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STABILITY OF THE SPLINE COLLOCATION METHOD FOR SECOND ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS[1]

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Abstract. Numerical stability of the spline collocation method for the 2nd order Volterra integro-differential equation is investigated and connection between this theory and corresponding theory for the 1st order Volterra integro-differential equation is established. Results of several numerical tests are presented.

Key words: The 2nd order Volterra integro-differential equation, stability of the spline collocation method

1. Introduction

We study the numerical stability of the spline collocation method for the 2nd order Volterra integro-differential equation (VIDE). Stability means here the boundedness of approximate solutions in the uniform norm in case of the test equation when the number of knots increases. Basic ideas in the numerical solution of Volterra integral equation (VIE) and VIDE are given in [2]. First results about stability of the collocation method by polynomial splines for Volterra integral equation are given in [3] and the most adequate ones seems to be given in [5]. A special case of smooth splines is treated in [4] and special case of piecewise polynomial splines, i.e. splines with possible discontinuities in knots, is presented in [6]. There is a standard reduction of the 1st order VIDE to VIE considering the derivative of the solution as a new unknown solution. But then the test equation with constant kernel transforms into an equation with nonconstant kernel and the results obtained for VIE are not directly extendable to the 1st order VIDE. Similar phenomena takes place if we try to reduce the problem of stability for the 2nd order VIDE to that for the 1st order VIDE. Another possibility is to present the 2nd order VIDE as a system consisting

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of a first order VIDE and a first order differential equation or as a first order VIDE in product space. The components of unknown in this product space are the solution of the initial 2nd order VIDE and its derivative. The notion of stability for this VIDE in product space means boundedness of approximate solutions for both components which is, however, different from the notion of stability given in Section 4. These two circumstances motivate our investigations.

2. The spline collocation method

Consider the 2nd order Volterra integro-differential equation

$$y''(t) = f(t, y'(t), y(t)) + \int_0^t \mathcal{K}(t, s, y'(s), y(s))ds, \quad t \in [0, T], \tag{2.1}$$

with initial conditions

$$y(0) = y_0, \quad y'(0) = y_1 .$$

The functions $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{K}\colon S \times \mathbb{R} \to \mathbb{R}$ (where $S = \{(t,s)\colon 0 \le s \le t \le T\}$), and numbers y_0 and y_1 are supposed to be given. In order to describe this method, let $0 = t_0 < t_1 < \ldots < t_N = T$ (with t_n depending on N) be a mesh on the interval $[0, T]$. Denote

$$h_n = t_n - t_{n-1}, \;\; \sigma_n = (t_{n-1}, t_n], \;\; n = 1, \ldots, N, \;\; \Delta_N = \{t_1, \ldots, t_{N-1}\} .$$

Let $\mathcal{P}_k$ denote the space of polynomials of degree not exceeding k. Then, for given integers $m \ge 1$ and $d \ge -1$, we define

$$\begin{aligned} S^d_{m+d}(\Delta_N) = \{u\colon \; u|_{\sigma_n} = u_n \in \mathcal{P}_{m+d}, \;\; n = 1, \ldots, N-1\,, \\ u_n^{(j)}(t_n) = u_{n+1}^{(j)}(t_n + 0), \;\; t_n \in \Delta_N,\; j = 0, 1, \ldots, d\} \end{aligned}$$

to be the space of polynomial splines of degree $m + d$ which for $d \ge 1$, are d-times continuously differentiable on $[0,T]$, for $d = 0$ are continuous on $[0,T]$ and for $d = -1$ may have jump discontinuities at the knots Δ_N.

An element $u \in S^d_{m+d}(\Delta_N)$ as a polynomial spline of degree not greater than $m + d$ for all $t \in \sigma_n$, $n = 1, \ldots, N$, can be represented in the form

$$u_n(t) = \sum_{k=0}^{m+d} b_{nk}(t - t_{n-1})^k. \tag{2.2}$$

To find coefficients b_{nk} we suppose that a fixed selection of collocation parameters $0 < c_1 < \ldots < c_m \le 1$ is given. Then we define collocation points $t_{nj} = t_{n-1} + c_j h_n$, $j = 1, \ldots, m$, $n = 1, \ldots, N$, forming a set $X(N)$. In order to determine the approximate solution $u \in S^d_{m+d}(\Delta_N)$ of the equation (2.1) we impose the following collocation conditions

$$u''(t) = f(t, u'(t), u(t)) + \int_0^t \mathcal{K}(t, s, u'(s), u(s))ds, \quad t \in X(N). \tag{2.3}$$

Starting the calculations by this method we assume also that we can use the initial values $u_1^{(j)}(0) = y^{(j)}(0)$, $j = 0, \ldots, d$, that is justified by the requirement $u \in C^d[0, T]$. Another possible approach is to use initial conditions $u_1(0) = y(0)$ and $u_1'(0) = y'(0)$ and more collocation points (if $d \geq 1$) to determine u_1. Thus, on every interval σ_n we have $d + 1$ conditions of smoothness and m collocation conditions to determine $m + d + 1$ parameters b_{nk}. This allows us to implement the method step-by-step going from an interval σ_n to the next one.

In this paper we will analyse the stability of the collocation method where the splines are at least continuously differentiable. Thus, we suppose in the sequel that $d \geq 1$.

3. The method in the case of a test equation

Let us consider the test equation

$$y''(t) = \alpha y(t) + \beta y'(t) + \lambda \int_0^t y(s)ds + f(t), \qquad t \in [0, T], \tag{3.1}$$

where α, β and λ may be any complex numbers. The equation (3.1) is called the basis test equation (see [1]) and it has been extensively used for studying stability properties of several methods. Assume that the mesh sequence $\{\Delta_N\}$ is uniform, i.e., $h_n = h = T/N$ for all n. Representing $t \in \sigma_n$ as $t = t_{n-1} + \tau h$, $\tau \in (0, 1]$, we have on σ_n the equality:

$$u_n(t_{n-1} + \tau h) = \sum_{k=0}^{m+d} a_{nk} \tau^k, \qquad \tau \in (0, 1], \tag{3.2}$$

where we passed to the new parameters $a_{nk} = b_{nk} h^k$.

The smoothness conditions (for any $u \in S_{m+d}^d(\Delta_N)$)

$$u_n^{(j)}(t_n - 0) = u_{n+1}^{(j)}(t_n + 0),\ j = 0, \ldots, d,\ n = 1, \ldots, N-1,$$

can be expressed in the form

$$a_{n+1,j} = \sum_{k=j}^{m+d} \frac{k!}{(k-j)!j!}\, a_{nk}, \quad j = 0, \ldots, d,\ n = 1, \ldots, N-1. \tag{3.3}$$

The collocation conditions (2.3), applied to the test equation (3.1), give

$$u''(t_{nj}) = f(t_{nj}) + \alpha y(t_{nj}) + \beta u'(t_{nj}) + \lambda \int_0^{t_{nj}} u(s)ds,$$
$$j = 1, \ldots, m, \;\; n = 1, \ldots, N\,. \tag{3.4}$$

From (3.2) we get

$$u_n(t_{nj}) = \sum_{k=0}^{m+d} a_{nk} c_j^k, \quad u_n'(t_{nj}) = \frac{1}{h} \sum_{k=1}^{m+d} a_{nk} k c_j^{k-1}$$

and

$$u_n''(t_{nj}) = \frac{1}{h^2} \sum_{k=2}^{m+d} k(k-1) a_{nk} c_j^{k-2}.$$

Now the equation (3.4) becomes

$$\begin{aligned}
\frac{1}{h^2} \sum_{k=0}^{m+d} k(k-1) a_{nk} k c_j^{k-2} &= \alpha \sum_{k=0}^{m+d} a_{nk} c_j^k + \beta \frac{1}{h} \sum_{k=0}^{m+d} k a_{nk} c_j^{k-1} \\
&\quad + \sum_{r=1}^{n-1} \lambda \int_{t_{r-1}}^{t_r} u_r(s) ds + \lambda \int_{t_{n-1}}^{t_{nj}} u_n(s) ds + f(t_{nj}) \\
= \alpha \sum_{k=0}^{m+d} a_{nk} c_j^k + \beta \frac{1}{h} \sum_{k=0}^{m+d} k a_{nk} c_j^{k-1} &+ \sum_{r=1}^{n-1} \lambda h \int_0^1 \Big(\sum_{k=0}^{m+d} a_{rk} \tau^k \Big) d\tau \\
&\quad + \lambda h \int_0^{c_j} \Big(\sum_{k=0}^{m+d} a_{nk} \tau^k \Big) d\tau + f(t_{nj}) \\
= \alpha \sum_{k=0}^{m+d} a_{nk} c_j^k + \beta \frac{1}{h} \sum_{k=0}^{m+d} k a_{nk} c_j^{k-1} &+ \sum_{r=1}^{n-1} \lambda h \Big(\sum_{k=0}^{m+d} \frac{1}{k+1} a_{rk} \Big) \\
&\quad + \lambda h \sum_{k=0}^{m+d} a_{nk} \frac{c_j^{k+1}}{k+1} + f(t_{nj}). \qquad (3.5)
\end{aligned}$$

Using the notation $\alpha_n = (a_{n0}, \ldots, a_{n,m+d})$, we write (3.5) as follows:

$$\begin{aligned}
&\sum_{k=0}^{m+d} a_{nk} k(k-1) c_j^{k-2} - \alpha h^2 \sum_{k=0}^{m+d} a_{nk} c_j^k - \beta h \sum_{k=0}^{m+d} a_{nk} k c_j^{k-1} \\
&\quad - \lambda h^3 \sum_{k=0}^{m+d} a_{nk} \frac{c_j^{k+1}}{k+1} = \lambda h^3 \langle q, \sum_{r=1}^{n-1} \alpha_r \rangle + h^2 f(t_{nj}), \qquad (3.6)
\end{aligned}$$

where $q = (1, 1/2, \ldots, 1/(m+d+1))$ and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^{m+d+1}$. The difference of the equations (3.6) with n and $n+1$ yields

$$\sum_{k=0}^{m+d} a_{n+1,k}k(k-1)c_j^{k-2} - \beta h \sum_{k=0}^{m+d} a_{n+1,k}kc_j^{k-1} - \alpha h^2 \sum_{k=0}^{m+d} a_{n+1,k}c_j^k$$

$$-\lambda h^3 \sum_{k=0}^{m+d} a_{n+1,k}\frac{c_j^{k+1}}{k+1} = \sum_{k=0}^{m+d} a_{nk}k(k-1)c_j^{k-2} - \beta h \sum_{k=0}^{m+d} a_{nk}kc_j^{k-1}$$

$$-\alpha h^2 \sum_{k=0}^{m+d} a_{nk}c_j^k - \lambda h^3 \sum_{k=0}^{m+d} a_{nk}\frac{c_j^{k+1}}{k+1} + \lambda h^3 \langle q, \sum_{r=1}^{n-1} \alpha_r \rangle$$

$$+h^2 f(t_{n+1,j}) - h^2 f(t_{nj}), \quad j = 1, \ldots, m, \quad n = 1, \ldots, N-1. \tag{3.7}$$

Now we may write equations (3.3) and (3.7) in the matrix form

$$(V - \beta h V_1 - \alpha h^2 V_2 - \lambda h^3 V_3)\,\alpha_{n+1} = \big(V_0 - \beta h V_1 - \alpha h^2 V_2$$
$$-\lambda h^3 (V_3 - V_4)\big)\,\alpha_n + h^2 g_n, \quad n = 1, \ldots, N-1\,, \tag{3.8}$$

with $(m+d+1) \times (m+d+1)$ matrices V, V_0, V_1, V_2, V_3, V_4 as follows:

$$V = \begin{pmatrix} E \\ C \end{pmatrix}, \qquad V_0 = \begin{pmatrix} A \\ C \end{pmatrix}, \qquad E = \begin{pmatrix} I & 0 \end{pmatrix}\,,$$

I being the $(d+1) \times (d+1)$ unit matrix, 0 is the $(d+1) \times m$ zero matrix,

$$C = \begin{pmatrix} 0 & 0 & 2 & 6c_1 & \ldots & (m+d)(m+d-1)c_1^{m+d-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 2 & 6c_m & \ldots & (m+d)(m+d-1)c_m^{m+d-2} \end{pmatrix},$$

A being a $(d+1) \times (m+d+1)$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 2 & \ldots & m+d \\ \cdot & \cdot & \cdot & \ldots & \ldots \\ 0 & \cdot & \cdot & 1 \ldots & \binom{m+d}{d} \end{pmatrix}, V_1 = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 2c_1 & \ldots & (m+d)c_1^{m+d-1} \\ \cdot & \cdot & \ldots & \ldots & \ldots \\ 0 & 1 & 2c_m & \ldots & (m+d)c_m^{m+d-1} \end{pmatrix},$$

$$V_2 = \begin{pmatrix} & & 0 & & \\ 1 & c_1 & c_1^2 & \ldots & c_1^{m+d} \\ \cdot & \ldots & \ldots & \ldots & \ldots \\ 1 & c_m & c_m^2 & \ldots & c_m^{m+d} \end{pmatrix}, \; V_3 = \begin{pmatrix} & & 0 & \\ c_1 & c_1^2/2 & \ldots & c_1^{m+d+1}/(m+d+1) \\ \ldots & \ldots & \ldots & \ldots \\ c_m & c_m^2/2 & \ldots & c_m^{m+d+1}/(m+d+1) \end{pmatrix},$$

V_4 having the first $d+1$ rows equal to 0 and the last m rows the vector q, and, finally, the $m+d+1$ dimensional vector

$$g_n = \big(0, \ldots, 0, f(t_{n+1,1}) - f(t_{n1}), \ldots, f(t_{n+1,m}) - f(t_{nm})\big)\,.$$

Thus $g_n = O(h)$ for $f \in C^1$.

Proposition 1. *The matrix $V-\beta hV_1-\alpha h^2V_2-\lambda h^3V_3$ is invertible for sufficiently small h.*

Proof. Since

$$\det V = \det\begin{pmatrix} (d+1)dc_1^{d-1} & \dots & (m+d)(m+d-1)c_1^{m+d-2} \\ (d+1)dc_2^{d-1} & \dots & (m+d)(m+d-1)c_2^{m+d-2} \\ \dots & \dots & \dots \\ (d+1)dc_m^{d-1} & \dots & (m+d)(m+d-1)c_m^{m+d-2} \end{pmatrix} = (d+1)dc_1^{d-1}$$

$$\times\dots\times(m+d)(m+d-1)c_m^{d-1}\det\begin{pmatrix} 1 & c_1 & \dots & c_1^{m-1} \\ \dots & \dots & \dots & \dots \\ 1 & c_m^d & \dots & c_m^{m-1} \end{pmatrix} \neq 0,$$

and $d \geq 1$, the matrix V is invertible. Such is also $V-\beta hV_1-\alpha h^2V_2-\lambda h^3V_3$ for small h. Although we have supposed, in general, that $d\geq 1$, let us remark that in cases $d=0$ and $d=-1$ we may argue similarly to the proof in [6] and show that $\det(V-\beta hV_1-\alpha h^2V_2-\lambda h^3V_3)\neq 0$, for small h. ■

Therefore, the equation (3.8) can be written in the form

$$\alpha_{n+1} = (V^{-1}V_0+W)\alpha_n + r_n, \quad n=1,\dots,N-1, \tag{3.9}$$

where $W=O(h)$ and $r_n=O(h^3)$ for $f\in C^1$.

4. Stability of the method

We have seen that the spline collocation method (2.3) for the test equation (3.1) leads to the recursion (3.9).

We distinguish the method with initial values $u_1^{(j)}(0)=y^{(j)}(0)$, $j=0,\dots,d$, and another method which uses $u_1(0)=y(0), u_1'(0)=y'(0)$ and additional collocation points $t_{0j}=t_0+c_{0j}h$, $j=1,\dots,d-1$, with fixed $c_{0j}\in(0,1]\setminus\{c_1,\dots,c_m\}$ on the first interval σ_1. Denote, in addition, $d_0=\max\{d-2,0\}$ for the method with initial values and $d_0=0$ for the method with additional initial collocation.

DEFINITION 1. We say that the spline collocation method is stable if for any $\alpha,\beta,\lambda\in\mathbb{C}$ and any $f\in C^{d_0}[0,T]$ the approximate solution u remains bounded in $C[0,T]$ in the process $h\to 0$.

Let us notice that the boundedness of $||u||_{C[0,T]}$ is equivalent to the boundedness of $||\alpha_n||$ in n and h in any fixed norm of $\mathbb{R}^{m+d+1}$.

The principle of uniform boundedness allows us to establish

Proposition 2. *The spline collocation method is stable if and only if*

$$||u||_{C[0,T]} \leq \mathrm{c}\,||f||_{C^{d_0}[0,T]} \quad \forall f\in C^{d_0}[0,T], \tag{4.1}$$

where the constant c *may depend only on* T, α, β, λ *and on parameters* c_j *and* c_{0j}.

In order to formulate and prove the results concerning the numerical stability properties of the polynomial spline collocation method, we need the following results for VIE (see [5]) and for the 1st order VIDE (see [7]). The step-by-step collocation method for VIE is supposed to determine the approximate solution in $S^d_{m+d}(\Delta_N)$ by the collocation conditions similarly to (2.3) at the points t_{nj}.

1. The stability for VIE depends on the matrix $\widetilde{M} = \widetilde{U}_0^{-1}\widetilde{U}$, where $\widetilde{U}_0$ and $\widetilde{U}$ are $(m+d+1) \times (m+d+1)$ matrices as follows:

$$\widetilde{U}_0 = \begin{pmatrix} E \\ \widetilde{G} \end{pmatrix}, \qquad \widetilde{U}_0 = \begin{pmatrix} A \\ \widetilde{G} \end{pmatrix}, \qquad \widetilde{G} = \begin{pmatrix} 1 & c_1 & \dots & c_1^{m+d} \\ \dots & \dots & \dots & \dots \\ 1 & c_m & \dots & c_m^{m+d} \end{pmatrix},$$

   E and A being defined as in V and V_0.
2. If all eigenvalues of $\widetilde{M}$ are in the closed unit disk and if those which lie on the unit circle have equal algebraic and geometric multiplicities, then the spline collocation method is stable.
3. If $\widetilde{M}$ has an eigenvalue outside of the closed unit disk, then the method is unstable (u has exponential growth: $\| u \|_\infty \geq \mathrm{c}\,\mathrm{e}^{KN}$, for some constants $K > 0$ and $\mathrm{c} > 0$).
4. If all eigenvalues of $\widetilde{M}$ are in the closed unit disk and there is an eigenvalue on the unit circle with different algebraic and geometric multiplicities, then the method is weakly unstable (u may have polynomial growth: $\| u \|_\infty \sim \mathrm{c}\, N^k, \mathrm{c} > 0, k \in \mathbb{N}$).
5. For fixed c_j the eigenvalues of $\overline{M} = U_0^{-1}U$ for the 1st order VIDE in the case m and $d+1$ and eigenvalues of $\widetilde{M}$ for VIE in the case m and d coincide and have the same algebraic and geometric multiplicities, except $\mu = 1$ whose algebraic multiplicity for VIDE is greater by one than for VIE. Here U_0 and U are $(m + d+1) \times (m+d+1)$ matrices as follows:

$$U = \begin{pmatrix} E \\ G \end{pmatrix}, \quad U_0 = \begin{pmatrix} A \\ G \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 1 & 2c_1 & \dots & (m+d)c_1^{m+d-1} \\ \dots\dots & \dots & \dots & & \dots \\ 0 & 1 & 2c_m & \dots & (m+d)c_m^{m+d-1} \end{pmatrix},$$

   E and A being defined as in V and V_0.

Theorem 1. *For fixed c_j the eigenvalues of M for the 2nd order VIDE in the case m and $d+2$ and eigenvalues of $\overline{M}$ for the 1st order VIDE in the case m and $d+1$ coincide and have the same algebraic and geometric multiplicities, except $\mu = 1$ whose algebraic multiplicity for the 2nd order VIDE is greater by one than for the 1st order VIDE.*

Proof. The eigenvalue problem for M is equivalent to the generalized eigenvalue problem for V_0 and V, i.e. $(M - \mu I)v = 0$ for $v \neq 0$ if and only if $(V_0 - \mu V)v = 0$

and $(M-\mu I)w=v$ takes place if and only if $(V_0-\mu V)w=Vv$. Denote $\nu=1-\mu$. Then for the 2nd order VIDE with the parameters m and $d+2$ we have

$$V_0-\mu V= \tag{4.2}$$

$$=\begin{pmatrix} \nu & 1 & 1 & 1 & \ldots\ldots & 1 \\ 0 & \nu & 2 & 3 & \ldots\ldots & m+d+1 \\ 0 & 0 & \nu & \binom{3}{2} & \ldots\ldots & \binom{m+d+2}{2} \\ \ldots\ldots & \ldots & & \ldots & \ldots\ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \nu\ \ldots & \binom{m+d+2}{d+2} \\ 0 & 0 & \nu\cdot 2 & \nu\cdot 6c_1 & \ldots\ldots & \nu(m+d+2)(m+d+1)c_1^{m+d} \\ \ldots\ldots & \ldots & & \ldots & \ldots\ldots & \ldots \\ 0 & 0 & \nu\cdot 2 & \nu\cdot 6c_m & \ldots\ldots & \nu(m+d+2)(m+d+1)c_m^{m+d} \end{pmatrix}.$$

Let $I_{i,p}$ be the diagonal matrix obtained from unit matrix, replacing the i-th diagonal element by the number p. Thus, the products $I_{i,p}A$ and $AI_{i,p}$ mean the multiplication of i-th row and i-th column of A, respectively, by p. The direct calculation and observation that $\binom{p}{q}\frac{q}{p}=\binom{p-1}{q-1}$, allows us to get from (4.2)

$$I_{d+3,d+2}\ldots I_{3,2}(V_0-\mu V)I_{3,1/2}\ldots I_{d+m+3,1/(m+d+2)}=\begin{pmatrix}\nu & \overline{q}\\ 0 & U_0-\mu U\end{pmatrix},$$

or

$$S(V_0-\mu V)S^{-1}=R\begin{pmatrix}\nu & \overline{q}\\ 0 & U_0-\mu U\end{pmatrix}, \tag{4.3}$$

where $S=I_{d+3,d+2}\ldots I_{3,2},\ R=I_{d+m+3,d+m+2}\ldots I_{d+4,d+3}$,

$$\overline{q}=\left(1,\frac{1}{2},\ldots,\frac{1}{m+d+2}\right).$$

Now (4.3) gives

$$\det(V_0-\mu V)=(d+3)\ldots(d+m+2)\nu\det(U_0-\mu U),$$

which permits to get the assertion about algebraic multiplicities of eigenvalues of M and $\overline{M}$. Similarly to [5] we can prove that the eigenvalue $\mu=1$ of M and $\overline{M}$ has geometric multiplicity m and similarly to [6] that geometric multiplicities of $\mu\neq 1$ as an eigenvalue of M and $\overline{M}$ coincide. ■

Proposition 3. *If M has an eigenvalue outside of the closed unit disk, then the spline collocation method is not stable with possible exponential growth of approximate solution.*

Proof. The structure of the proof is similar to that of Prop. 5 in [5] and we will deal only with main moments. Consider an eigenvalue μ of $M+W$ such that $|\mu| \geq 1+\delta$ with some fixed $\delta > 0$ for any sufficiently small h. For $\alpha_1 \neq 0$, being an eigenvector of $M+W$, we have here

$$(V - \beta h V_1 - \alpha h^2 V_2 - \lambda h^3 V_3)\alpha_1 = h^2 g_0, \tag{4.4}$$

where $g_0 = (a_{10}, \dots, a_{1d}, f(t_{11}), \dots, f(t_{1m}))$, $a_{1j} = \dfrac{h^j y^{(j)}(0)}{j!}$, $j = 0, \dots, d$. Because of

$$\begin{aligned} &y''(0) = \alpha y(0) + \beta y'(0) + f(0)\,, \\ &y^{(j)}(0) = \alpha y^{(j-2)}(0) + \beta y^{(j-1)}(0) + \lambda y^{(j-3)}(0) + f^{(j-2)}(0),\ j = 3, \dots, d, \end{aligned} \tag{4.5}$$

the vector α_1 determines via (4.4) and (4.5) the values

$$f^{(j)}(0),\ \ j = 0, \dots, d-1,\ \ f(t_{11}), \dots, f(t_{1m}).$$

We take f on $[0,h]$ as the polynomial interpolating the values

$$f^{(j)}(0),\ \ j = 0, \dots, d-2,\ \ f(t_{1j}),\ \ j = 1, \dots, m,$$

and $f^{(j)}(h) = 0$, $j = 0, \dots, d_0$ (if $c_m = 1$, then $f^{(j)}(h) = 0$, $j = 1, \dots, d_0$). In the case of the method of additional knots let f be on $[0,h]$ the interpolating polynomial by the data $f(0)$, $f(t_{0j})$, $j = 0, \dots, d-1$, $f(t_{1j})$, $j = 1, \dots, m$, and $f^{(j)}(h) = 0$, (here $d_0 = 0$ and if $c_m = 1$, then $f(t_{1m}) = f(h)$ is already given and we drop the requirement $f(h) = 0$). In both cases we ask f to be on $[nh, (n+1)h]$, $n \geq 1$, the interpolating polynomial by the values $f^{(j)}(nh) = 0$ and $f^{(j)}((n+1)h) = 0$, $j = 0, \dots, d_0$ (if $c_m = 1$, then for $j = 1, \dots, d_0$), and also $f(t_{n+1,j}) = f(t_{1j})$, $j = 1, \dots, m$. This guarantees that $f \in C^{d_0}[0,T]$ and $r_n = 0$, $n \geq 1$. The interpolant f can be represented on $[t_n, t_{n+1}]$ by the formula:

$$f(t) = f(t_n + \tau h) = \sum_{i=0}^{\kappa} \left(\sum_{l=0}^{k_i} h^{s_l} p_{il} f^{(s_l)}(\xi_l) \right) \prod_{r=0}^{i-1} (\tau - b_r) \tag{4.6}$$

with b_r being c_j or c_{0j}, ξ_l being t_{nj} or t_j, $0 \leq s_l \leq d_1$, $k_i \leq i$, constants p_{il} depending on c_j and c_{0j}. In the case of initial conditions $\kappa = m + d + d_0 - 1$ ($\kappa = m + d + d_0 - 2$ if $c_m = 1$) and in the case of additional knots $\kappa = m + d + 1$ ($\kappa = m + d$, if $c_m = 1$) on the interval $[0,h]$ and $\kappa = m + 2d_0 + 1$ ($\kappa = m + 2d_0$ if $c_m = 1$) on the interval $[nh, (n+1)h]$, $n \geq 1$.

Replacing h by h/k, $k = 1, 2, \dots$, and keeping $||\alpha_1|| = h^2/k^2$, we have $||g_0||_\infty$ bounded which means that $f(t_{1j})$, $j = 1, \dots, m$, and $h^j y^{(j)}(0)/k^j$, $j = 0, \dots, d$, or $h^j f^{(j)}(0)/k^j$, $j = 0, \dots, d_0$, are bounded too in the process $k \to \infty$. Thus (4.6) gives

$$||f||_{C^{d_0}[0,T]} \leq \mathrm{c}\, k^{d_0}. \tag{4.7}$$

On the other hand, $||\alpha_{n+1}|| \geq (1+\delta)^n ||\alpha_1||$ yields

$$||\alpha_{kN}|| \geq \frac{h}{k}(1+\delta)^{kN-1} \tag{4.8}$$

and (4.1) cannot be satisfied. The inequalities (4.7) and (4.8) mean also the exponential growth of approximate solution if we keep the norm of f bounded in C^{d_0}. ■

5. Examples and numerical tests

Let us consider some special cases of d and m.

Case $d = 1$, $m \geq 1$ being arbitrary. We have

$$V = \begin{pmatrix} 1\,0\,\ldots\,0 \\ C \end{pmatrix}, \quad V_0 = \begin{pmatrix} 1\,1\,\ldots\,1 \\ C \end{pmatrix}$$

and $\det(V_0 - \mu V) = (1-\mu)^{m+2} \det C_0$ where C_0 is obtained from C omitting first two columns. This means that the method is always stable.

Case $d = 2$, $m = 1$ *(cubic splines)*. The equation $\det(V_0 - \mu V) = 0$ besides $\mu = 1$ has the solution $\mu = 1 - 1/c_1$. The method is stable if and only if $1/2 \leq c_1 \leq 1$.

Case $d = 2$, $m = 2$. Now the equation $\det(V_0 - \mu V) = 0$ has the root $\mu = 1$ with geometric multiplicity 2. From the solution

$$\mu(c_1, c_2) = 1 - \frac{c_1 + c_2 + 1}{c_1 c_2}$$

it follows that the method is stable if and only if $c_1 + c_2 \geq 1$. In numerical tests we explored the 2nd order integro-differential equation

$$\begin{cases} y''(t) = y(t) + y'(t) + \int_0^t y(s)ds - \sin(t) - \cos(t) - e^t\,, \\ y(0) = 1, \quad y'(0) = 1, \quad t \in [0,1]\,. \end{cases}$$

This equation has the exact solution $y(t) = (\sin t + \cos t + e^t)/2$. As an approximate value of $||u||_\infty$ we actually calculated $\max_{1\leq n\leq N} \max_{0\leq k\leq 10} \left|u_n(t_{n-1} + kh/10)\right|$. The results are presented in Tables 1–4. From these numerical examples we can observe a good conformity of theoretical results presented in the proceeding sections and numerical results given in this section.

Table 1. Case $d = 1$, $m = 1$ (quadratic splines).

N	4	16	64	256	4096
$c_1 = 0.5$	2.053593	2.050242	2.050041	2.050028	2.050028
$c_1 = 1.0$	2.112955	2.060136	2.052332	2.050591	2.050062

Table 2. Case $d = 1$, $m = 2$ (Hermite cubic splines).

N	4	16	64	256	4096
$c_1 = 0.4$ $c_2 = 0.6$	2.047625	2.049880	2.050018	2.050027	2.050028
$c_1 = 0.7$ $c_2 = 1.0$	2.042264	2.049630	2.050004	2.050026	2.050028

Table 3. Case $d = 2$, $m = 1$ (cubic splines).

N	4	16	64	256	512
$c_1 = 0.4$	2.047252	2.049817	61.720406	$1.60 \cdot 10^{33}$	$1.20 \cdot 10^{77}$
$c_1 = 0.5$	2.047590	2.049861	2.050017	2.050027	2.050027
$c_1 = 1.0$	2.055555	2.050364	2.050048	2.050028	2.050028

Table 4. Case $d = 2$, $m = 2$.

N	4	64	256	512
$c_1 = 0.2$ $c_2 = 0.5$	2.049254	$7.65 \cdot 10^{26}$	$2.89 \cdot 10^{139}$	$1.21 \cdot 10^{292}$
$c_1 = 0.3$ $c_2 = 0.7$	2.049935	2.050027	2.050028	2.050028
$c_1 = 0.5$ $c_2 = 1.0$	2.050015	2.050028	2.050028	2.050028

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Antros eilės Volterra integro-diferencialinių lygčių splainų kolokacijos metodo stabilumas

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Straipsnyje nagrinėjamas antros eilės Volteros integro-diferencialinių lygčių splainų kolokacijos metodo skaitinis stabilumas ir nustatytas ryšys tarp šios teorijos ir atitinkamos pirmos eilės Volterra integro-diferencialinių lygčių teorijos. Pateikti keleto skaitinių eksperimentų rezultatai.