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DIFFUSION OF POPULATION UNDER THE INFLUENCE INDUSTRIALIZATION IN A TWIN-CITY ENVIRONMENT

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Abstract. A mathematical model of a living population in a twin-city is proposed. Here populations are migrating from one place to another for their resource and settlement under the influence of industrialization. The long term effect of industrialization on the movement of human population is considered in two adjoining cities. It is shown that the steady state distribution of population is positive, continuous, monotonic and the system is stable under certain set of conditions. Further, numerical solution of the steady state distributions of population and industrialization are shown by taking particular values of the parameters.

Key words: Diffusion of population, industrialization, twin-city, steady state distribution, stability

1. Introduction

Modelling is very useful in understanding the behaviour of any environmental system. The models, in fact, represent the system in an abstract form, and provide necessary information about the system. It is a sequential and iterative process which helps in conceptualization, synthesis, simulation and analysis of the synthesis [1, 13, 14, 15, 16].

Every population is characterized with such characteristics as dispersion, fluctuation, sex ratio, birth rate and death rate [12, 14]. Population growth in a particular region is directly related to the continuous changes taking place in that environment as the environment is never static and keeps on changing from time to time due to several reasons of which some are natural while others are man-made [5, 6, 8, 9, 10, 11].

One of the man-made reasons responsible for population migration from one place to another, is industrialization [2], which has positive effects in terms of employment and resources, while some major negative effects in the form of air and water pollution. Moreover, the effluents from industries contain many chemicals that are toxic to living organisms. Industrial effluents may also contain some radioactive substances which cause many deadly diseases [17].

An interesting problem, in a twin city the continuous movement of population takes place due to the influence of different level of industrialization. In this paper we therefore propose a mathematical model to understand the long time effect of industrialization in two adjoining cities.

2. Mathematical Model

We consider a linear environment $0 \leq x \leq L_2$, consisting of two adjoining cities $0 \leq x \leq L_1$ and $L_1 \leq x \leq L_2$ with L_1 as the interface between the two cities. Here the regions are divided either by a river, highway or any other geographical or topographical condition. Let $I_i(x,t)$ and $N_i(x,t)$ be the respective industrialization and population densities at the location x at time t in i-th region, where $i = 1, 2$ (see Fig. 1).

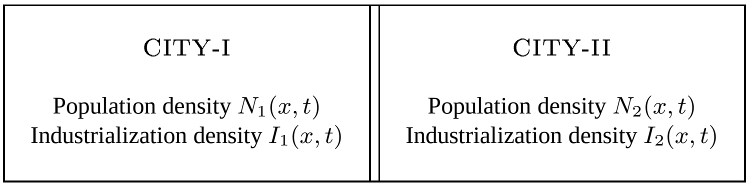

Figure 1. Twin city environment.

It is assumed that $I_i(x,t)$ grows logistically in both regions, with growth rate a_i and carrying capacity C_i. Further, we assume that in i-th city the population grows in absence of industrialization in a general logistical manner with growth rate r_i, carrying capacity K_i, growth factor β_i and in presence of industrialization, the growth rate of population varies with interaction rate α_i (i.e. which includes all positive and negative effects of industrialization on population). Here, D_{1i} and D_{2i} are the diffusion coefficients of industrialization and population respectively in the i-th region. It is also assumed that, when the industrialization reaches its highest level (i.e at the carrying capacity), then the population in that region also reaches its carrying capacity.

Keeping in view all these assumptions, we get the following mathematical model:

$$\begin{cases} \dfrac{\partial I_i}{\partial t} == a_i I_i \left(1 - \dfrac{I_i}{C_i}\right) + D_{1i} \dfrac{\partial^2 I_i}{\partial x^2}\,, \\ \dfrac{\partial N_i}{\partial t} = r_i N_i \left(1 - \left(\dfrac{N_i}{K_i}\right)^{\beta_i}\right) + \alpha_i N_i \left(1 - \dfrac{I_i}{C_i}\right) + D_{2i} \dfrac{\partial^2 N_i}{\partial x^2}\,, \end{cases} \tag{2.1}$$

where $a_i, C_i, r_i, K_i, \alpha_i, D_{1i}, D_{2i} > 0$, $i = 1, 2;\ \ 0 < x < L_2$.

We are interested only in coexistence of the population and industrialization at the equilibrium (N_i^*, I_i^*). Thus we want to find $N_i^* > 0$, $I_i^* > 0$, such that

$$\begin{cases} a_i I_i^* \left(1 - \dfrac{I_i^*}{C_i}\right) = 0\,, \\ r_i N_i^* \left(1 - \left(\dfrac{N_i^*}{K_i}\right)^{\beta_i}\right) + \alpha_i N_i^* \left(1 - \dfrac{I_i^*}{C_i}\right) = 0\,, \end{cases}$$

which implies $N_i^* = K_i$ and $I_i^* = C_i$. Therefore, the nonzero equilibrium point is given by the carrying capacities of the industrialization and population, respectively in that habitat.

We also assume the continuity and flux matching conditions at the interface $x = L_1$. The continuity conditions at the interface $x = L_1$ for this system are

$$I_1(L_1, t) = I_2(L_1, t), \quad N_1(L_1, t) = N_2(L_1, t) \tag{2.2}$$

and the continuous flux matching conditions at the interface $s = L_1$ for $I_i(x, t)$ and $N_i(x, t)$ are written as

$$D_{11}\frac{\partial I_1(L_1, t)}{\partial x} = D_{12}\frac{\partial I_2(L_1, t)}{\partial x}\,, \qquad D_{21}\frac{\partial N_1(L_1, t)}{\partial x} = D_{22}\frac{\partial N_2(L_1, t)}{\partial x}\,. \tag{2.3}$$

The model is studied under no-flux boundary conditions i.e.

$$\frac{\partial I_1(0, t)}{\partial x} = 0, \quad \frac{\partial I_2(L_2, t)}{\partial x} = 0\,, \qquad \frac{\partial N_1(0, t)}{\partial x} = 0, \quad \frac{\partial N_2(L_2, t)}{\partial x} = 0\,. \tag{2.4}$$

Finally, the model is completed by assuming some positive initial distribution

$$\begin{aligned} &I_1(x, 0) = f_1(x) > 0, \quad N_1(x, 0) = g_1(x) > 0, \quad 0 < x < L_1\,, \\ &I_2(x, 0) = f_2(x) > 0, \quad N_2(x, 0) = g_2(x) > 0, \quad L_1 < x < L_2\,. \end{aligned} \tag{2.5}$$

We first study the existence and stability behaviour of system (2.1) in homogeneous habitat, the effect of patchiness will be investigated later.

3. Model in a Homogeneous Habitat

The corresponding model of (2.1) in a single homogeneous habitat without diffusion, i.e.

$$a_i = a, \;\; r_i = r, \;\; K_i = K, \;\; C_i = C, \;\; \alpha_i = \alpha, \;\; \beta_i = \beta, \quad i = 1, 2$$

becomes

$$\begin{cases} \dfrac{dI}{dt} = aI\left(1 - \dfrac{I}{C}\right), \\ \dfrac{dN}{dt} = rN\left(1 - \left(\dfrac{N}{K}\right)^{\beta}\right) + \alpha N\left(1 - \dfrac{I}{C}\right). \end{cases} \tag{3.1}$$

Here the positive equilibrium is (K, C), which is locally asymptotically stable if

$$\left(\frac{\alpha K}{C}\right)^2 \leqslant 4ar\beta\,. \tag{3.2}$$

This can be easily verified by using Lyapunov's direct method and taking the following positive definite function:

$$V(t) = \frac{1}{2}\left((I - C)^2 + (N - K)^2\right).$$

Example 1. By taking the following values of the parameters

$$a = 0.05,\ r = 0.06,\ \alpha = 0.005,\ \beta = 2,\ C = 1000,\ K = 80,000,$$

we see that condition (3.2) holds true.

4. Steady State Problem in Twin City Environment

We denote the steady state of the industrial density by $u_i(x)$ and the population density by $v_i(x)$ in the i-th city, for $i = 1, 2$.

Then the steady state problem becomes:

$$\begin{cases} D_{1i}\dfrac{d^2u_i}{dx^2} + a_iu_i\left(1 - \dfrac{u_i}{C_i}\right) = 0\,, \\ D_{2i}\dfrac{d^2v_i}{dx^2} + r_iv_i\left(1 - \left(\dfrac{v_i}{K_i}\right)^{\beta_i}\right) + \alpha_iv_i\left(1 - \dfrac{u_i}{C_i}\right) = 0\,. \end{cases} \tag{4.1}$$

The continuity and flux matching conditions at the interface $x = L_1$ are given by

$$\begin{aligned} &D_{11}\frac{du_1}{dx}(L_1) = D_{12}\frac{du_2}{dx}(L_1), \quad D_{21}\frac{dv_1}{dx}(L_1) = D_{22}\frac{dv_2}{dx}(L_1)\,, \\ &u_1(L_1) = u_2(L_1), \quad v_1(L_1) = v_2(L_1)\,, \end{aligned} \tag{4.2}$$

and the no-flux boundary conditions are

$$\begin{aligned} \frac{du_1}{dx}(0) = 0, \quad \frac{du_2}{dx}(L_2) = 0, \\ \frac{dv_1}{dx}(0) = 0, \quad \frac{dv_2}{dx}(L_2) = 0\,. \end{aligned} \tag{4.3}$$

By using a similar analysis as in [3, 4, 7], we can prove the following theorem.

Theorem 1. *The steady state distribution u_i and v_i are positive, continuous and monotonic in the whole region.*

Now we will find the stability conditions of the system by using Lyapunov's direct method.

Theorem 2. *The steady state system* (4.1) *for $\beta_i = 1,\ i = 1, 2$ is locally asymptotically stable if*

$$u_i \geqslant \frac{C_i}{\alpha_i}, \quad v_i \geqslant \frac{K_i}{2}\,, \tag{4.4}$$

$$\left(\frac{\alpha_i K_i}{C_i}\right)^2 \leqslant 4a_i r_i\,. \tag{4.5}$$

Proof. Linearizing the steady state system (4.1) by using

$$eI_i(x,t) = u_i(x) + n_i(x,t), \ \ N_i(x,t) = v_i(x) + m_i(x,t)$$

we get a system of linear equations

$$\begin{cases} \dfrac{\partial n_i}{\partial t} = a_i n_i \left(1 - \dfrac{2u_i}{C_i}\right) + D_{1i}\dfrac{\partial^2 n_i}{\partial x^2}\,, \\ \dfrac{\partial m_i}{\partial t} = r_i m_i \left(1 - \dfrac{2v_i}{K_i}\right) + \alpha_i \left[m_i\left(1 - \dfrac{u_i}{C_i}\right) - n_i \dfrac{v_i}{C_i}\right] + D_{2i}\dfrac{\partial^2 m_i}{\partial x^2}\,. \end{cases}$$

We use the following positive definite function

$$V(t) = \sum_{i=1}^{2} \int_{L_{i-1}}^{L_i} \frac{1}{2}\left(n_i^2 + m_i^2\right)\,dx.$$

By using the boundary and flux matching conditions at the interface L_1, we obtain

$$\sum_{i=1}^{2} \int_{L_{i-1}}^{L_i} D_{1i} n_i \frac{\partial^2 n_i}{\partial x^2}\,dx = -\sum_{i=1}^{2} \int_{L_{i-1}}^{L_i} D_{1i}\left(\frac{\partial n_i}{\partial x}\right)^2 dx\,,$$

$$\sum_{i=1}^{2} \int_{L_{i-1}}^{L_i} D_{2i} m_i \frac{\partial^2 m_i}{\partial x^2}\,dx = -\sum_{i=1}^{2} \int_{L_{i-1}}^{L_i} D_{2i}\left(\frac{\partial m_i}{\partial x}\right)^2 dx\,.$$

Hence, the system will be asymptotically stable if

$$\left(1 - \frac{2u_i}{C_i}\right) \leqslant 0, \ \ r_i\left(1 - \frac{2v_i}{K_i}\right) + \alpha_i\left(1 - \frac{u_i}{C_i}\right) \leqslant 0\,, \tag{4.6}$$

$$\left[\alpha_i \frac{v_i}{C_i}\right]^2 \leqslant 4a_i\left(1 - \frac{2u_i}{C_i}\right)\left[r_i\left(1 - \frac{2v_i}{K_i}\right) + \alpha_i\left(1 - \frac{u_i}{C_i}\right)\right]\,. \tag{4.7}$$

It can be easily verified that, (4.6) is automatically satisfied if inequalities (4.4) are true. Moreover from (4.7) by using a simple concept that if $f(x) \leqslant g(x)$ then $\max f(x) \leqslant \min g(x)$, we get (4.5). Hence the theorem is proved. ■

5. Discussion and Numerical Simulation

There are several twin cities in India, for example Secundrabad–Hyderabad, Chandigarh–Mohali, Delhi–Gaziabad. First pair of cities are separated by highways, second pair by inter state border and third pair by river. We can also find many examples round the globe.

Moreover in our study the industrialization means not only small and big industries, it also includes all man-made projects, e.g. marketing complex, housing complex. Hence it is very difficult to get complete or concrete data of population and industrialization at different time and space in both the cities.

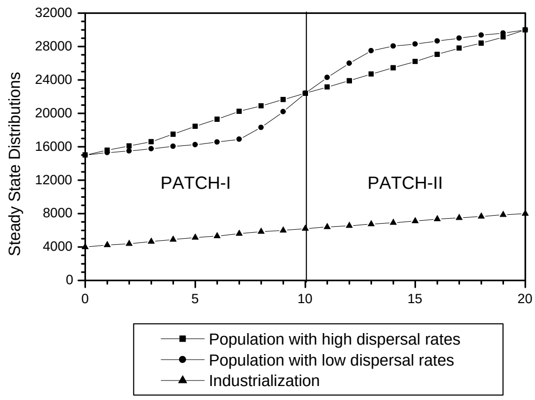

Figure 2. Steady state distributions

For better understanding, in this section we study, the steady state distributions of the population and industrialization in two adjoining regions (see Fig. 2) with flux matching conditions at interface and no-flux boundary conditions as stated in (4.1) – (4.3). A particular set of parameters is shown in Table 1, the region parameters are the following:

$$\beta = 1.2, \quad L_1 = 10, \quad L_2 = 20\,.$$

It can be easily verified that the stability conditions of Theorem 2 are satisfied for these values of parameters. We see in Fig. 2, that both population and industrialization distributions are continuous and monotonic from one end to the other end of the habitat. Moreover, if the dispersal rates D_{21} and D_{22} are very high, then the steady state distribution of the population is almost linear and when dispersal rates are very low then the population distribution is almost at the level of carrying capacities of the respective cities except near to the interface of cities where abrupt changes take place (see Fig. 2).

Table 1. Values of the parameters for Fig. 2.

Parameters	Patch 1	Fig. 2	Patch 2	Fig. 2
Growth rate of Industrialization	a_1	0.02	a_2	0.03
Growth rate of Population	r_1	0.04	r_2	0.03
Carrying capacity of Population	K_1	15000	K_2	30000
Carrying capacity of Industrialization	C_1	4000	C_2	8000
Interaction rate	α_1	0.005	α_2	0.003
Migration coeffi cient of Industrialization	D_{11}	0.5	D_{12}	0.5
For high dispersal rate of population	D_{21}	0.8	D_{22}	0.7
For low dispersal rate of population	D_{21}	0.015	D_{22}	0.02

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Populiacijos difuzija miestų-dvynių aplinkoje, esant industralizacijos poveikiui

J. Dhar, H. Singh

Straipsnyje pasi¯ulytas populiacijos dinamikos miestuose-dvyniuose matematinis modelis. Daroma preilaida, kad populiacija migruoja iš vienos vietos į kitą industralizacijos poveikyje. Tiriamas ilgalaikis industralizacijos poveikis žmonių judėjimui dviejuose gretimai esančiuose miestuose. Įrodyta, kad esant išpildytoms tam tikroms sąlygoms, nusistovėjęs rėžimas yra tolydus, monotoniškas ir stabilus. Taip pat pateiktas skaitiniais metodais gautas stacionarusis pasiskirstymas, esant pasirinktam parametrų rinkiniui.