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# GENERALIZED RIESZ METHOD AND CONVERGENCE ACCELERATION

I. TAMMERAID

*Department of Mathematics, Tallinn University of Technology*

Tallinn, 19086, Estonia

E-mail: itammeraid@edu.ttu.ee

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**Abstract.** Several propositions on $\lambda$-boundedness for generalized Riesz method $(\Re, P_n)$, where $P_n$ are linear bounded operators from Banach space $X$ into $X$, are proved. These results are applied to study convergence acceleration using generalized Riesz method and generalized Zygmund method.

**Key words:** linear summability methods, convergence acceleration, Tauberian remainder theorems

## 1. Introduction and Lemmas

Let $X, Y$ be Banach spaces and $\mathcal{L}(X,Y)$ be a space of all bounded linear operators from $X$ into $Y$. A sequence $x = (\xi_k)$ $(\xi_k \in X)$ is called $\lambda$-bounded if

$$\exists \lim \xi_k = \xi \ \wedge \ \beta_k = \lambda_k (\xi_k - \xi) \ \wedge \ \beta_k = O(1),$$

whereas $\lambda = (\lambda_k)$ with $0 < \lambda_k \nearrow$.

Let $m_X^\lambda$ be the set of all $\lambda$-bounded sequences. If $\lambda_k = O(1)$, then $m_X^\lambda = c_X$, while $c_X$ is a set of convergent sequences with $\xi_k \in X$. A sequence $x = (\xi_k)$ is called *summable* (see [3, 13]) by a generalized method $\mathcal{A} = (A_{nk})$, $A_{nk} \in \mathcal{L}(X,Y)$ if $y = (\eta_n)$ with

$$\eta_n = \sum_{k=0}^{\infty} A_{nk}\xi_k \tag{1.1}$$

is convergent. Let $\mu = (\mu_k)$ with $0 < \mu_k \nearrow$. The transformation $\mathcal{A}$ is called accelerating $\lambda$-boundedness if $\mathcal{A} m_X^\lambda \subset m_Y^\mu$ with $\lim \mu_k/\lambda_k = \infty$.

A method $\mathcal{A} = (A_{nk})$ with $(A_{nk} \in \mathcal{L}(X,X))$ is called *regular* if $\mathcal{A} c_X \subset c_X$ and

$$\lim_n \eta_n = \lim_k \xi_k,$$

while $(\xi_k) \in c_X$ and $\eta_n$ is defined by (1.1). Let $I \in \mathcal{L}(X, X)$ denote identity operator.

Kornfeld [6] proved that any regular numerical method of summability can not accelerate the convergence. Nevertheless of nonexistence of a regular summability method improving $\lambda$-boundedness in applied mathematics linear methods are used to accelerate the convergence (see [10]). Such an acceleration is possible if we use some subsets of $m_X^\lambda$. Also for acceleration it is possible to use nonregular methods or in some cases the pseudosummability (see [10]).

In [12] Kornfeld's result is generalized for several methods $\mathcal{A} = (A_{nk})$ with $\big(A_{nk} \in \mathcal{L}(X, X)\big)$. In [9] certain numerical summability methods are compared by rates of convergence. In [2] are presented main results of convergence acceleration using nonlinear methods.

Let us denote by $(\Re, P_n)$ or shortly by $\Re$ the generalized Riesz method (see [1, 7]), defined by

$$A_{nk} = R_{nk} = \begin{cases} R_n P_k & (k = 0, 1, \ldots, n), \\ \theta & (k > n), \end{cases}$$

where $P_k$, $R_n \in \mathcal{L}(X, X)$, while $R_n$ is determined by

$$R_n \sum_{k=0}^{n} P_k \zeta = \zeta \quad (\zeta \in X,\ n \in \mathbf{N}_0). \tag{1.2}$$

Let $\{B_n\}$ be a sequence of operators $B_n \in \mathcal{L}(X, X)$ satisfying the conditions

$$(n+1)\,\|B_{n+1} - B_n\| = O\left(\|B_n\|\right) \quad (n \in \mathbf{N}) \tag{1.3}$$

and

$$B_0 = \theta, \quad B_n \neq \theta \quad (n \in \mathbf{N}).$$

Let us denote by $(\mathcal{Z}, B_n)$ or shortly by $\mathcal{Z}$ the generalized Zygmund method as a generalized Riesz method $(\Re, P_n)$ with $P_n = B_{n+1} - B_n$.

**Lemma 1.** ([7]). *The conditions*

$$\lim \|R_n\| = 0 \tag{1.4}$$

*and*

$$\|R_n\| \sum_{k=0}^{n} \|P_k\| = O(1) \tag{1.5}$$

*imply that the method* $\Re$ *is regular.*

**Lemma 2.** *If* $B_n^{-1} \in \mathcal{L}(X, X)$ $(n \in \mathbf{N})$,

$$\lim \left\|B_n^{-1}\right\| = 0 \tag{1.6}$$

*and*

$$\left\|B_{n+1}^{-1}\right\| \sum_{k=0}^{n} \frac{\|B_k\|}{k+1} = O(1), \tag{1.7}$$

*then* $(\mathcal{Z}, B_n)$ *is regular.*

*Proof.* Using the definition of the method $(\mathcal{Z}, B_n)$ and Lemma 1 we get that (1.6) implies (1.4). Using (1.3) and (1.7) we get

$$\|R_n\| \sum_{k=0}^{n} \|P_k\| = \left\|B_{n+1}^{-1}\right\| \sum_{k=0}^{n} \|B_{k+1} - B_k\|$$
$$= \left\|B_{n+1}^{-1}\right\| \sum_{k=0}^{n} O(1) \frac{\|B_k\|}{k+1} = O(1).$$

Thus the condition (1.5) is satisfied and by Lemma 1 the method $(\mathcal{Z}, B_n)$ is regular. ■

**Lemma 3.** ([12]). *If $\mathcal{A} = (A_{nk})$ with $(A_{nk} \in \mathcal{L}(X, X))$ is a regular triangular matrix method satisfying the condition*

$$\sum_{k=0}^{n} A_{nk} = I \quad (n \in \mathbf{N}_0),$$

*then $\mathcal{A}$ can not accelerate the convergence.*

Using Lemmas 2 and 3 we get the next corollary.

*Corollary 1.* The method $\mathcal{Z}$, satisfying the conditions (1.6) and (1.7), can not accelerate the convergence.

In the sequel $\lambda_k \nearrow \infty$, $\mu_k \nearrow \infty$ and $\tau_k \nearrow \infty$.

**Lemma 4.** ([11]). *The conditions* (1.4), (1.5) *and*

$$\mu_n \|R_n\| \sum_{k=0}^{n} \frac{\|P_k\|}{\lambda_k} = O(1)$$

*are sufficient for the inclusion $\Re m_X^\lambda \subset m_X^\mu$.*

*Corollary 2.* If $B_n^{-1} \in \mathcal{L}(X, X)$ $(n \in \mathbf{N})$, then the conditions (1.6), (1.7) and

$$\mu_n \left\|B_{n+1}^{-1}\right\| \sum_{k=0}^{n} \frac{\|B_k\|}{(k+1)\lambda_k} = O(1)$$

are sufficient for the inclusion $\mathcal{Z} m_X^\lambda \subset m_X^\mu$.

T. Sõrmus (see [8]) studied an inverse problem and proved so-called Tauberian theorem for generalized Riesz method $\Re$ in the case $\lambda_n = O(1)$ and $\mu_n = O(1)$ and got the next result.

**Lemma 5.** *Let $P_k^{-1}$, $R_n^{-1} \in \mathcal{L}(X, X)$ and generalized Riesz method $\Re$ be regular. If $x = (\xi_k)$ is $(\Re, P_n)$–summable to $\eta$ and*

$$P_k^{-1} R_k^{-1} \Delta\xi_k = o(1),$$

*whereas $\Delta\xi_k = \xi_k - \xi_{k-1}$, then $\xi_k \to \eta$.*

The following so-called Tauberian remainder theorem for generalized method of summability $\mathcal{A}$ is proved in [12].

**Lemma 6.** *Let $\mathcal{A} = (A_{nk})$ with $A_{nk} \in \mathcal{L}(X,X)$ be a regular triangular generalized method of summability satisfying the conditions*

$$\lambda_n \sum_{k=0}^{n} \lambda_k^{-1} \| A_{nk} \| = O(1), \tag{1.8}$$

$$\lambda_k \left\| \sum_{\nu=0}^{k} A_{n\nu} \right\| \| \Delta\xi_k \| = O(\| A_{nk} \|) \qquad (k \le n, \quad n \in \mathbf{N}_0), \tag{1.9}$$

$$\lambda_n \left\| \sum_{k=0}^{n} A_{nk} \; - I \right\| \|\xi_n\| = O(1). \tag{1.10}$$

*Then $\mathcal{A}x \in m_X^\lambda$ implies $x \in m_X^\lambda$.*

**Lemma 7.** ([5]) *If a numerical regular Riesz method $(\Re, P_k)$ with $P_k > 0$ and*

$$R_k^{-1} = O\left(R_{k-1}^{-1}\right)$$

*is preserving $\lambda$-boundedness and*

$$\tau_k R_k^{-1} \Delta\xi_k = O(P_k),$$

*then $\mathcal{R}x \in m_{\mathbf{R}}^\lambda$ implies $x \in m_{\mathbf{R}}^\lambda$, whereas*

$$1 \le \frac{\lambda_k}{\tau_k} \uparrow, \quad \lambda_k \tau_k \uparrow, \quad \mu_k = \sqrt{\lambda_k \tau_k}.$$

## 2. Main Results

**Proposition 1.** *The conditions* (1.4), (1.5) *and*

$$\lambda_n \|R_n\| \sum_{k=0}^{n} \frac{\|P_k\|}{\lambda_k} = O(1) \tag{2.1}$$

*imply*

$$\frac{\lambda_n \|R_n\|}{\lambda_\nu \|R_\nu\|} = O(1) \;\; (\nu \le n), \tag{2.2}$$

*and*

$$\lambda_n \|R_n\| = o(1). \tag{2.3}$$

*Proof.* As

$$\sum_{k=0}^{n} \frac{\|P_k\|}{\lambda_k} \ge \sum_{k=0}^{\nu} \frac{\|P_k\|}{\lambda_k} \ge \frac{1}{\lambda_\nu} \sum_{k=0}^{\nu} \|P_k\| \ge \frac{1}{\lambda_\nu \|R_\nu\|} \|R_\nu\| \sum_{k=0}^{\nu} \|P_k\|$$

and (1.2) implies

$$\|R_\nu\| \sum_{k=0}^{\nu} \|P_k\| \geq 1, \tag{2.4}$$

then

$$\sum_{k=0}^{n} \frac{\|P_k\|}{\lambda_k} \geq \frac{1}{\lambda_\nu \|R_\nu\|}. \tag{2.5}$$

Using (2.1) and (2.5) we get (2.2). The condition (2.2) implies

$$\lambda_n \|R_n\| = O(1).$$

Therefore there exists such $m > 0$, that

$$\frac{1}{\lambda_n} \geq m \|R_n\|. \tag{2.6}$$

From (2.6) and (2.4) it follows that

$$\frac{\|P_n\|}{\lambda_n} \geq m \|R_n\| \|P_n\| \geq m \frac{\|P_n\|}{\sum\limits_{k=0}^{n} \|P_k\|}.$$

From (1.4) and (2.4) we get

$$\sum_{k=0}^{n} \|P_k\| \to \infty.$$

Therefore the series $\sum_{k=0}^{\infty} \|P_k\| / \lambda_k$ is divergent. Using (2.1) we get (2.3). ■

*Corollary 3.* The conditions $R_n^{-1} \in \mathcal{L}(X, X)$, (1.4), (1.5), (2.1) and

$$\left\|R_n^{-1}\right\| = O\left(\left\|R_{n-1}^{-1}\right\|\right)$$

imply

$$\lambda_n = O(\lambda_{n-1}).$$

*Remark 1.* In case of numbers a result, similar to Proposition 1, was proved by Kangro (see [4]).

**Proposition 2.** *If*

$$\lambda_n \|R_n\| \sum_{k=0}^{n} \frac{\|P_k\|}{\lambda_k} = o(1),$$

*then* $\Re$ *is accelerating* $\lambda$*-boundedness.*

*Proof.* Let $x \in m_X^\lambda$ and $\{\eta_n\}$ is defined by (1.1) – (1.2). Since

$$\sum_{k=0}^{n} R_n P_k = I$$

and using (2) we get

$$\begin{aligned}\lambda_n \|\eta_n - \eta\| &= \lambda_n \left\| \sum_{k=0}^{n} R_n P_k \xi_k - \sum_{k=0}^{n} R_n P_k \xi \right\| \\ &= \lambda_n \left\| \sum_{k=0}^{n} R_n P_k (\xi_k - \xi) \right\| \le \lambda_n \sum_{k=0}^{n} \|R_n\| \, \|P_k\| \, \|\xi_k - \xi\| \\ &\le \lambda_n \|R_n\| \sum_{k=0}^{n} \frac{\|P_k\|}{\lambda_k} \lambda_k \|\xi_k - \xi\| \le \lambda_n \|R_n\| \sum_{k=0}^{n} \frac{\|P_k\|}{\lambda_k} O(1) \\ &\le o(1) O(1) = o(1),\end{aligned}$$

then the method $(\Re, P_n)$ is accelerating $\lambda$-boundedness. ■

*Corollary 4.* If $B_n^{-1} \in \mathcal{L}(X, X)$ $(n \in \mathbf{N})$ and

$$\lambda_n \left\|B_{n+1}^{-1}\right\| \sum_{k=0}^{n} \frac{\|B_k\|}{(k+1)\lambda_k} = o(1),$$

then the method $\mathcal{Z}$ is accelerating $\lambda$-boundedness.

Next we formulate the following problem:

*Problem 1.* Is it possible to find a simple nonregular generalized method, satisfying the conditions of Proposition 2 or Corollary 4?

In the sequel we get Tauberian remainder theorems for the methods $\Re$ and $\mathcal{Z}$.

**Proposition 3.** *Let the conditions* (1.4), (1.5), (2.1) *and*

$$\lambda_k \left\|R_n R_k^{-1}\right\| \|\Delta \xi_k\| = O(\|R_n P_k\|) \quad (k \le n, \quad n \in \mathbf{N}_0) \tag{2.7}$$

*be satisfied. Then* $\Re x \in m_X^\lambda$ *implies* $x \in m_X^\lambda$.

*Proof.* We will get a proof of this assertion using Lemma 6. By Lemma 1 the conditions (1.4) and (1.5) imply that the method $\Re$ is regular. Using (2.1) we get that the condition (1.8) is satisfied. The condition (2.7) implies that (1.9) is fulfilled. Using (1.2) we get that (1.10) is fulfilled. So $\Re x \in m_X^\lambda$ implies $x \in m_X^\lambda$. ■

*Corollary 5.* Let $B_n^{-1} \in \mathcal{L}(X, X)$ $(n \in \mathbf{N})$ and the conditions (1.4), (1.7), (1.8),

$$\lambda_n \left\|B_{n+1}^{-1}\right\| \sum_{k=0}^{n} \frac{\|B_k\|}{(k+1)\lambda_k} = O(1)$$

and

$$\lambda_k \left\|B_{n+1}^{-1} B_{k+1}\right\| \|\Delta \xi_k\| = O\left(\left\|B_{n+1}^{-1}(B_{k+1} - B_k)\right\|\right) \quad (k \le n, \quad n \in \mathbf{N}_0)$$

be satisfied. Then $\mathcal{Z}x \in m_X^\lambda$ implies $x \in m_X^\lambda$.

*Remark 2.* As the form and the assertion of Tauberian theorems essentially depend on the method used for proof it would be interesting to compare the assertions of Lemma 5, Proposition 3 and Lemma 7.

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**Apibendrintas Riesz metodas ir konvergavimo pagreitis**

I. Tammeraid

Straipsnyje įrodyta keletas teiginių, susijusių su $\lambda$ aprėžtumu apibendrintam Rieszo metodui, kai $P_n$ yra tiesinis aprėžtasis operatorius Banacho erdvėje $X$. Šie rezultatai taikomi tiriant Rieszo ir apibendrinto Zygmundo metodų konvergavimo pagreitį.