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ON AN EXACT DESCRIPTION OF THE SCHOTTKY GROUPS OF SYMMETRIES[1]

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Abstract. Exact description of the Schottky groups of symmetries is given for certain special configurations of multiply connected circular domains. It is used in the representation of the solution of the Schwarz problem which is applied at the study of effective properties of composite materials.

Key words: symmetries, Schottky group, Schwarz boundary value problem, composite materials

1. INTRODUCTION

Description of special subgroups of the group of conformal mappings on the complex plane is a classical problem. The first essential results in this direction were obtained at the end of XIX - beginning of XX centuries by F. Schottky, H. Schwarz, H. Poincaré, L. Fuchs, E. Picard, A. Hurwitz, F. Klein and others. These results formed the base of the theory of the groups of conformal mappings, the theory of automorphic functions and Poincaré θ-series (see [3]). Further results and modern view on this subject are presented in the monographs [4, 9].

The theory of conformal mapping constitutes a very suitable tool for the study of two-dimensional problems of mathematical physics. Recently an interest has arised to describe special groups of conformal mappings, which belong to so called class of the Schottky groups. It should be noted, for instance, the application of such groups to the constructive representation of conformal mappings of multiply connected domains onto canonical domains (see, *e.g.* [1, 2, 8]), to the analytic solution of the Schwarz boundary value problem

$$\operatorname{Re} F(t) = f(t), \quad t \in L, \tag{1.1}$$

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or more general Riemann-Hilbert boundary value problem

$$\mathrm{Re}\,\overline{\lambda(t)}F(t) = f(t), \quad t \in L, \tag{1.2}$$

for a multiply connected circular domain (see [7]). They are also used for the study of certain special cases of $\mathbb{R}$-linear boundary value problem

$$\phi^+(t) = a(t)\phi^-(t) + b(t)\overline{\phi^+(t)} + c(t), \quad t \in L, \tag{1.3}$$

for multiply connected domains. Investigations of all these problems based on the method of functional equations are described in the recent monograph [8]. These results constitute the ground for a new constructive approach to the study of boundary value problems of mathematical physics.

Therefore the study of general properties of Schottky groups generated by symmetries with respect to a number of circles becomes an actual problem. It is also important to give an exact representation of elements of such groups for certain special cases since these groups are used in formulas for the solutions of the problems (1.1), (1.2), (1.3). They can be applied for solving problems of filtration, composite materials, porous media (see the description of these applications, *e.g.*, in [6, 8]).

2. Notation and general results

2.1. Groups of symmetries

We consider representation of elements of so called Schottky groups (or Schottky-type groups). The formal definition of the Schottky group is as follows:

DEFINITION 1. Let $Q_1, Q_2, \ldots, Q_n$ and $Q'_1, Q'_2, \ldots, Q'_n$ be two families of circles. Let the circles of each family be situated outside each other (*i.e.* the circles of each family are nonoverlapping). Let $\mathbf{T}_j$ be a (fractional)-linear transform with respect to z or $\overline{z}$ which map Q_j onto Q'_j and interior of the circles of each Q_j onto exterior of Q'_j. This transform generates the group $\mathcal{K}_j$. The composition of these groups $\mathcal{K}_j$, $j = 1, 2, \ldots, n$, is called Schottky group generated by the mappings $\mathbf{T}_j$, $j = 1, 2, \ldots, n$.

Intensive study of such groups was done in twenties and thirties of the XX century. It appeared that in most cases the Schottky group has quite complicated structure and not too many general properties can be formulated.

We consider here a special case of the Schottky groups when the generators $\mathbf{T}_j$ are simply the symmetries with respect to the circles Q_j (and thus $Q_j = Q'_j$). We obtain an exact description of the elements of the corresponding Schottky groups for a number of particular cases. Let

$$Q_j = Q_j(a_j, r_j) := \{z \in \mathbb{C} : |z - a_j| = r_j\}, \quad j = 1, 2, \ldots, n,$$

be a family of circles on the complex plane (with centers a_j and with radii r_j). Let us introduce the following mappings (see [8, p. 125]):

$$z^*_{(j_m,j_{m-1},\dots,j_1)} := \left(z^*_{(j_{m-1},\dots,j_1)}\right)^*_{(j_m)}, \tag{2.1}$$

where $z^*_{(j)} = \frac{r_j^2}{\overline{(z-a_j)}} + a_j$ is the symmetry with respect to the circle Q_j. Hence, $z^*_{(j_m,j_{m-1},\dots,j_1)}$ is the composition of the successive symmetries with respect to the circles $Q_{j_1},\dots Q_{j_{m-1}},Q_{j_m}$. In the sequence $j_m, j_{m-1},\dots,j_1$ no two neighbouring numbers are equal. The number m is called the *level* of the mapping $z^*_{(j_m,j_{m-1},\dots,j_1)}$. When m is even, these mappings are Möbius transformations. If m is odd then we have anti-Möbius transformations, *i.e.* Möbius transformations with respect to $\overline{z}$. Thus these mappings can be written in the form

$$\phi_k(z) = (\alpha_k z + \beta_k)\,/\,(\gamma_k z + \delta_k)\,, \quad m \text{ is even},$$

$$\phi_k(\overline{z}) = (\alpha_k \overline{z} + \beta_k)\,/\,(\gamma_k \overline{z} + \delta_k)\,, \quad m \text{ is odd},$$

where $\alpha_k\delta_k - \beta_k\gamma_k = 1$. Here

$$\begin{aligned} &\phi_0(z) = z,\ \ \phi_1(\overline{z}) = z^*_{(1)},\ \ \phi_2(\overline{z}) = z^*_{(2)},\ \ \dots,\ \ \phi_m(\overline{z}) = z^*_{(m)}, \\ &\phi_{m+k}(z) = z^*_{((k+1),1)},\quad k \ge 1\,. \end{aligned} \tag{2.2}$$

The functions ϕ_k generate a Schottky group $\mathcal{K}$ (see [3]). In the following we denote by $\mathcal{G}$ the subgroup of $\mathcal{K}$ consisting of the mappings ϕ_k of an even order, and by $\mathcal{F}$ the subgroup of $\mathcal{K}$ consisting of the mappings ϕ_k of an odd order. The following general properties of the successive symmetries are well-known (see [3]).

Properties of successive symmetries

1. Each (fractional)-linear transform $w = \frac{\alpha z+\beta}{\gamma z+\delta}$ of the complex plane $\mathbb{C}$ is equivalent to an even number of symmetries with respect to certain circles.
2. (Fractional)-linear transforms of the complex plane $\mathbb{C}$ (which are not identity, $w \equiv z$) have at most two fixed points. Thus it is true for the elements of the subgroup $\mathcal{G}$.
3. Any transform $w = \frac{\alpha z+\beta}{\gamma z+\delta}$ can be represented in one of the following forms

   $$\frac{w-\zeta_1}{w-\zeta_2} = K\frac{z-\zeta_1}{z-\zeta_2}, \quad \text{or} \quad w-\zeta_1 = K(z-\zeta_1),$$

   where ζ_1, ζ_2 are the fixed points of the transform, and the coefficient K is a complex number satisfying the relation

   $$\sqrt{K} + \frac{1}{\sqrt{K}} = \alpha + \delta\,.$$

4. If $K = Ae^{i\theta}$ ($A > 0, \theta \in [0, 2\pi)$) then the transform w
   a) is called hyperbolic, if $K = A$;
   a) is called elliptic, if $K = e^{i\theta}$;
   a) is called loxodromic, if $K = Ae^{i\theta}$, $\theta \neq 0$.

5. Let the transform $w = \frac{\alpha z+\beta}{\gamma z+\delta}$ has two fixed points ζ_1, ζ_2 and is represented in the form
$$\frac{w-\zeta_1}{w-\zeta_2} = K\frac{z-\zeta_1}{z-\zeta_2}.$$
Then the m-th iteration of this transform, *i.e.* the transform
$$w^{(m)} := \underbrace{w \circ w \circ \ldots \circ w}_{m-times}$$
has the same fixed points and is represented in the form
$$\frac{w^{(m)}-\zeta_1}{w^{(m)}-\zeta_2} = K^m\frac{z-\zeta_1}{z-\zeta_2}\,. \tag{2.3}$$
It should be mentioned that the above properties hold only for the symmetries of even level. As for the symmetries of odd level the situation is more complicated.
6. The set of fixed points of the transform of an odd level
$$w = \frac{\alpha\overline{z}+\beta}{\gamma\overline{z}+\delta}$$
can be either the whole complex plane $\mathbb{C}$, or a circle, or two points, or a point, or even an empty set. Really, fixed points ($z = x+iy$) have to satisfy the following system of real equations
$$\begin{cases} a(x^2+y^2)+bx+cy+d=0, \\ b_1x+c_1y+d_1=0\,. \end{cases} \tag{2.4}$$
Then the property 6 follows immediately. It is not difficult to see that all possibilities for the fixed set are achieved by certain transformations of an odd level.

2.2. Schwarz operator and groups of symmetries

Our interest to obtain an exact description of elements of Schottky groups of symmetries is motivated by the application of such groups at the study of composite materials. Thus, the properties of two-dimensional composite materials with cylindric inclusions are described in terms of the solutions of certain boundary value problems for harmonic functions in a multiply connected circular domain. Such models are described in the monograph [8]. These solutions are represented in term of certain Schottky groups of symmetries.

To clarify this situation let us give such a formula for one of the most simple boundary value problems, which describes the composite materials, namely, for the Schwarz boundary value problem (1.1). Let us consider mutually disjoint discs
$$\mathbb{D}_j := \mathbb{D}(a_j, r_j) = \{z \in \mathbb{C} : \ |z-a_j| < r_j\}, \ \ j = 1, 2, \ldots, n$$
on the complex plane $\mathbb{C}$. Let $D := \widehat{\mathbb{C}} \setminus \bigcup\limits_{j=1}^{n} \mathbb{D}_j$ (see Fig. 1). We choose the orientation of the boundary $Q := \bigcup\limits_{j=1}^{n} Q_j = \partial\mathbb{D}_j$ in such a way, that the domain containing

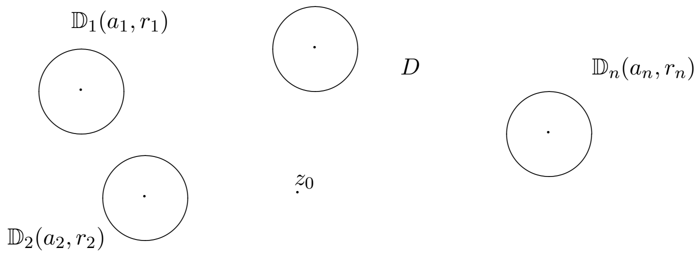


Figure 1. A multiply connected domain.

∞ is on the left side. We give here the formulation of the Schwarz problem in this specified type of domains in order to be precise at the representation of the solutions. In fact, the Schwarz problem can be posed for any Jordan domain.

The *Schwarz problem* for the domain D is to find a function F, analytic in D and continuous in $\mathrm{cl}\, D$, such that its boundary values satisfy the relation

$$\begin{cases} \operatorname{Re} F(t) = f(t), \quad t \in Q = \partial D, \\ \operatorname{Im} F(z_0) = 0, \end{cases} \tag{2.5}$$

where f is a given function on Q, z_0 is a given point in D. The operator $\mathbf{T}$, which assigns to each pair (f, z_0) the solution of the Schwarz problem (2.5), is called the *Schwarz operator* of the domain D.

In the case of the Hölder-continuous function f and the multiply connected domain D being of the above described type the Schwarz operator is delivered by the formula [8, p. 135]

$$\begin{aligned}(\mathbf{T}f)(z) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int\limits_{Q_j} f(\zeta) \Bigg\{ \sum_{\phi_j \in \mathcal{G}, j \neq 0} \left[\frac{1}{\zeta - \phi_j(z_0)} - \frac{1}{\zeta - \phi_j(z)} \right] \\ + \left(\frac{r_j}{\zeta - a_j} \right)^2 \sum_{\phi_j \in \mathcal{F}} \left[\frac{1}{\overline{(\zeta - \phi_j(\overline{z}))}} - \frac{1}{\overline{(\zeta - \phi_j(\overline{z_0}))}} \right] - \frac{1}{\zeta - z} \Bigg\} d\zeta \\ + \sum_{j=1}^{n} \int\limits_{Q_j} f(\zeta) \frac{\partial A}{\partial \nu}(\zeta)\, d\zeta + \sum_{m=1}^{n} A_m \left[\log(z - a_m) + \psi_m(z) \right] + i\varsigma,\end{aligned} \tag{2.6}$$

where

$$A_m = \sum_{j=1}^{n} \int\limits_{Q_j} f(\zeta) \frac{\partial \alpha_j}{\partial \nu}(\zeta)\, d\zeta, \quad j = 1, 2, \ldots, n,$$

α_j is a harmonic measure of the domain $\mathbb{D}_j$, the functions $A(z)$, $\psi_m(z)$ are uniquely defined by certain additional relations (see [8]), ν is an external normal vector to the corresponding circle, ς is an arbitrary real constant. This formula represents the Schwarz operator in any compact subset of the domain D.

3. Representation of elements of the Schottky groups of symmetries

In this Section we give a number of results concerning the representation of elements of some special Schottky groups of symmetries. We start with the most simple case of symmetries with respect to two circles.

3.1. Symmetries with respect to two circles

Let

$$\mathbb{D}_j := \mathbb{D}(a_j, r_j) = \{z \in \mathbb{C} : |z - a_j| < r_j, \quad j = 1, 2\}$$

be two nonoverlapping discs on the complex plane $\mathbb{C}$ (*i.e.* $|a_1 - a_2| \geq r_1 + r_2$) (see Fig. 2). Then the transform $w = z^*_{(1,2)}$ can be delivered by the formula

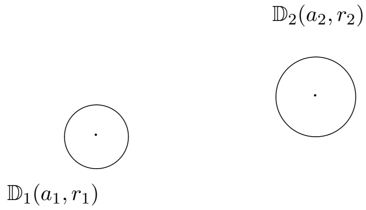


Figure 2. Symmetries with respect to two discs.

$$w = \frac{Az + B}{Cz + D},$$

where

$$\begin{aligned} &A = a_2(\overline{a}_1 - \overline{a}_2) + r_2^2, \quad B = a_1 a_2(\overline{a}_1 - \overline{a}_2) + r_1^2 a_2 - r_2^2 a_1, \\ &C = \overline{a}_1 - \overline{a}_2, \quad D = a_1(\overline{a}_2 - \overline{a}_1) + r_1^2. \end{aligned} \tag{3.1}$$

This transform has two fixed points ζ_1, ζ_2 and satisfies the relation

$$\frac{w - \zeta_1}{w - \zeta_2} = K \frac{z - \zeta_1}{z - \zeta_2},$$

where

$$\zeta_1 = \frac{M + N}{r_1 r_2}, \quad \zeta_2 = \frac{M - N}{r_1 r_2}, \quad K = \frac{L - N}{L + N}, \tag{3.2}$$

$$M = r_2^2 - r_1^2 + \big(\overline{a}_1 - \overline{a}_2\big)(a_1 + a_2),$$

$$N = \Big((r_2^2 - r_1^2)^2 + |a_1 - a_2|^4 - 2|a_1 - a_2|^2(r_1^2 + r_2^2)\Big)^{1/2},$$

$$L = r_1^2 + r_2^2 - |a_1 - a_2|^2.$$

In the same notation the transform consisting of $2m$-symmetries

$$w^{(m)} := z^*_{(1,2,1,2,\ldots,1,2)} = \underbrace{\left(\left(\ldots\left(z^*_{(1,2)}\right)^*_{(1,2)}\ldots\right)^*_{(1,2)}\right)^*_{(1,2)}}_{\mathrm{m-times}}$$

satisfies the relation

$$\frac{w^{(m)}-\zeta_1}{w^{(m)}-\zeta_2} = K^m \frac{z-\zeta_1}{z-\zeta_2}.$$

3.2. Symmetries with respect to three circles

Let

$$\mathbb{D}_j := \mathbb{D}(a_j, r_j) = \{z \in \mathbb{C} : |z-a_j| < r, \quad j = 1,2,3\}$$

be three nonoverlapping discs of equal radii r on the complex plane $\mathbb{C}$ (*i.e.* $|a_k - a_j| \geq 2r, k \neq j$) (see Fig.3).

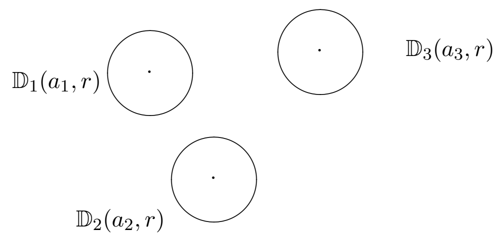


Figure 3. Symmetries with respect to three discs.

Then the transform $w = z^*_{(1,2,3)}$ can be delivered by the formula

$$w = \frac{A\overline{z}+B}{C\overline{z}+D},$$

where

$$\begin{aligned} A &= r^2(a_1 - a_2 + a_3) + a_3(\overline{a}_2 - \overline{a}_3)(a_1 - a_2), \\ B &= r^4 - r^2\left[a_3\overline{a}_1 + \overline{a}_1(a_1 - a_2) - a_3(\overline{a}_2 - \overline{a}_3)\right] - \overline{a}_1 a_3(a_1 - a_2)(\overline{a}_2 - \overline{a}_3), \\ C &= r^2 + (a_1 - a_2)(\overline{a}_2 - \overline{a}_3), \\ D &= r^2(\overline{a}_2 - \overline{a}_3 - \overline{a}_1) - \overline{a}_1(a_1 - a_2)(\overline{a}_2 - \overline{a}_3). \end{aligned} \tag{3.3}$$

The formulae for the transforms $w = z^*_{(1,3,2)}, z^*_{(2,1,3)}, \ldots, z^*_{(3,2,1)}$ can be obtained from (3.3) by interchanging of indexes.

It follows from [5] that the effective characteristics of the composites possess extreme values in the case of percolation, *i.e.* when the discs $\mathbb{D}_j$ touch each others

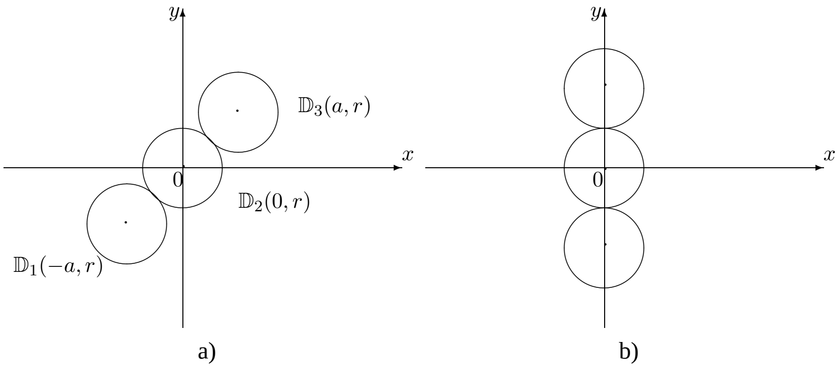


Figure 4. Arrangement of discs: a) three discs in line, b) three discs in line along imaginary axes.

and constitute a chain-type set. In the case of an external field in the direction of the real axes the extreme configuration for three discs is the following: $\mathbb{D}_j$ are situated in a line along the real or imaginary axes.

In order to present exact formulae for the corresponding transforms we consider firstly the situation when the discs lay along certain line (see Fig.4a). Namely, let

$$\mathbb{D}_1 := \{z \in \mathbb{C} : |z+a| < r\}, \mathbb{D}_2 := \{z \in \mathbb{C} : |z| < r\}, \mathbb{D}_3 := \{z \in \mathbb{C} : |z-a| < r\},$$

where $a \in \mathbb{C}$, $r > 0$, $|a| = 2r$. In this case the composition of the successive symmetries $z^*_{(1,2,3)}$ has the following representation:

$$z^*_{(1,2,3)} = \frac{a|a|^2\overline{z} + r^4 - |a|^2r^2 + |a|^4}{(r^2 + |a|^2)\overline{z} + \overline{a}|a|^2}. \tag{3.4}$$

The formulae for the transforms $w = z^*_{(1,3,2)}, z^*_{(2,1,3)}, \ldots, z^*_{(3,2,1)}$ can be obtained from (3.3) by interchanging indexes.

Further we describe the transforms in the case of optimal effective characteristics (for the external field oriented along the real axes). Let the discs $\mathbb{D}_j, j = 1, 2, 3,$ be situated along the imaginary axes and touch each other (see Fig. 4b), *i.e.* $a = 2ri$. Then

$$\begin{aligned} z^*_{(1,2,3)} &= \frac{8ir\overline{z} + 13r^2}{5\overline{z} - 8ir}, & z^*_{(1,3,2)} &= \frac{-4ir\overline{z} - 7r^2}{-7\overline{z} + 12ir}, \\ z^*_{(2,1,3)} &= \frac{-12ir\overline{z} - 7r^2}{-7\overline{z} + 4ir}, & z^*_{(2,3,1)} &= \frac{12ir\overline{z} - 7r^2}{-7\overline{z} - 4ir}, \\ z^*_{(3,1,2)} &= \frac{4ir\overline{z} - 7r^2}{-7\overline{z} - 12ir}, & z^*_{(3,2,1)} &= \frac{-8ir\overline{z} + 13r^2}{5\overline{z} + 8ir}. \end{aligned} \tag{3.5}$$

Let the discs $\mathbb{D}_j, j = 1, 2, 3$ be situated along the real axes and touch each other, *i.e.* $a = 2r$ (see Fig. 5). Then

$$z^*_{(1,2,3)} = \frac{8r\overline{z} + 13r^2}{5\overline{z} + 8r}, \qquad z^*_{(1,3,2)} = \frac{-4r\overline{z} - 7r^2}{-7\overline{z} - 12r}, \tag{3.6}$$

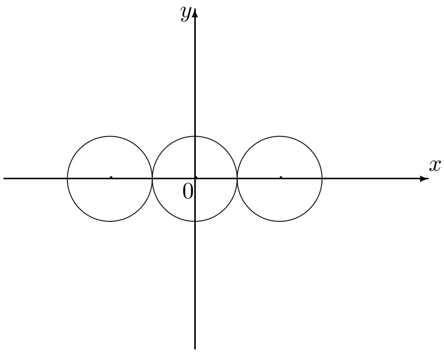
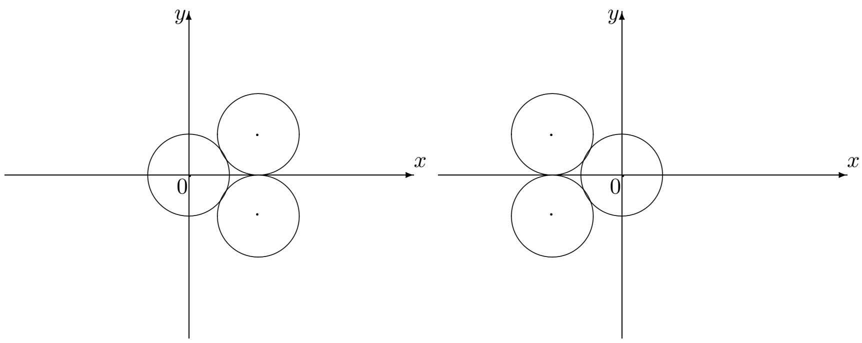


Figure 5. Three discs in line along real axes.

$$z^*_{(2,1,3)} = \frac{-12r\overline{z} - 7r^2}{-7\overline{z} - 4r}, \quad z^*_{(2,3,1)} = \frac{12r\overline{z} - 7r^2}{-7\overline{z} + 4r},$$
$$z^*_{(3,1,2)} = \frac{4r\overline{z} - 7r^2}{-7\overline{z} + 12r}, \quad z^*_{(3,2,1)} = \frac{-8r\overline{z} + 13r^2}{5\overline{z} - 8r}.$$

From the point of view of applications dealing with composite materials (see, *e.g.*, [5, 8]) it is also interesting to consider the case of discs which constitute so called "packages" of discs. Let us present two results for such configuration.



a) b)

Figure 6. Special packages: a) package of three discs I, b) package of three discs II.

Let $\mathbb{D}_j := \mathbb{D}(a_j, r_j),\ j = 1, 2, 3$, where

$$r_j = r, \ \ a_1 = 0, \ \ a_2 = 2re^{i\frac{\pi}{6}}, \ \ a_3 = 2re^{-i\frac{\pi}{6}}\ ,$$

i.e. the centers of the discs lay at the vertex of the right triangle (see Fig.6a). Then the transform $w = z^*_{(1,2,3)}$ can be delivered by the formula

$$w = \frac{2ir\overline{z} - (1 + 2\sqrt{3}i)r^2}{(-1 + 2\sqrt{3}i)\overline{z} - 2ir}.$$

Let $\mathbb{D}_j := \mathbb{D}(a_j, r_j), j = 1, 2, 3$, where (see Fig. 6b)

$$r_j = r, \quad a_1 = 0, \quad a_2 = 2re^{i\frac{5\pi}{6}}, \quad a_3 = 2re^{-i\frac{5\pi}{6}}.$$

Then the transform $w = z^*_{(1,2,3)}$ can be delivered by the formula

$$w = \frac{2ir\overline{z} + (-1 + 2\sqrt{3}i)r^2}{(-1 - 2\sqrt{3}i)\overline{z} - 2ir}.$$

3.3. Symmetries with respect to four circles

In the case of four discs we consider the only situation with four discs of equal radii symmetrically situated with respect to the origin:

$$\mathbb{D}_1 := \{ z \in \mathbb{C} : |z - a| < r \}, \quad \mathbb{D}_2 := \{ z \in \mathbb{C} : |z + \overline{a}| < r \},$$
$$\mathbb{D}_3 := \{ z \in \mathbb{C} : |z + a| < r \}, \quad \mathbb{D}_4 := \{ z \in \mathbb{C} : |z - \overline{a}| < r \},$$

where $a \in \mathbb{C},\ r > 0,\ |\mathrm{Re}\, a| \geq r,\ |\mathrm{Im}\, a| \geq r$ (see Fig.7).

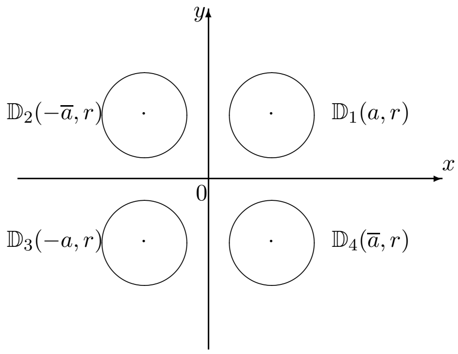


Figure 7. Four symmetrically situated discs.

In this case the composition of the successive symmetries $w = z^*_{(1,2,3,4)}$ is delivered by the formula

$$w = \frac{Az + B}{Cz + D},$$

with

$$A = r^4 + r^2a^2 - r^2\overline{a}^2 + a\overline{a}^3 - a^3\overline{a}a^2\overline{a}^2 + \overline{a}^4, \tag{3.7}$$
$$B = r^2\overline{a}^3 + r^2a\overline{a}^2 - r^2a^3 + a^4\overline{a} - a^2\overline{a}^3,$$
$$C = -a^3 - a^2\overline{a} + a\overline{a}^2 + \overline{a}^3,$$
$$D = r^4 + r^2\overline{a}^2 - r^2a^2 + r^2a\overline{a} - a^2\overline{a}^2 + a^4.$$

The transform $w = z^*_{(1,2,3,4)}$ satisfies the following relation

$$\frac{w-\zeta_1}{w-\zeta_2} = K\frac{z-\zeta_1}{z-\zeta_2}, \tag{3.8}$$

where

$$\zeta_1 = \frac{a^4 - 2a^2r^2 + a\overline{a}r + 2\overline{a}^2r^2 - a\overline{a}^3 - \overline{a}^4 + \sqrt{F}}{2(\overline{a}^2 - a^2)(\overline{a}+a)}, \tag{3.9}$$

$$\zeta_2 = \frac{a^4 - 2a^2r^2 + a\overline{a}r + 2\overline{a}^2r^2 - a\overline{a}^3 - \overline{a}^4 - \sqrt{F}}{2(\overline{a}^2 - a^2)(\overline{a}+a)},$$

$$F = (a^4 - 2a^2r^2 + a^3\overline{a} + a\overline{a}r + 2\overline{a}^2r^2 - a\overline{a}^3 - \overline{a}^4)^2$$
$$- 4(\overline{a}^2 - a^2)\left(a^3r^2 - a^4\overline{a} - a\overline{a}^2r^2 + a^2\overline{a}^3 - \overline{a}^3r^2\right),$$

$$K = \frac{A - C\zeta_1}{A - C\zeta_2},$$

A, C are given in (3.7).

The most interesting case for applications is when four discs constitute the package, *i.e.* $a = r + ir$. Then the transform $w = z^*_{(1,2,3,4)}$ has the following form:

$$w = \frac{zr(-4+7i) + 8r^2}{-8z + r(4+7i)}. \tag{3.10}$$

The transform $w = z^*_{(1,2,3,4)}$ satisfies the relation (3.8), its fixed points ζ_1, ζ_2 are given by

$$\zeta_1 = \frac{-1+\sqrt{3}i}{2}r, \quad \zeta_2 = \frac{1-\sqrt{3}i}{2}r,$$

and the coefficient K in (3.8) is given by the formula

$$K = 97 + 8\sqrt{3}.$$

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Apie Schottky simetrijos grupių tikslų apibrėžimą

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Darbe pateiktas Schottky simetrijos grupių apibrėžimas tam tikros specialios konfiguracijos daugiajungėms skritulinėms sritims. Jis yra panaudotas gaunant Švarco uždavinio, kuris pritaikomas nagrinėjant efektyvias kompozicijų savybes, sprendinio išraišką.