1$ to (3.10) we obtain the estimate (2.8) with $k = 0$ for $u - y$. ■

Proof of Theorem 2.2 Let the conditions of Theorem 2.2 be fulfilled. It follows from (2.2) and Theorem 2.1 that we have to prove (2.11) only for

$k = 0, m > 1 - \nu$. Using the equality $(I - P_N S)^{-1} = I + (I - P_N S)^{-1} P_N S$, we rewrite the error (3.9) in the form

$$v - z = P_N z - z + (I - P_N S)^{-1} P_N S(P_N z - z), \ N \geq N_0. \tag{3.11}$$

Due to continuity and boundedness of $K(t,s)$ on Δ_T, S is bounded, as an operator from $L^1(0,T)$ to $C[0,T]$ (see (1.10) and (3.7)). Using this, on the base of (1.2), (1.6), (2.7), (3.1), (3.8) and (3.11) we obtain that

$$|u(t) - y(t)| = \left| \int_0^t [v(s) - z(s)] ds \right| \leq c \int_0^T |(P_N z)(s) - z(s)| ds, \tag{3.12}$$

where $0 \leq t \leq T$ and c is a constant not depending on N. It follows from $z \in C^{m,\nu}[0,T]$ and Lemma 3.2 for $m > 1 - \nu$ that

$$\int_0^T |(P_N z)(s) - z(s)| \, ds \leq \int_0^{t_1} |(P_N z)(s) - z(s)| \, ds + \sum_{j=2}^{N} \int_{t_{j-1}}^{t_j} |(P_N z)(s) - z(s)| \, ds$$
$$\leq c_1 \left(h_N^{2-\nu} + h_N^{m+1} \sum_{j=2}^{N} t_{j-1}^{1-\nu-m} \right),$$

where h_N is defined by (2.3). This, together with (2.1), (2.2) and

$$t_{j-1} \geq (j-1) \min\{ t_j - t_{j-1} : \ j = 2, \ldots, N \},$$

yields that

$$\int_0^T |(P_N z)(s) - z(s)| ds \leq c_2 \big(N^{-(2-\nu)} + N^{-(2-\nu)} \sum_{j=2}^{N} (j-1)^{1-\nu-m} \big) \tag{3.13}$$

with a constant c_2 not depending on N. Further, we have

$$\sum_{j=2}^{N} (j-1)^{1-\nu-m} \leq c_3 \begin{cases} N^{2-\nu-m} & \text{if} \quad m < 2 - \nu, \\ 1 + \log N & \text{if} \quad m = 2 - \nu, \\ 1 & \text{if} \quad m > 2 - \nu, \end{cases} \tag{3.14}$$

with a constant c_3 not depending on N. Combining the results (3.12)-(3.14), it is easy to see that (2.11) holds for $k = 0, \ m > 1 - \nu$. ■

4. NUMERICAL EXPERIMENTS

In this section we test the convergence behavior numerically. We consider problem (1.1) – (1.2), where

$$T = 1,\ \ y_0 = 0,\ \ p(t) = -1\,,$$
$$K_1(t,s) = -(t-s)^{-\nu},\ K_2(t,s) = -(t-s)^{-\nu+1}\ ,$$
$$q(t) = (2-\nu)t^{1-\nu} + t^{2-\nu} + t^{3-2\nu}\int_0^1 (1-x)^{-\nu}x^{2-\nu}\,dx$$
$$+ (2-\nu)t^{3-2\nu}\int_0^1 (1-x)^{-\nu+1}x^{1-\nu}\,dx, \quad \nu \in \{-1/3, 1/2\}\,.$$

In this case the exact solution of problem (1.1) – (1.2) is $y(t) = t^{2-\nu}$ and it is easy to check that the assumptions of Theorem 2.2 about p, q, K_1 and K_2 hold with arbitrary $m \in \mathbb{N}$.

This problem is solved numerically using method (2.4) – (2.7). Some of results obtained are presented in Tables 1, 2 for $m = 2$ and for the collocation parameters $\eta_1 = 1/4$, $\eta_2 = 3/4$ (see (2.6)). In fact, in Tables 1, 2 for different grids $\Pi_{N,1}$, $\Pi_{N,5}^{(1)}$, $\Pi_{N,5}^{(2)}$, the error (compare (2.10))

$$\varepsilon_N^{(l)} = \{\max |u^{(l)}(\tau_{jk}) - y^{(l)}(\tau_{jk})| : \ k = 1,\dots,9; j = 1,\dots,N\},$$

and the ratio $\varrho_N^{(l)} = \varepsilon_{N/2}^{(l)}/\varepsilon_N^{(l)}$ for $l = 0$ and $l = 1$ are given. In order to calculate the error (2.10) we have taken $t = \tau_{jk}$ where $\tau_{jk} = t_{j-1} + k(t_j - t_{j-1})/10$, $k = 1,\dots,9$; $j = 1,\dots,N$. Further, $\Pi_{N,1}$ is the uniform grid of $[0,1]$ with $h_N = N^{-1}$ (see (2.3)), $\Pi_{N,5}^{(1)}$ and $\Pi_{N,5}^{(2)}$ are the quasi-uniform grids Π_N, with $\theta = 5$ in (2.1), defined as follows:

$$\Pi_{N,\theta}^{(l)} = \{t_0,\dots,t_N : \ \ t_j = \frac{2j}{N}T_l,\ j = 0,\dots,\frac{N}{2};$$
$$t_{j+N/2} = T_l + 2j\frac{T - T_l}{N},\ j = 1,\dots,\frac{N}{2}\ \},$$
$$T_l = \frac{\theta^{1-l}}{\theta+1}T,\ \ l = 0,\ l = 1\,.$$

From Theorem 2.2 for $m = 2$ we can derive the following convergence results. In case of $\nu = -1/3$ the ratio $\varrho_N^{(0)}$ ought to be approximately $4(= 2^m)$, and the ratio $\varrho_N^{(1)}$ ought to be approximately $2.5(\approx 2^{1-\nu})$. In case of $\nu = 1/2$ the ratio $\varrho_N^{(0)}$ ought to be approximately $2.8(\approx 2^{2-\nu})$, and the ratio $\varrho_N^{(1)}$ ought to be approximately $1.4(\approx 2^{1-\nu})$.

From Tables 1, 2 we can see that the numerical results are in good agreement with the theoretical estimates of Theorem 2.2.

Table 1.
Numerical examples in case $\nu = -\frac{1}{3}$

	$\Pi_{N,1}$		$\Pi^{(1)}_{N,5}$		$\Pi^{(2)}_{N,5}$	
N	$\varepsilon^{(0)}_N$	$\varrho^{(0)}_N$	$\varepsilon^{(0)}_N$	$\varrho^{(0)}_N$	$\varepsilon^{(0)}_N$	$\varrho^{(0)}_N$
4	9.7E-4	4.2	2.8E-3	4.4	2.0E-3	4.4
8	2.4E-4	4.1	6.6E-4	4.2	4.2E-4	4.7
16	5.8E-5	4.0	1.6E-4	4.1	9.6E-5	4.4
32	1.5E-5	4.0	4.1E-5	4.0	2.3E-5	4.2
64	3.7E-6	3.9	1.0E-5	4.0	5.5E-6	4.1
128	9.5E-7	3.9	2.6E-6	3.9	1.4E-6	4.1
256	2.4E-7	3.9	6.6E-7	3.9	3.4E-7	4.0
512	6.1E-8	3.9	1.7E-7	3.9	8.4E-8	4.0
N	$\varepsilon^{(1)}_N$	$\varrho^{(1)}_N$	$\varepsilon^{(1)}_N$	$\varrho^{(1)}_N$	$\varepsilon^{(1)}_N$	$\varrho^{(1)}_N$
4	1.6E-2	2.4	3.0E-2	2.2	1.7E-2	2.8
8	6.5E-3	2.4	1.3E-2	2.4	5.3E-3	3.2
16	2.6E-3	2.5	5.1E-3	2.5	1.5E-3	3.5
32	1.0E-3	2.5	2.1E-3	2.5	4.1E-4	3.7
64	4.2E-4	2.5	8.2E-4	2.5	1.1E-4	3.8
128	1.7E-4	2.5	3.3E-4	2.5	3.9E-5	2.8
256	6.6E-5	2.5	1.3E-4	2.5	1.5E-5	2.5
512	2.6E-5	2.5	5.2E-5	2.5	6.1E-6	2.5

Table 2.
Numerical examples in case $\nu = \frac{1}{2}$

	$\Pi_{N,1}$		$\Pi^{(1)}_{N,5}$		$\Pi^{(2)}_{N,5}$	
N	$\varepsilon^{(0)}_N$	$\varrho^{(0)}_N$	$\varepsilon^{(0)}_N$	$\varrho^{(0)}_N$	$\varepsilon^{(0)}_N$	$\varrho^{(0)}_N$
4	3.2E-3	2.5	6.3E-3	2.3	1.8E-3	3.4
8	1.2E-3	2.7	2.4E-3	2.6	4.4E-4	4.1
16	4.2E-4	2.8	9.0E-4	2.7	1.1E-4	4.0
32	1.5E-4	2.8	3.2E-4	2.8	3.1E-5	3.5
64	5.4E-5	2.8	1.2E-4	2.8	1.1E-5	3.0
128	1.9E-5	2.8	4.1E-5	2.8	3.9E-6	2.8
256	7.0E-6	2.8	1.5E-5	2.8	1.4E-6	2.8
512	2.5E-6	2.8	5.3E-6	2.8	5.0E-7	2.8
N	$\varepsilon^{(1)}_N$	$\varrho^{(1)}_N$	$\varepsilon^{(1)}_N$	$\varrho^{(1)}_N$	$\varepsilon^{(1)}_N$	$\varrho^{(1)}_N$
4	5.0E-2	1.2	5.8E-2	1.1	3.1E-2	1.4
8	3.7E-2	1.3	4.6E-2	1.3	2.2E-2	1.4
16	2.7E-2	1.4	3.4E-2	1.3	1.6E-2	1.4
32	1.9E-2	1.4	2.5E-2	1.4	1.1E-2	1.4
64	1.4E-2	1.4	1.8E-2	1.4	8.0E-3	1.4
128	9.8E-3	1.4	1.3E-2	1.4	5.7E-3	1.4
256	6.9E-3	1.4	8.9E-3	1.4	4.0E-3	1.4
512	4.9E-3	1.4	6.3E-3	1.4	2.8E-3	1.4

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Silpnai singuliarių Voltero integralinių-diferencialinių lygčių aproksimavimas kolokacijų metodu

I. Parts, A. Pedas

Darbe nagrinėjamas silpnai singuliarių Voltero integralinių-diferencialinių lygčių skaitinio artinio radimo algoritmas. Integralai priklauso ne tik nuo sprendinio, bet ir nuo jo pirmosios išvestinės. Ištirtas kolokacijų metodo tikslumas, kai naudojami netolygūs ir artimi tolygiems tinklai. Teoriniai įverčiai patvirtinti skaičiavimo eksperimento rezultatais.