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ON THE EXISTENCE AND UNIQUENESS OF TWO-FLUID CHANNEL FLOWS

J. SOCOLOWSKY

University of Applied Sciences, Engineering Department – Mathematics Group

PSF 2132, D-14737 Brandenburg/Havel, GERMANY

E-mail: socolowsky@fh-brandenburg.de

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Abstract. Viscous two-fluid channel flows arise in different kinds of coating technologies. The corresponding mathematical models represent two-dimensional free boundary value problems for the Navier-Stokes equations. In this paper the solvability of the related stationary problems is discussed and computational results are presented.
Furthermore, it is shown that depending on the flow parameters like viscosity or density ratios and on the fluxes there can happen nonexistence of steady-state solutions. For other parameter sets the solution is even unique.

Key words: Free boundary value problems, viscous channel flows, two-fluid flows, Navier-Stokes equations

1. Introduction

In this contribution we consider the plane stationary flow of two viscous incompressible fluids (with kinematic viscosities $\nu_i > 0$ and densities $\varrho_i > 0,\ i = 1, 2$) through a special uniform channel (cf. Fig.1). Emphasize that the corresponding problem will be formulated in dimensionless form. The concrete transition to that formulation can be found in [11]. The flow is steady-state and has some features of a slot coating process. The channel is horizontal, unbounded in both directions and contains a semi-infinite inner wall (cf. Fig.1). The lower wall $S_1 := \{\mathbf{x} \in \mathbb{R}^2 : -\infty < x_1 < +\infty,\ x_2 = 0\}$ is moving with constant velocity $\mathbf{R} = (R, 0)^T$ $(R \geqslant 0)$. The upper wall $S_2 := \{\mathbf{x} \in \mathbb{R}^2 : -\infty < x_1 < +\infty,\ x_2 = 1\}$ is at rest. Furthermore, the partial inner wall $S_3 := \{\mathbf{x} \in \mathbb{R}^2 : -\infty < x_1 < 0,\ x_2 = h_1\ (0 < h_1 < 1)\}$ is given. Thus, in fact we have two separated parallel channels for negative values of x_1. Both viscous fluids are flowing out of the two channels and behind the point $Q(0, h_1)$ they are joining and creating a free interface $\Gamma := \{\mathbf{x} \in \mathbb{R}^2 : 0 < x_1 < +\infty,\ x_2 = \psi(x_1)\}$ where ψ is unknown a priori and has to be found. It is supposed that the free interface Γ separates from the inner wall S_3 at its endpoint Q.

By $G_1 := \{\mathbf{x} \in \mathbb{R}^2 : 0 < x_2 < h_1,$ if $-\infty < x_1 \leqslant 0,$ and $0 < x_2 < \psi(x_1),$ if $0 < x_1 < +\infty\}$ we denote the fbw domain of the lower fluid. By G_2 we denote the fbw domain of the upper fluid $G_2 := \{\mathbf{x} \in \mathbb{R}^2 : h_1 < x_2 < 1,$ if $-\infty < x_1 \leqslant 0,$ and $\psi(x_1) < x_2 < 1,$ if $0 < x_1 < +\infty\}$. Finally, by $G := G_1 \cup G_2$ we mean the union of both fluid layers.

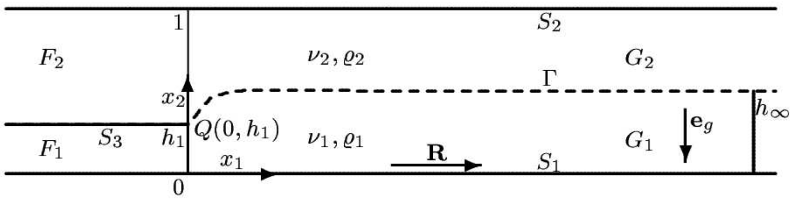

Figure 1. Two–fluid channel flow with partial inner wall.

The direction $\mathbf{e}_g$ of the gravitational force is the vector $\mathbf{e}_g = (0, -1)^T$. We study the two-fluid fbw within the channel G caused by pressure gradients downstream and by the motion of the lower channel wall. This means mathematically that the positive flux F_i in each liquid layer G_i $(i = 1, 2)$ is prescribed and the *final* fluid layer thicknesses h_∞ and $(1 - h_\infty)$ are to be determined. Note, that our (mathematical) fluxes F_i are in fact the real physical fluxes divided by the constant densities of the fluids.

An interpretation of such a fbw could be the fbw of two liquids coming from different reservoirs (i.e. slots or chambers) and fbwing commonly in one channel after their unification. In slot coaters such fbws occur on some parts of the coater. The corresponding motion as well as the final layer thicknesses are important there.

Let h_∞ with $0 < h_\infty < 1$ be the constant limit of $\psi(x_1)$ at infinity. The problem under consideration has the following form: find a vector of velocity $\mathbf{v} = (v_1(x_1, x_2), v_2(x_1, x_2))^T$, a pressure $p(x_1, x_2)$ and a function $\psi(x_1)$ satisfying in the domain G the Navier-Stokes system of equations

$$\begin{cases} (\mathbf{v} \cdot \nabla)\, \mathbf{v} - \nu \triangle \mathbf{v} + \frac{1}{\varrho}\, \nabla p = g\, \mathbf{e}_g\,, \\ \nabla \cdot \mathbf{v} = 0\,, \end{cases} \tag{1.1}$$

and the boundary and integral conditions

$$\mathbf{v}|_{S_0} = \mathbf{R} = (R, 0)^T, \quad \mathbf{v}|_{S_2} = \mathbf{0}, \quad \mathbf{v}|_{S_3^\pm} = \mathbf{0}\,, \tag{1.2}$$

$$\begin{cases} [\mathbf{v}]|_\Gamma = \mathbf{0}, \quad \mathbf{v} \cdot \mathbf{n}|_{\Gamma^-} = 0, \quad [\mathbf{t} \cdot \mathbf{S}(\mathbf{v})\, \mathbf{n}]|_\Gamma = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}x_1}\, \dfrac{\psi'(x_1)}{\sqrt{1 + \psi'(x_1)^2}} = \dfrac{1}{\sigma}\, [-p + \mathbf{n} \cdot \mathbf{S}(\mathbf{v})\, \mathbf{n}]|_\Gamma, \\ \lim_{x_1 \to +\infty} \psi(x_1) = h_\infty, \quad \int_{\delta_1(\widehat{q})} v_1(\widehat{q}, x_2)\, \mathrm{d}x_2 = F_1, \end{cases} \tag{1.3}$$

$$\int_{\delta_2(\widehat{q})} v_1(\widehat{q}, x_2)\,\mathrm{d}x_2 = F_2. \tag{1.4}$$

In problem (1.1) – (1.4) the symbol $\delta_i(\widehat{q})$ denotes the intersection of G_i with the vertical line $x_1 = \widehat{q}$ and $\sigma > 0$ is the surface tension at Γ. We further emphasize that from a physical point of view in (1.3), (1.4) only positive values of F_i make sense.

In problem (1.1) – (1.4) the following notations have been used: $\mathbf{n}$ and $\mathbf{t}$ are unit vectors normal and tangential to Γ and oriented as x_2, x_1, respectively. By $\mathbf{a} \cdot \mathbf{b}$ we mean the inner product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2)^T$ is the gradient operator, $\nabla p =$ grad p, $\nabla \cdot \mathbf{v} =$ div $\mathbf{v}$, $\varrho|_{G_m} = \varrho_m$ $(m = 1, 2)$ is the restriction of ϱ to G_m (analogously for ν). $\triangle = \nabla^2$ denotes the Laplace operator. By $\mathbf{S}(\mathbf{v})$ we denote the deviatoric stress tensor, i.e. a matrix with elements

$$S_{ij}(\mathbf{v}) = \varrho\nu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \quad i, j = 1, 2\,.$$

The symbol $[w]|_\Gamma$ expresses the jump of w crossing the free interface Γ, i.e.

$$[w(\mathbf{x}_0)]\,|_\Gamma := \lim_{\mathbf{y}\to\mathbf{x}_0} w(\mathbf{y}) - \lim_{\mathbf{x}\to\mathbf{x}_0} w(\mathbf{x}), \qquad (\mathbf{x}_0 \in \Gamma,\ \mathbf{y} \in G_1,\ \mathbf{x} \in G_2),$$

and the symbol $w|_{\Gamma^-}$ denotes the limit from below at the interface Γ, more precisely

$$w(\mathbf{x}_0)|_{\Gamma^-} := \lim_{\mathbf{y}\to\mathbf{x}_0} w(\mathbf{y}), \qquad (\mathbf{x}_0 \in \Gamma,\ \mathbf{y} \in G_1).$$

An analogous statement is true for $S_3^\pm$ (and also for $\Sigma_3^\pm$ in Section 4). Note that the left-hand side of $(1.3)_2$ (i.e. of the second equation in (1.3)) is equal to the curvature $K(x_1)$ of Γ. The fluid layer thickness h_∞ has also to be determined. Obviously, it should hold $0 < h_\infty < 1$.

Mathematical problems for the stationary flows of a viscous incompressible fluid with a free boundary were investigated by many authors. Numerous references on this field can be found, e.g., in the bibliographies of [4, 6, 12, 13]. Coating flows with the static or dynamic contact angles were studied in [1, 2, 5, 10, 11, 14, 15]. In all these papers considering either compact or semi-infinite free boundary value problems the same general scheme developed in [3, 9] has been used.

Let us shortly recall this scheme and apply it to problem (1.1)–(1.4). The starting problem is divided into two problems: the boundary value problem for the Navier-Stokes system of equations in a fixed domain and the problem of finding the free boundary Γ from the equation

$$K(x_1) = \sigma^{-1}[-p(\mathbf{x}) + \mathbf{n} \cdot \mathbf{S}(\mathbf{v})\mathbf{n}]|_\Gamma \tag{1.5}$$

with the corresponding boundary conditions. The solution of the free boundary problem can be found by the method of successive approximations. At every step of successive approximations the Navier-Stokes system is solved in a fixed domain. The obtained solution is substituted into the right-hand side of (1.5) and by solving this equation one obtains the next iteration for the free boundary Γ. Thus, one gets a new domain in which the Navier-Stokes system has to be solved again. So, this scheme can be illustrated by the diagram

$$\Gamma^{(0)} \to G^{(0)} \to (\mathbf{v}^{(1)}, p^{(1)}) \to \Gamma^{(1)} \to G^{(1)} \to (\mathbf{v}^{(2)}, p^{(2)}) \to \dots$$

Note that in this method at every step of successive approximations the construction of $(\mathbf{v}, p)$ is separated from the construction of the free boundary Γ. On the other hand, for free boundary problems in which the unknown flow domain is unbounded in two directions the described scheme is not applicable (cf. [4, 6, 7] and others).

2. Function Spaces

Let B be an arbitrary domain in $\mathbb{R}^2$ and $N \subset \overline{B}$ a manifold of dimension less than 2. The symbol $\varrho_N(\mathbf{x})$ denotes (in this section only) the distance $\operatorname{dist}(\mathbf{x}, N) := \inf_{\mathbf{y}\in N} |\mathbf{x}-\mathbf{y}|$. Let $\beta = (\beta_1, \beta_2)$ be a multiindex with

$$|\beta| = \beta_1 + \beta_2 \quad \text{and} \quad D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} \ (\beta_i \in \mathbb{N} \cup \{0\}).$$

The symbol $\lfloor r \rfloor$ will denote the integer part of r. $C^r(B)$ ($r > 0$, non-integer) denotes the Hölder space of functions defined in a domain $B \subset \mathbb{R}^2$ with a finite norm

$$|u|_B^{(r)} = \sum_{|\beta|<r} \sup_{\mathbf{x}\in B} |D^\beta u| + \sum_{|\beta|=\lfloor r \rfloor} \sup_{\mathbf{x},\mathbf{y}\in B} \frac{|D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{r-\lfloor r \rfloor}}.$$

Let $\dot{C}_s^r(B, N)$ be the weighted Hölder space of functions defined in $B \backslash N$ and having a finite norm

$$|u|_{\dot{C}_s^r(B,N)} = \sum_{|\beta|<r} \sup_{\mathbf{x}\in B\backslash N} \varrho_N^{|\beta|-s}(\mathbf{x}) |D^\beta u(\mathbf{x})| + \sum_{|\beta|=\lfloor r \rfloor} \sup_{\mathbf{x}\in B\backslash N} \varrho_N^{r-s}(\mathbf{x}) \sup_{|\mathbf{x}-\mathbf{y}|<\frac{1}{2}\varrho_N(\mathbf{x})} \frac{|D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{r-\lfloor r \rfloor}}.$$

$C_s^r(B, N)$ ($r > s > 0; r, s$ non-integer) denotes the space of functions with a finite norm

$$|u|_{C_s^r(B,N)} := |u|_B^{(s)} + \sum_{s<|\beta|<r} \sup_{\mathbf{x}\in B\backslash N} \varrho_N^{|\beta|-s}(\mathbf{x}) |D^\beta u(\mathbf{x})| + \sum_{|\beta|=\lfloor r \rfloor} \sup_{\mathbf{x}\in B\backslash N} \varrho_N^{r-s}(\mathbf{x}) \sup_{|\mathbf{x}-\mathbf{y}|<\frac{1}{2}\varrho_N(\mathbf{x})} \frac{|D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{r-\lfloor r \rfloor}}.$$

Clearly, $\dot{C}_s^r(B, N)$ is a subspace of $C_s^r(B, N)$ consisting of functions vanishing on N together with their derivatives of order up to $\lfloor s \rfloor$. For $s < 0$ assume $C_s^r(B, N) := \dot{C}_s^r(B, N)$.

Finally we define the weighted Hölder spaces to which the generalized solutions to the problem (1.1)–(1.4) belong. We use the following notations:

$$G^0 := \{\mathbf{x} \in G : |x_1| < 2\}, \quad G^+ := \{\mathbf{x} \in G : x_1 > 1\},$$
$$G^- := \{\mathbf{x} \in G : x_1 < -1\}, \quad J^0 := (0, 2), \quad J^+ := (1, +\infty).$$

For an arbitrary real number $z > 0$ we define the space

$$C^r_{s,z}(G) = \big\{u(\mathbf{x}), u|_{G^0} \in C^r_s(G^0, Q^*), \exp(zx_1)u(\mathbf{x})|_{G^+} \in C^r(G^+),$$
$$\exp(-zx_1)u(\mathbf{x})|_{G^-} \in C^r(G^-)\big\}$$

with the norm:

$$\| \, u \, \|^{r,z}_{G,s} := |u|_{C^r_s(G^0,Q)} + |\exp(zx_1)u|^{(r)}_{G^+} + |\exp(-zx_1)u|^{(r)}_{G^-} \, .$$

For functions $f(x_1)$ defined in $\mathbb{R}^1_+$ we introduce the space $C^r_{s,z}(\mathbb{R}^1_+)$ with the norm

$$\| \, f \, \|^{r,z}_{\mathbb{R}^1_+,s} = |f|_{C^r_s(J^0,0)} + |f(x_1)\exp(zx_1)|^{(r)}_{J^+} \, .$$

The spaces of vector-fields $\mathbf{u}$ are denoted by bold letters. The corresponding norms are the sum of the norms of the coordinate functions.

3. Analytical Results

By straightforward calculations one can determine the exact Poiseuille fbws

$$\{\mathbf{v}^{(-)}(\mathbf{x}), \; p^{(-)}(\mathbf{x})\}, \quad \mathbf{x} \in G^-_i, \quad i = 1, 2 \, .$$

The corresponding velocities do not depend on x_1. In G^-_1 (i.e. if $0 \leqslant x_2 \leqslant h_1$) one obtains

$$\begin{cases} v^{(-)}_1(\mathbf{x}) = \left(\dfrac{3R}{h^2_1} - \dfrac{6F_1}{h^3_1}\right) x^2_2 + \left(-\dfrac{4R}{h_1} + \dfrac{6F_1}{h^2_1}\right) x_2 + R, \\ v^{(-)}_2(\mathbf{x}) \equiv 0, \\ p^{(-)}(x_1, x_2) = 2\nu_1\varrho_1 \left(\dfrac{3R}{h^2_1} - \dfrac{6F_1}{h^3_1}\right) x_1 - \varrho_1 g x_2 + k_1 \, . \end{cases} \tag{3.1}$$

In G^-_2 (i.e., if $h_1 \leqslant x_2 \leqslant 1$) one gets, respectively,

$$\begin{cases} v^{(-)}_1(x_1, x_2) = -\dfrac{6F_2}{(1-h_1)^3} x^2_2 + \dfrac{6(1+h_1)F_2}{(1-h_1)^3} x_2 - \dfrac{6h_1F_2}{(1-h_1)^3}, \\ v^{(-)}_2(\mathbf{x}) \equiv 0, \\ p^{(-)}(x_1, x_2) = -\dfrac{12\nu_2\varrho_2F_2}{(1-h_1)^3} x_1 - \varrho_2 g x_2 + k_2. \end{cases} \tag{3.2}$$

It is well-known that the pressure p can be determined only up to an additive constant in channel fbws (cf. k_1, k_2).

In [6, 7] the Poiseuille fbw $\{\mathbf{v}^{(+)}, p^{(+)}\}$ for the united channel G^+ was determined by straightforward calculations, too. The corresponding fbw fields are given by the following formulae [cf. also equations (32) – (34) in [7] (p. 206, 207) or equations $(\mathrm{A}.11')$, $(\mathrm{A}.12')$ in [6] (p. 41)]

$$\begin{cases} v_1^{(+)}(x_2) = \begin{cases} 0.5a_1x_2^2 + b_1x_2 + R, & 0 \leqslant x_2 \leqslant h_\infty \\ 0.5a_2(x_2^2-1) + b_2(x_2-1), & h_\infty \leqslant x_2 \leqslant 1 \end{cases}, \\ v_2^{(+)}(x_2) \equiv 0, \\ p^{(+)}(\mathbf{x}) = \begin{cases} p_0x_1 - \varrho_1 g + k, & 0 \leqslant x_2 \leqslant h_\infty \\ p_0x_1 - \varrho_2 g(x_2-1) - \varrho_1 h^2 g + k, & h_\infty \leqslant x_2 \leqslant 1 \end{cases}, \end{cases} \tag{3.3}$$

where the coefficients have the representations

$$a_1 = -3\tfrac{F_1 - Rh_\infty}{h_\infty^2} - 3\tfrac{F_2}{r(1-h_\infty^2)},\ b_1 = (2+h_\infty)\tfrac{F_1 - Rh_\infty}{h_\infty^2} + h_\infty\tfrac{F_2}{r(1-h_\infty^2)},$$

$$a_2 = -3r\tfrac{F_1 - Rh_\infty}{h_\infty^2} - 3\tfrac{F_2}{1-h_\infty^2},\quad b_2 = r(2+h_\infty)\tfrac{F_1 - Rh_\infty}{h_\infty^2} + h_\infty\tfrac{F_2}{1-h_\infty^2},$$

and $r := \dfrac{\varrho_1\nu_1}{\varrho_2\nu_2}$ in this section. For the pressure gradient, i.e. $\dfrac{\partial p}{\partial x_1} = p_0$, it holds $p_0 = a_1\nu_1\varrho_1 = a_2\nu_2\varrho_2$.

Note that in [6] the viscous two-fluid flow through a perturbed uniform channel (without a partial inner wall) was studied by different functional-analytic methods (cf. also [8]). An essential part of the determination of $\{\mathbf{v}^{(+)}, p^{(+)}\}$ consisted in the calculation of the value h_∞ from the following 5th degree polynomial equation (cf. also equation $(A.14)$ in [6], p. 43).

$$\begin{aligned} &r(r-1)Rh_\infty^5 + [-4r(r-1)R - r(r-1)F_1 - (r-1)F_2]\,h_\infty^4 \\ &\quad + [r(6r-5)R + 2r(2r-3)F_1 - 2rF_2]\,h_\infty^3 + [2r(-2r+1)R \\ &\quad + 3r(-2r+3)F_1 + 3rF_2]h_\infty^2 + \left[r^2R + 4r(r-1)F_1\right]h_\infty - r^2F_1 = 0. \end{aligned} \tag{3.4}$$

Note that the final thickness h_∞ is a function of F_1, F_2, R and the rheological parameters of the fluids. It can have up to three different values within $(0,1)$ for the same parameter set (cf. [6, 7]). Let $\widehat{h}_\infty$ be one of these values. Furthermore, by $\widehat{\psi}(x_1)$ we denote the associated infinitely differentiable solution of the following boundary value problem

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x_1}\dfrac{\psi'(x_1)}{\sqrt{1+\psi'(x_1)^2}} + \dfrac{g(\varrho_1-\varrho_2)}{\sigma}\,\psi(x_1) = \dfrac{g(\varrho_1-\varrho_2)}{\sigma}\,\widehat{h}_\infty, \\ \psi(0) = h_1, \quad \lim\limits_{x_1\longrightarrow+\infty}\psi(x_1) = \widehat{h}_\infty, \end{cases} \tag{3.5}$$

which can be obtained from the second line of (1.3) by setting $\mathbf{v} = \mathbf{0}$ and $p = \mathrm{const}$ as the initial solution for $F_1 = F_2 = R = 0$. Let $\xi = \xi(x_1)$ be a smooth cut-off function vanishing for $|x_1| \leqslant 1$ and being equal to 1 for $|x_1| \geqslant 2$. Finally, suppose that $\varrho_1 > \varrho_2$ is fulfilled. This assumption is physically sensefull. Now we can formulate the main result of this section.

Theorem 1. *There exist positive real numbers s_0, M_0 and $z_0 \leqslant \sqrt{\frac{g\,(\varrho_1-\varrho_2)}{\sigma}}$ such that for arbitrary $s \in (0,s_0), z \in (0,z_0), \max\left[F_1,F_2,R\right] < M_0$ and for parameters $\widehat{h}_\infty, F_1, F_2, R$ satisfying the condition*

$$\left|h_1 - \widehat{h}_\infty\right| < \sqrt{\frac{2\sigma}{g\left(\varrho_1 - \varrho_2\right)}}, \tag{3.6}$$

where $\widehat{h}_\infty$ *is one of the roots to equation* (3.4)*, the free boundary value problem* $(1.1) - (1.4)$ *has a unique solution* $\{\mathbf{v}, p, \psi\}$ *which can be represented in the form*

$$\begin{cases} \mathbf{v} = \xi(-x_1)\mathbf{v}^{(-)} + \xi(x_1)\mathbf{v}^{(+)} + \mathbf{w}, \quad \psi(x_1) = \widehat{\psi}(x_1) + \omega(x_1), \\ p = \xi(-x_1)p^{(-)} + \xi(x_1)p^{(+)} + p_0(\mathbf{x}), \end{cases} \tag{3.7}$$

where $\{\mathbf{v}^{(-)}, p^{(-)}\}$ *denotes the Poiseuille flow from equations* $(3.1), (3.2)$ *in both channels as* $x_1 \longrightarrow -\infty$ *and* $\{\mathbf{v}^{(+)}, p^{(+)}\}$ *is the basic solution* (3.3) *for* $x_1 \longrightarrow +\infty$*. Moreover,* $\mathbf{w} \in \mathbf{C}^{s+2}_{s,z}(G), p_0 \in C^{s+1}_{s-1,z}(G^0 \cup G^+), \nabla p_0 \in C^s_{s-2,z}(G)$ *and* $\omega \in C^{3+s}_{1+s,z}(\mathbb{R}^1_+)$ *hold.*

The proof of this theorem can be realized in the same way as in [10] applying the above mentioned scheme. We omit here the proof. The condition (3.6) is a consequence of solving the boundary value problem (3.5) and the physical restriction $\varrho_1 > \varrho_2$ is also essential for the applied method. The weight parameter s_0 in Theorem 3.1 can be estimated studying a model problem for the Stokes system in a neighbourhood of Q in the same way as in [10]. The exponential behaviour of $\mathbf{w}, p_0, \omega$ at infinity is well-known (cf. [4, 10]).

4. Computational Results

For computational purposes it was necessary to truncate the theoretical unbounded fbw domain from Fig.1.

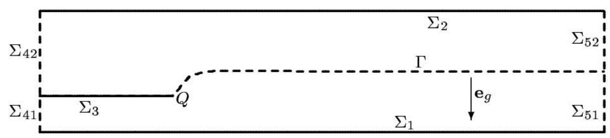

Figure 2. Computational (truncated) flow domain.

Therefore, one gets an artificial inlet $\Sigma_4 = \Sigma_{41} \cup \Sigma_{42}$ (i.e. an infbw region in both channels) and an artificial outlet $\Sigma_5 = \Sigma_{51} \cup \Sigma_{52}$ far enough from the separation point Q. We obtain the following two free boundary value problems

$$\begin{cases} (\mathbf{v} \cdot \nabla)\, \mathbf{v} - \nu \triangle \mathbf{v} + \frac{1}{\varrho} \nabla p = g\, \mathbf{e}_g, \\ \nabla \cdot \mathbf{v} = 0\,, \end{cases} \tag{4.1}$$

$$\mathbf{v}|_{\Sigma_1} = (1, 0)^T, \;\; \mathbf{v}|_{\Sigma_2} = \mathbf{0}, \;\; \mathbf{v}|_{\Sigma_3^\pm} = \mathbf{0}, \;\; \mathbf{v}|_{\Sigma_{4k}} = \mathbf{v}^{(4,k)}, \; (k = 1, 2), \tag{4.2}$$

$$\begin{cases} [\mathbf{v}]|_\Gamma = \mathbf{0}, \quad \mathbf{v}\cdot\mathbf{n}|_{\Gamma^-} = 0, \quad [\mathbf{t}\cdot\mathbf{S}(\mathbf{v})\,\mathbf{n}]|_\Gamma = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}x_1}\,\dfrac{\psi'(x_1)}{\sqrt{1+\psi'(x_1)^2}} = \dfrac{1}{\sigma}\,[-p + \mathbf{n}\cdot\mathbf{S}(\mathbf{v})\,\mathbf{n}]|_\Gamma, \end{cases} \tag{4.3}$$

with either

$$v_1|_{\Sigma_{5k}} = v_1^{(5)}, \qquad v_2|_{\Sigma_{5k}} = 0,\ (k=1,2), \tag{4.4$_\mathrm{D}$}$$

or

$$\mathbf{t}\cdot\mathbf{T}(\mathbf{v})\,\mathbf{n}|_{\Sigma_{5k}} = 0 = \left.\frac{\partial v_1}{\partial x_1}\right|_{\Sigma_{5k}}, \qquad v_2|_{\Sigma_{5k}} = 0,\ (k=1,2). \tag{4.4$_\mathrm{N}$}$$

At Σ_4 we pose Dirichlet boundary conditions (4.2_4) (i.e. the fourth equation in (4.2)) where $\mathbf{v}^{(4)}$ is in fact the Poiseuille flow $\mathbf{v}^{(-)}$ from (3.1), (3.2). At the outlet Σ_5 we set either Dirichlet boundary conditions (4.4_D) with $v_1^{(5)} = v_1^{(+)}$ taken from (3.3) or Neumann boundary conditions (4.4_N) for the downstream velocity v_1.

We were especially interested in the case, when h_∞ has three different values in $(0,1)$ for given fluxes F_1, F_2. This happens if $F_1 = 0.41$ and $F_2 = 0.01$ hold (cf. [6, 7]). The associated remaining parameters for this first example are $\nu_1 = 10.0$, $\nu_2 = 2.0$, $\varrho_1 = 1.0$, $\varrho_2 = 0.5263, \sigma = 0.001$. The partial inner wall Σ_3 is located at $h_1 = 0.5$. The inflow region is situated at $x_1 = 0.0$, the separation point Q at $x_1 = 4.0$ and the outflow region was chosen at $x_1 = 17.0$.

The numerical simulations have been performed with the help of a FORTRAN code that uses both the FEM and the method of support lines (or spines) for the discretization of the flow domain (cf. [11]). The discretization has been performed using 643 nodes, 288 triangular elements and 19 spines. Thus, the total number of unknowns was 2345. More details on the discretization of similar problems can be found in [6]. All computations presented below were realized on a PENTIUM III personal computer with 450 MHz. The time per iteration was about 20 seconds.

In the first computation we posed Dirichlet boundary conditions for v_1 at the outlet and the x_2 - value h_∞ of Γ at $x_1 = 17.0$ has not been fixed. Its starting value has been $h_\infty^{(0)} = 0.6321$, i.e. one of the three exact solutions to the problem without inner wall in [6]. The position of Γ after 5 iterations is presented in Fig.3.

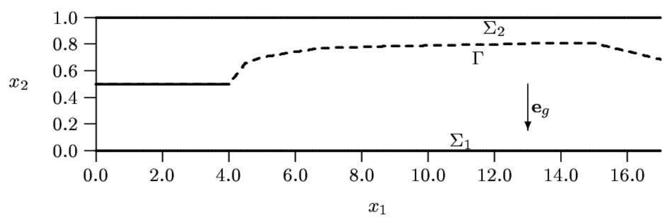

Figure 3. Computed free interface for $h_\infty^{(0)} = 0.6321$.

When taking $h_\infty^{(0)} = 0.8031$ the following figure arises (see Fig.4) In the third

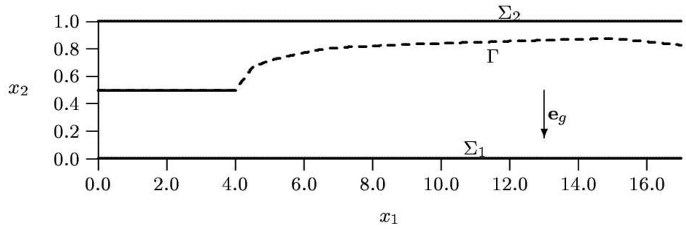

Figure 4. Computed free interface for $h_\infty^{(0)} = 0.8031$.

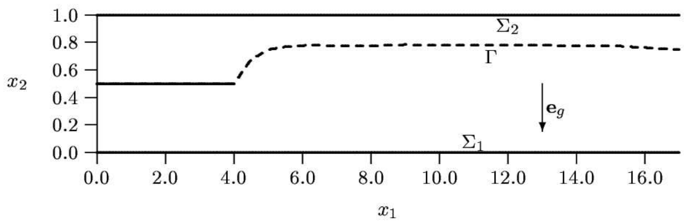

Figure 5. Computed free interface of a two-fluid channel flow with Neumann boundary conditions.

computation (cf. Fig.5) of the first example we have used Neumann boundary conditions for v_1 at the outlet and the position h_∞ of Γ at the outlet has also not been fixed. Its starting value for the iteration scheme has been $h_\infty^{(0)} = 0.5$. Figure 5 shows the computed position of the free interface Γ after 30 iterations.

Even if choosing the position h_1 of the inner wall very close to one of the three exact values of h_∞, namely $h_1 = 0.8$, the computational results did not become better. Figure 6 represents the corresponding situation.

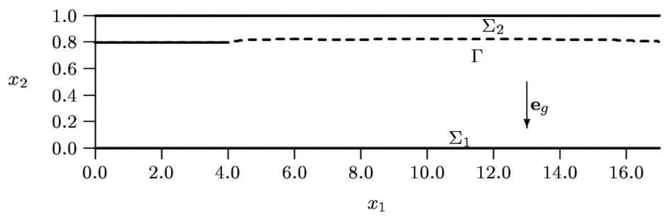

Figure 6. Computed free interface of a two-fluid channel flow with a different inner wall.

In all these computations of the first example we could not reach convergence of the iteration scheme. Moreover, one can recognize that in Figs.3 – 6 the free interface Γ

turns off in front of the outlet. It cannot find a *final* thickness h_∞. This was typical for all similar computations. Therefore, it seems to us that there is no solution of problems $(4.1) - (4.4_\mathrm{D})$ and $(4.1) - (4.4_\mathrm{N})$ for the above mentioned parameter set.

Let us introduce a second parameter set:

$$F_1 = 0.534, \quad F_2 = 0.266, \quad \nu_1 = 166.667, \quad \nu_2 = 250.0\,,$$
$$\varrho_1 = 1.0, \quad \varrho_2 = 0.9, \quad \sigma = 0.0001\,.$$

The location of the partial inner wall Σ_3 is $h_1 = 0.3$ (cf. Fig.7).

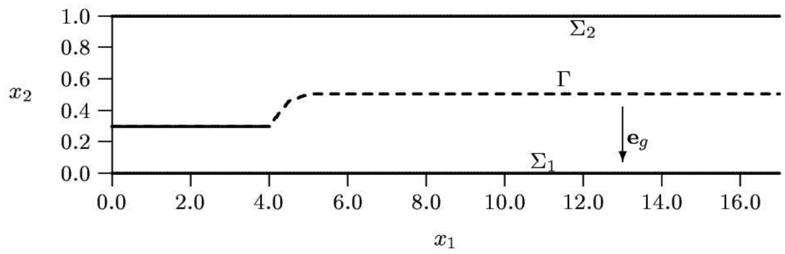

Figure 7. Computed free interface for a second parameter set.

It is well-known (cf. [6, 7]) that for this parameter set the uniform channel problem without inner wall possesses a unique solution which leads to the layer thickness $h_\infty = 0.5027$. We could show that our problem $(4.1) - (4.4_\mathrm{N})$ has also a unique solution with the same h_∞. The iteration scheme converges independently of the starting value $h_\infty^{(0)}$. Figure 7 shows the computed free interface after 30 iterations. One can see the uniform behaviour of Γ. The last picture (Fig.8) represents the velocity moduli at the nodes located on the free interface.

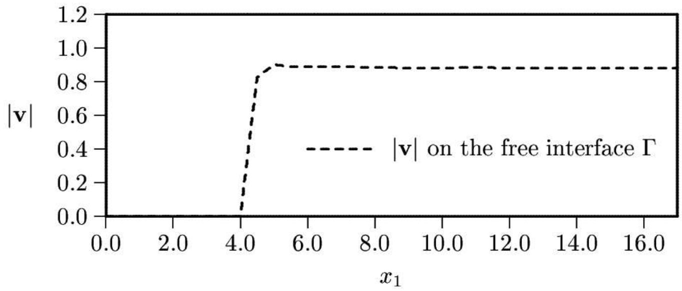

Figure 8. Velocities at the free interface for a two-fluid channel flow.

Note finally, that the pressure converges very well except at the neighbourhood of the separation point Q where the pressure admits a singularity.

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Apie dviejų tekančių kanale skysčių srauto egzistavimą ir vienatį

J. Socolowsky

Dviejų, tekančių kanale, klampių skysčių srauto uždavinys iškyla taikant įvairias skirtingų rūšių paviršių padengimo technologijas. Atitinkamas matematinis modelis išreiškiamas dvimačiu kraštiniu uždaviniu su laisvu paviršiumi Navje-Stokso lygtims.
Straipsnyje nagrinėjamas santykinai stacionaraus uždavinio išsprendžiamumas ir pateikiami skaičiavimo rezultatai. Be to parodoma, kad priklausomai nuo srovės parametrų kaip ir nuo klampumo ir tankio santykio stacionarūs sprendiniai gali neegzistuoti. Su kitais parametrais egzistuoja tiksliai vienas sprendinys.