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CONVERGENCE ACCELERATION AND LINEAR METHODS[1]

I. TAMMERAID

Department of Mathematics, Tallinn Technical University

Tallinn, 19086, Estonia

E-mail: itammeraid@edu.ttu.ee

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ABSTRACT

Two λ-convergence propositions for linear methods $\mathcal{A} = (A_{nk})$, while A_{nk} are linear bounded operators from Banach space X into Banach space Y, are presented. These results are applied to study convergence acceleration of linear methods.

Key words: convergence acceleration, summability methods, linear methods

1. INTRODUCTION

Let X, Y be Banach spaces and $\mathcal{L}(X, Y)$ be a space of all bounded linear operators from X into Y. A sequence $x = (\xi_k)$ $(\xi_k \in X)$ is called correspondingly λ-convergent or λ-bounded if

$$\exists \lim \xi_k = \xi \ \wedge \ \beta_k = \lambda_k (\xi_k - \xi) \ \wedge \ \exists \lim \beta_k = \beta$$

or $\exists \lim \xi_k = \xi \ \wedge \ \beta_k = \lambda_k (\xi_k - \xi) \ \wedge \ \beta_k = O(1)$, whereas $\lambda = (\lambda_k)$ with $0 < \lambda_k \nearrow$. Let c_X^λ and m_X^λ be respectively the sets of all λ-convergent or λ-bounded sequences. If $\lambda_k = O(1)$, then $c_X^\lambda = m_X^\lambda = c_X$, while c_X is a set of convergent sequences with $\xi_k \in X$. A sequence $x = (\xi_k)$ is called summable (see [4; 8; 12; 14]) by a generalized method $\mathcal{A} = (A_{nk})$, $A_{nk} \in \mathcal{L}(X, Y)$ if $y = (\eta_n)$ with

$$\eta_n = \textstyle\sum_k A_{nk} \xi_k \tag{1.1}$$

is convergent. Unless indicated otherwise a sum $\sum_k$ will always be understood as $\sum_{k=0}^\infty$ and a limit lim or $\lim_n$ will be understood as $\lim_{n\to\infty}$. Let $\mu = (\mu_k)$ with $0 < \mu_k \nearrow$. The transformation $\mathcal{A}$ is called correspondingly accelerating λ-convergence or λ-boundedness if $\mathcal{A} c_X^\lambda \subset c_Y^\mu$ or $\mathcal{A} m_X^\lambda \subset m_Y^\mu$ with $\lim \mu_k / \lambda_k = \infty$.

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A method $\mathcal{A} = (A_{nk})$ with $(A_{nk} \in \mathcal{L}(X, X))$ is called regular if $\mathcal{A}c_X \subset c_X$ and

$$\lim_n \eta_n = \lim_k \xi_k,$$

while $(\xi_k) \in c_X$ and η_n is defined by (1.1). Let $I \in \mathcal{L}(X, X)$ denote identity operator. In the sequel $\lambda_k \nearrow \infty$ and $\mu_k \nearrow \infty$.

In this paper we study a possibility to use generalized linear methods for convergence acceleration. G. Kangro (see [5]) proved in number case the necessary and sufficient conditions for the inclusion $\mathcal{A}c_K^\lambda \subset c_K^\mu$ and as a corollary he proved that a regular triangular number-matrix method can not accelerate λ-convergence. I. Kornfeld [6] generalized G. Kangro's result for any regular number-matrix method. We generalize G. Kangro's results for linear methods $\mathcal{A} = (A_{nk})$, while A_{nk} are linear bounded operators from Banach space X into Banach space Y.

2. MAIN RESULTS

The Proposition 2.1 (see [12]) and the Proposition 2.2 give a possibility to solve several problems of the convergence acceleration using linear methods.

Proposition 2.1. *Let $A_{nk} \in \mathcal{L}(X, Y)$, $\mathcal{A} = (A_{nk})$ and $e(\zeta) := (\zeta, \zeta, \zeta, \ldots)$ with $\zeta \in X$. If*

$$\exists \lim_n A_{nk} = A_k \quad (k \in \mathbf{N}_0) \quad \text{in norm,} \tag{2.1}$$

then the conditions

$$\mathcal{A}e_X(\zeta) \in m_Y^\mu, \tag{2.2}$$

$$\sum_k \lambda_k^{-1} \|A_k\| < \infty, \tag{2.3}$$

$$\mu_n \sum_k \lambda_k^{-1} \|A_{nk} - A_k\| = O(1) \tag{2.4}$$

are necessary and sufficient for

$$\mathcal{A}m_X^\lambda \subset m_Y^\mu. \tag{2.5}$$

Corollary 2.1. (see [12]). If X is a B-space and $\mathcal{A} = (A_{nk})$ with $A_{nk} \in \mathcal{L}(X, X)$ is a regular triangular matrix method satisfying the condition

$$\sum_{k=0}^{n} A_{nk} = I,$$

then $\mathcal{A}$ can not accelerate λ-boundedness.

Proposition 2.2. *Let* $A_{nk} \in \mathcal{L}(X,Y)$, $\mathcal{A} = (A_{nk})$ *and*

$$e_k(\zeta) := (0, \dots, 0, \zeta, 0, \dots), \quad e_\lambda(\zeta) := \left(\lambda_1^{-1}\zeta, \lambda_2^{-1}\zeta, \lambda_3^{-1}\zeta, \dots\right)$$

with $\zeta \in X$. *The conditions*

$$\mathcal{A}e_k(\zeta) \in c_Y^\mu, \tag{2.6}$$

$$\mathcal{A}e(\zeta) \in c_Y^\mu, \tag{2.7}$$

$$\mathcal{A}e_\lambda(\zeta) \in c_Y^\mu, \tag{2.8}$$

$$\sup_{\|\varsigma_k\| \le 1} \sum_{k=0}^{p} \left\| \lambda_k^{-1} A_{nk} \varsigma_k \right\| = O(1), \quad (\varsigma_k \in X,\ n, p \in \mathbf{N}_0) \tag{2.9}$$

$$\sup_{\|\varsigma_k\| \le 1} \sum_{k=0}^{p} \left\| \mu_n \lambda_k^{-1} (A_{nk} - A_k) \varsigma_k \right\| = O(1), (\varsigma_k \in X,\ n, p \in \mathbf{N}_0) \tag{2.10}$$

are necessary and sufficient for the inclusion

$$\mathcal{A}c_X^\lambda \subset c_Y^\mu. \tag{2.11}$$

Proof. If $\zeta \in X$, then $e_k(\zeta), e(\zeta), e_\lambda(\zeta) \in c_X^\lambda$. So the conditions (2.6) – (2.8) are necessary for the inclusion (2.11). That means the conditions

$$\exists \lim_n A_{nk}\varsigma = A_k\varsigma \ \wedge\ \exists \lim_n \mu_n (A_{nk} - A_k)\varsigma \quad (\varsigma \in X), \tag{2.12}$$

$$\exists \lim_n \sum_k A_{nk}\varsigma = A\varsigma \ \wedge\ \exists \lim_n \mu_n \left(\sum_k A_{nk} - A \right) \varsigma \ \ (\varsigma \in X), \tag{2.13}$$

$$\exists \lim_n \sum_k \lambda_k^{-1} A_{nk}\varsigma = A^\lambda\varsigma \ \wedge\ \exists \lim_n \mu_n \left(\sum_k \lambda_k^{-1} A_{nk} - A^\lambda \right) \varsigma \quad (\varsigma \in X), \tag{2.14}$$

with $\mathcal{A}^\lambda = \left(\lambda_k^{-1} A_{nk}\right)$ are necessary for the inclusion (2.11). On the strength of (2.13) the quantity η_n defined by (1.1) may be presented in the form

$$\eta_n = \textstyle\sum_k \lambda_k^{-1} A_{nk}\beta_k + \displaystyle\sum_k A_{nk}\xi \tag{2.15}$$

with $\beta_k = \lambda_k(\xi_k - \xi)$. Taking into consideration the condition (2.13) and the presentation (2.15) we get $(\eta_n) \in c_Y \ \Leftrightarrow\ \mathcal{A}^\lambda(\beta_k) \in c_Y$. According to the results of K. Zeller (see [14]) the conditions

$$\exists \lim_n \lambda_k^{-1} A_{nk}\varsigma = \lambda_k^{-1} A_k\varsigma \quad (\varsigma \in X,\ k \in \mathbf{N}_0), \tag{2.16}$$

$$\exists \lim_n \sum_k \lambda_k^{-1} A_{nk}\varsigma = A^\lambda\varsigma \quad (\varsigma \in X) \tag{2.17}$$

and (2.9) are necessary and sufficient for $\mathcal{A}^{\lambda}c_X \subset c_Y$, while if the conditions (2.9), (2.16) and (2.17) are satisfied we have

$$\eta = \lim_n \eta_n = A^{\lambda}\beta + \sum_k \lambda_k^{-1} A_k \left(\beta_k - \beta\right) + A\xi. \tag{2.18}$$

It is easy to see that (2.12) implies (2.16) and (2.14) implies (2.17). The condition (2.9) can be inferred from

$$\sum_k \lambda_k^{-1} \| A_{nk} \| = O(1). \tag{2.19}$$

Using (2.15) and (2.18) we get

$$\eta_n - \eta = \textstyle\sum_k \lambda_k^{-1} A_{nk}\beta_k + \displaystyle\sum_k A_{nk}\xi - A^{\lambda}\beta - \sum_k \lambda_k^{-1} A_k \left(\beta_k - \beta\right) - A\xi$$

and

$$\begin{aligned} \mu_n\Big(\eta_n - \eta\Big) &= \sum_k \mu_n \lambda_k^{-1}(A_{nk} - A_k)(\beta_k - \beta) + \mu_n(\sum_k A_{nk} - A)\xi \\ &+ \mu_n(\sum_k \lambda_k^{-1} A_{nk} - A^{\lambda})\beta. \end{aligned}$$

By the conditions (2.13) and (2.14) the limits of two last summands on the right hand side exist. That is why the condition

$$\mathcal{A}^{\lambda,\mu} c_X^0 \subset c_Y \tag{2.20}$$

with $\mathcal{A}^{\lambda,\mu} = (\mu_n \lambda_k^{-1}(A_{nk} - A_k))$ and $c_X^0 := \{x = (\xi_k) \,|\, (\xi_k \in X) \wedge (\lim \xi_k = 0)\}$ is necessary and sufficient for the inclusion (2.11) (if (2.6) – (2.9) are satisfied). According to G. Kangro [4] the conditions (2.10) and (2.12) are necessary and sufficient for the inclusion (2.20). This completes the proof.■

3. COROLLARIES AND REMARKS

Remark 3.1. If we take $X = Y = \mathbf{K}$ and $A_{nk} = a_{nk}I \;\; (a_{nk} \in \mathbf{K})$ then we get from Proposition 2 G. Kangro's (see [5]) corresponding result for number case. We generalized G. Kangro's method.

Corollary 3.1. If the conditions (2.1), (2.3) and (2.4) are satisfied then the conditions (2.6) – (2.8) are necessary and sufficient for (2.11).

Proof. The condition (2.6) implies (2.10) and (2.19) implies (2.9). The conditions (2.1), (2.3) and (2.4) imply (2.19). ■

Example 3.1. Let $X = C[a,b]$ be a B-space of real value continuous functions of a real variable, while the norm in $C[a,b]$ will be $\|\xi\| := \max_{a \le t \le b} |\xi(t)|$. Let the functions $K_{nk}(t,s)$ be continuous on $[a,b] \times [a,b]$. Let the integral operators $K_{nk} : C[a,b] \to C[a,b]$ be defined by

$$(K_{nk}\zeta)(t) := \int_a^b K_{nk}(t,s)\,\zeta(s)\,ds \quad (t \in [a,b]).$$

The operators K_{nk} are linear and bounded, while (see [7])

$$\|K_{nk}\| = \max_{a \le t \le b} \int_a^b |K_{nk}(t,s)|\,ds.$$

So $K_{nk} \in \mathcal{L}(X,X)$. If $\lim_{n\to\infty} K_{nk} = K_k$ in norm$\sum_k \lambda_k^{-1} \|K_k\| < \infty$, $\mu_n \sum_k \lambda_k^{-1} \|K_{nk} - K_k\| = O(1)$, then the conditions (limits in norm)

$$\exists \lim_n K_{nk}\varsigma = K_k\varsigma \;\wedge\; \exists \lim_n \mu_n (K_{nk} - K_k)\varsigma \quad (\varsigma \in X),$$

$$\exists \lim_n \sum_k K_{nk}\varsigma = K\varsigma \;\wedge\; \exists \lim_n \mu_n \left(\sum_k K_{nk} - K\right)\varsigma \;\; (\varsigma \in X),$$

$$\exists \lim_n \sum_k \lambda_k^{-1} K_{nk}\varsigma = K^\lambda\varsigma \;\wedge\; \exists \lim_n \mu_n \left(\sum_k \lambda_k^{-1} K_{nk} - K^\lambda\right)\varsigma \;\; (\varsigma \in X),$$

are necessary and sufficient for $\mathcal{K}c^\lambda_{C[a,b]} \subset c^\mu_{C[a,b]}$ with $\mathcal{K} = (K_{nk})$.

Corollary 3.2. If X is a B-space and $\mathcal{A} = (A_{nk})$ with $A_{nk} \in \mathcal{L}(X,X)$ is a regular triangular matrix method satisfying the condition $\sum_{k=0}^n A_{nk} = I$, then $\mathcal{A}$ can not accelerate λ-convergence.

Proof. This assertion can be proved analogically as Corollary 2.1.■

In number case Corollary 3.2 was proved by G. Kangro [5]. I. Kornfeld [6] generalized this G. Kangro's result for any regular number-matrix method.

Remark 3.2. Nevertheless of nonexistence of a regular summability method improving λ-convergence in applied mathematics linear methods are used to accelerate the convergence (see [11]). This is possible if we use some subsets of c_X^λ or m_X^λ. Also for acceleration it is possible to use nonregular methods or in some cases the pseudo-summability (see [11]).

Remark 3.3. J. Wimp (see [13]) asserted that linear methods are limited in their usefulness primarily because the class of sequences for which the methods are regular is too large. The experience indicates that the size of the domain of regularity of a transformation and its efficiency seem to be inversely related. So we cannot expect too much from any linear summability method.

Remark 3.4. J.P. Delahaye (see [3]) asserted that in some problems of numerical analysis or optimization we are faced with nonconvergent sequences and sometimes we want to know what kind of nonconvergence it is.

Remark 3.5. The main results on nonlinear methods of convergence acceleration are obtained by Claude Brezinski (see [2; 3]).

Remark 3.6. Sometimes having the information about the transformed sequence $\mathcal{A}x$ we can get the information about x using the Tauberian theorems (see [9; 10] and [12]).

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Konvergavimo pagreitis ir tiesiniai metodai

I. Tammeraid

Pateikti du λ – konvergavimo teiginiai tiesiniams metodams $\mathcal{A} = (A_{nk})$, kai A_{nk} yra tiesiniai aprėžti operatoriai iš Banacho erdvės X į Banacho erdvę Y. Šie teiginiai taikomi tiriant tiesinių metodų konvergavimo pagreitį.