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ABOUT THE SOLUTION IN CLOSED FORM OF GENERALIZED MARKUSHEVICH BOUNDARY VALUE PROBLEM IN THE CLASS OF ANALYTICAL FUNCTIONS

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Abstract. The paper is devoted to the investigation of the problem of obtaining piecewise analytical functions $F(z) = \left\{F^+(z), F^-(z)\right\}$ with the jump line L, vanishing on the infinity and satisfying on L the boundary condition

$$F^+\left[(\alpha(t)\right] = G(t)\, F^-(t) + b(t)\,\overline{F^-(t)} + g(t), t \in L,$$

where $\alpha(t)$ is the preserving orientation homeomorphism of L onto itself and $G(t)$, $b(t)$, $g(t)$ are given on L functions of Holder class and $G(t) \neq 0$ on L.

The algorithm for the solution of this problem was obtained and particular cases, when it is solvable in closed form are determined.

Key words: bianalytical function, boundary value problem, plane with slots, index

1. The Formulation of the Problem

Let T^+ be a bounded simply connected region on the plane of the complex variable $z = x + iy$, bounded by the simple closed Liapunov's contour L, and $T^- = \bar{C} \setminus (T^+ \cup L)$. For determination we shall suppose, that the point $z = 0$ belongs to T^+. Let us denote by $\alpha(t)$ the function, mapping the contour L onto itself with the preservation of the rule and having the derivative, satisfying the Holder condition $H(L)$. We shall use notations from [8].

Let us consider the following problem. It is required to find all piecewise analytical functions $F(z) = \{F^+(z), F^-(z)\}$ from the class $A(T^\pm) \cap H(L)$, vanishing on infinity and satisfying on L the following boundary condition

$$F^+\left[\alpha(t)\right] = G(t)\, F^-(t) + b(t)\,\overline{F^-(t)} + g(t), \quad t \in L, \tag{1.1}$$

where $G(t)$, $b(t)$, $g(t)$ are given on L functions of the class $H(L)$, and $G(t) \neq 0$ on L.

It should be noted that the problem in form (1.1) in case $\alpha(t) \equiv t$ firstly was formulated in 1946 by A.I. Markushevitch [5]. So we shall call the problem formulated above as the Markushevitch boundary value problem or, in short, the problem M, and the corresponding *homogeneous* problem ($g(t) \equiv 0$) as the problem M^0. If $\alpha(t) \neq t$ we shall call this problem *generalized* Markushevitch boundary value problem for analytical functions, or, in short, the GM problem.

During the last 50 years many original works have been devoted to the problem (1.1) (see, for example [2, 4, 6, 7, 9, 11] and the bibliography there). Even in the first works [1, 10], devoted to the investigation of the problem GM it was established, that if the condition

$$G(t) \neq 0, \quad t \in L, \tag{$*$}$$

is fulfilled, it is the *Noeter* problem.

In the author's work [7] the constructive algorithm for solution of the problem M was obtained in the general case, i.e. if only one of the conditions (*) is fulfilled. In this article we shall obtain the constructive algorithm for solution of the problem GM and show the cases, when the problem GM can be solved in a closed form (in quadratures).

2. Solution of the Markushevitch Problem in a Closed Form for Rational Coefficients

Let the region T^+ be the unity circle, i.e. $T^+ = \{z : |z| < 1\}$. Then, as it was proved in [7], if $\kappa = Ind\, G(t) \geq 0$ the problem M is equivalent to the following integral equation of Fredholm type:

$$F^-(t) + \int_L K(t,\tau)\overline{F(\tau)d\tau} = Q(t) + X^-(t)P_{\kappa-1}(t), \tag{2.1}$$

where

$$K(t,\tau) = \frac{\tau^2 X^-(t)}{2\pi i}\left\{\left[\frac{b(\tau)}{X^+(\tau)} - \frac{b(t)}{X^+(t)}\right]\frac{1}{\tau - t} + \frac{b(t)}{\tau X^+(t)}\right\}, \tag{2.2}$$

$$Q(t) = -\frac{g(t)}{2\alpha(t)} + \frac{X^-(t)}{2\pi i}\int_L \frac{g(\tau)}{X^+(\tau)}\frac{d\tau}{\tau - t}, \tag{2.3}$$

$X^+(t)$, $X^-(t)$ are canonical functions of the Riemann boundary value problem with the coefficient $G(t)$ and $P_{\kappa-1}(z)$ is the polynomial of the degree not higher than $\kappa - 1$ with arbitrary complex coefficients.

If $\kappa < 0$, by the following conditions

$$\int_L \frac{b(\tau)\overline{F^-(t)} + g(t)}{X^+(t)} t^{k-1} d\tau = 0, \qquad k = 1, \ldots, -\kappa, \tag{2.4}$$

the problem M is also equivalent to the integral equation of the form (2.1), where we put $P_{\kappa-1}(z) \equiv 0$.

Now it is easy to notice, that if the coefficients $G(t)$ and $b(t)$ are rational functions, then the kernel $K(t,\tau)$, determined by the formula (2.2), will be *degenerate* (see e.g. [8], p.181), i.e.

$$K(t,\tau) = \sum_{j=1}^{N} r_j(t) q_j(\tau), \tag{2.5}$$

where $r_j(t)$, $q_j(t)$ are determined rational functions of there arguments. Hence, in this case integral equation (2.1) has the solution *in closed form* (see for example, [3], p.37). But then the boundary value problem M also has the solution in closed form.

Thus, the following statement is valid.

Theorem 1. *If* $L = \{t : |t| = 1\}$ *and the coefficients* $G(t)$ *and* $b(t)$ *are rational functions, then the problem* M *is equivalent to the integral equation of Fredholm type* (2.1) *with degenerate kernel, and, consequently, it can be solved in a closed form (in quadratures).*

Remark 1. The statement of the theorem 1 can be also obtained from the fact, then in the case of rational coefficients and unity circle $T^+ = \{z : |z| < 1\}$ the solution of boundary value problem M, as it is known (see, for example, [4], p.223), is equivalent to the solution of the two-dimensional vector-matrix Riemann boundary value problem, where coefficient is non-singular matrix with rational elements. The latter problem also can be solved in a closed form (see, for example, [11], p.40).

3. Solution of the Markushevitch Boundary Value Problem in Rational Images of the Unity Circle

Let for finite simple connected region T^+, bounded by the simple closed Liapunov's contour, the rational function exists

$$z = \omega(\zeta), \quad \zeta = \xi + iv, \tag{3.1}$$

mapping the unity circle $K_1 = \{\zeta : |\zeta| < 1\}$ conformally on this region. So we shell call the region T^+ *the rational image of the unity circle*.

The following statement is valid.

Theorem 2. *If* T^+ *is the rational image of the unity circle and the coefficients* $G(t)$, $b(t)$ *are rational functions, then the problem* M *can be solved in closed form.*

Proof. Introducing the following notations:

$$f^{\pm}(\zeta) = F^{\pm}(z) = F^{\pm}(\omega(\zeta)),$$
$$G_1(\zeta) = G(\omega(\zeta)), \quad b_1(\zeta) = b(\omega(\zeta)), \quad g_1(\zeta) = g(\omega(\zeta)),$$

we can rewrite the boundary condition of M in this way

$$f^+(\tau) = G_1(\tau)f^-(\tau) + b_1(\tau)\overline{f^-(\tau)} + g_1(\tau), \quad |\tau| = 1, \tag{3.2}$$

The equality (3.2) is the boundary condition of the usual Markushevitch boundary value problem with the rational coefficients in the class of the analytical functions, permitting the poles in the points $\zeta = \infty$ and $\zeta = a_i, \quad i = 1, 2, \ldots, m$, where $a_i \in \{\zeta : |\zeta| > 1\}$. Reasoning further as in the work [7], we can prove, that the problem (3.2) is equivalent to the Fredholm's integral equation of the form (2.1) with the degenerate kernel (i.e. of the form (2.5)). Therefore, the problem (3.2) can be solved in closed form, which means that the problem M can also be solved in quadratures. ■

4. Solution of the Generalized Markushevitch Boundary Value Problem

As it is known (see, for example, [2], p. 153), the regions T^+ and T^- can be conformally mapped on two mutually supplementary for the full plane regions T_1^+ and T_2^- with the common boundary L_1, so, that the analytical functions $F^+(z)$ and $F^-(z)$ will be transferred to the functions $F_1^+(z)$ and $F_2^-(z)$ defined in T_1^+ and T_2^- correspondingly, and on L_1 the boundary condition of the following form will be fulfilled

$$F_1^+(\zeta) = G_1(\zeta)\, F_1^-(\zeta) + b_1(\zeta)\, \overline{F_1^-(\zeta)} + g_1(\zeta), \quad \zeta \in L_1, \tag{4.1}$$

where

$$\zeta = \omega^-(t) = \omega^+ \left[\alpha(t)\right], \, G_1(\zeta) = G\left[(\sigma(\zeta)\right], \, b_1(\zeta) = b\left[\sigma(\zeta)\right], \, \sigma\left[\omega^-(t)\right] \equiv t.$$

Here the functions $\omega^{\pm}(z)$, mapping the regions $T^{\pm}$ onto $T_1^{\pm}$ conformally, can be uniquely determined as the solution of the following Riemann boundary value problem (see, for example [2], p.154)

$$\omega^+\left[\alpha(t)\right] = \omega^-(t), \quad \omega^-(z) = z + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots, z \to \infty. \tag{4.2}$$

It is important to notice, that the indexes of the problems (1.1) and (4.1) are equal, i.e. $Ind_{L_1} G_1(\zeta) = Ind_L G(t)$. Therefore, the generalized boundary value problem (1.1) is equivalent to the usual boundary value problem (4.1) both in the sense of solvability, and the number of linear independent solutions of the corresponding homogeneous problems.

Solving the problem (4.1) by the method, proposed in [7], we can obtain the functions $F_1^+(z)$ and $F_1^-(z)$. By the formula

$$F^+(z) = F_1^+\left[\omega^+(z)\right], \quad F^-(z) = F_1^-\left[\omega^-(z)\right]$$

we can obtain the solutions of the problem (1.1). We conclude that the following statement is valid.

Theorem 3. *The generalized Markushevitch boundary value problem* (1.1) *permits the solutions in closed form, when we can solve in closed form the Riemann boundary value problem* (4.2) *and the usual Markushevitch boundary value problem* (4.1).

Example 1. Let $T^+\{z : |z| < 1\}$ and $L = \{t : |t| = 1\}$. It is required to find all piecewise analytical functions $F(z) = \{F^+(z), F^-(z)\}$, belonging to the class $A(T^\pm) \cap H(L)$, vanishing on the infinity and satisfying on L the following boundary condition:

$$F^+(-t) = t^2 F^-(t) + \frac{1}{t}\overline{F^-(t)} + \frac{2}{t^2}. \tag{4.3}$$

Solution. Here $\alpha(t) = -t$, $b(t) = \dfrac{1}{t}$, $g(t) = \dfrac{2}{t^2}$. The functions $\omega^+(z) = -z$, $\omega^-(z) = z$ are solutions of the Riemann boundary value problem (4.2). Consequently, in this case the boundary value problem (4.1) will have the form

$$F_1^+(t) = t^2 F_1^-(t) + \frac{1}{t}\overline{F_1^-(t)} + \frac{2}{t^2}, \quad |t| = 1.$$

It should be noted, that here

$$\kappa = Ind\left(t^2\right) = 2, X^+(z) = 1, X^-(z) = \frac{1}{z^2}, P_{\kappa-1}(z) = C_0 + C_1 z,$$

where C_0 and C_1 are arbitrary complex constants. Hence, taking into consideration formulas (2.1) and (2.2), we get in this case:

$$K(t,\tau) \equiv 0, \quad Q(t) = \frac{2}{t^4},$$
$$F_1^-(z) = \frac{C_1}{z} + \frac{C_0}{z^2} - \frac{2}{z^4},$$
$$F_1^+(z) = (C_0 + \bar{C}_1) + (C_1 + \bar{C}_0)z - 2z^3.$$

Then the following functions will be the solution of the problem (4.3):

$$F_1^-(z) = \frac{C_1}{z} + \frac{C_0}{z^2} - \frac{2}{z^4},$$
$$F^+(z) = (C_0 + \bar{C}_1) - (C_1 + \bar{C}_0)z + 2z^3.$$

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Apie apibendrintojo Markuševičiaus uždavinio sprendimą analizinių funkcijų klasėje

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Darbe pateikiamas algoritmas Markuševičiaus uždavinio, kai ieškomos dalimis analizinės funkcijos $F(z) = \{F^+(z), F^-(z)\}$ nykstančioje begalybėje, savo šuolių linijoje L tenkinančios sąlygą

$$F^=[\alpha(t)] = G(t)\bar{F}(t) + b(t)\overline{F^-(t)} + g(t), \quad t \in L,$$

kur $G(t), b(t), g(t)$ – apibrėžtos kontūre L funkcijos Golderio klasės, o $\alpha(t)$ – homemorfizmas kontūro į save. Atvejui $\alpha(t) \equiv t$ uždavinį suformulavo A.I. Markuševičius 1946 m. Įrodyta, kad uždavinio sprendimas suvedamas į integralinės antrosios rūšies Fredholmo tipo lygties sprendimą. Pateikiamas pavyzdys, iliustruojantis gautus teorinius rezultatus.