Mathematical Modelling and Analysis
Volume 10 Number 2, 2005, pages 173–190
 ISSN 1392-6292

INCREASING OF ACCURACY FOR ENGINEERING CALCULATION OF HEAT TRANSFER PROBLEMS IN TWO LAYER MEDIA

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Received September 21, 2004; revised March 12, 2005

Abstract. In this paper we study the simple algorithms for modelling the heat transfer problem in two layer media. The initial model which is based on a partial differential equation is reduced to ordinary differential equations (ODEs). The increase of accuracy is shown if instead of first order ODE initial value problem ([4, 5]) the second order differential equations is taken. Such a procedure allows us to obtain a simple engineering algorithm for solving heat transfer equations in two layered domain of Cartesian, cylindrical (with axial symmetry) and spherical coordinates (with radial symmetry). In a stationary case the exact finite difference scheme is obtained.

Key words: heat transfer, layered media, numerical methods, mathematical modelling

1. The Mathematical Model

We shall consider the partial differential equations [4]:

$$\gamma_k \frac{\partial u_k}{\partial t} = \frac{1}{p(x)} \frac{\partial}{\partial x}\big(\lambda_k p(x) \frac{\partial u_k}{\partial x}\big) - q_k(x,t) \tag{1.1}$$

in multilayered domain Ω with N layers

$$\Omega = \{x : \ x \in [x_{k-1}, x_k],\ k = \overline{1,N}\},$$

where $x_0 = 0$, $x_N = L$, $h_k = x_k - x_{k-1}$, $u_k = u_k(x,t)$ is the absolute temperature $[K]$ in the layer $[x_{k-1}, x_k]$, $\gamma_k = c_k \rho_k$, $c_k[\frac{J}{kg.K}]$, $\rho_k[\frac{kg}{m^3}]$, $\lambda_k[\frac{W}{m.K}]$ are corresponding constants of specifi c thermal capacity, density and coeffi cients of heat conductivity in every layer, $t[s]$ is the time, $q_k = q_k(x,t)$ is the function of thermal sources, $x[m]$ is the space coordinate, $p = p(x)$ is given function depending on the system of coordinates: $p = 1$ in the Cartesian coordinates, $p = x$ in cylindrical coordinates with an axial symmetry, $p = x^2$ in spherical coordinates with a radial symmetry.

Adding continuity conditions on surfaces $x = x_k, k = \overline{1, N-1}$

$$\begin{cases} u_k(x_k,t) = u_{k+1}(x_k,t) \\ \lambda_k \dfrac{\partial u_k(x_k,t)}{\partial x} = \lambda_{k+1} \dfrac{\partial u_{k+1}(x_k,t)}{\partial x}, \end{cases} \tag{1.2}$$

boundary conditions on the surfaces $x = x_0 = 0,\ x = x_n = L$

$$\begin{cases} \lambda_1 p(0) \dfrac{\partial u_1(0,t)}{\partial x} = \alpha_0 (u_1(0,t) - T_0) \\ \lambda_N p(L) \dfrac{\partial u_N(L,t)}{\partial x} = f(u_N(L,t)) \end{cases} \tag{1.3}$$

and the initial condition at $t = 0$

$$u_k(x,t) = \phi(x), k = \overline{1, N} \tag{1.4}$$

we obtain the initial–boundary value problem (1.1-1.4) for the heat transfer equation. The nonlinear function $f(u_N)$ in the boundary condition (1.3) describes the radiation from heaters and convection, for example

$$f(u_N(L,t)) = \alpha_L (T_L - u_N(L,t)) + \epsilon\sigma(T_*^4 - u_N^4(L,t)),$$

where $\alpha_0, \alpha_L[\frac{W}{m^3.K}]$ are the coefficients of heat transfer, ϵ is the coefficient of emissivity ($\epsilon \in [0,1]$), $\sigma = 5.6703.10^{-8}[\frac{W}{m^2.K^4}]$ is the Stefan-Boltzmann constant, T_0, T_L, T_* are the constants of the temperatures outside the media and outside the heaters, $\phi = \phi(x)$ is the given initial temperature.

If $\alpha_0 = \alpha_L = \infty$, then we have the first kind boundary conditions in the form

$$u_1(0,t) = T_0, \quad u_N(L,t) = T_L.$$

If $p(0) = 0$ then the first boundary condition (1.3) is omitted and we can consider the symmetry condition

$$\frac{\partial u_1(0,t)}{\partial x} = 0. \tag{1.5}$$

In the case of homogeneous media we consider the following partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{p(x)} \frac{\partial}{\partial x} \big(p(x) \frac{\partial u}{\partial x}\big) - q(x,t), \tag{1.6}$$

where the constants of heat transfer parameters c, ρ, λ are normalizing magnitudes and $c\rho/\lambda$ is used as appropriate factor to the time t and function q.

In every layer the heat equation (1.1) can be presented in the following form

$$\frac{\partial}{\partial x} \big(\lambda_k \frac{\partial u_k(x,t)}{\partial x}\big) = F_k, \quad k = \overline{1, N}, \tag{1.7}$$

where $F_k = \gamma_k \dot{u}_k + q_k,\ \dot{u}_k = \dfrac{\partial u_k}{\partial t}$.

2. The Exact 3-Points Finite-Difference Scheme

We use the method of finite volumes [3] for approximation of the differential problem. We consider $N+1$ grid points in the $x-$ direction

$$0 = x_0 < x_1 < \ldots < x_N = L.$$

Then the exact finite-difference scheme for a given function F_k is defined in the form [2]

$$\lambda_1 a_1(u_1 - u_0) - \alpha_0(u_0 - T_0) = \bar{R}_0^+, \tag{2.1}$$

$$\lambda_{k+1}a_{k+1}(u_{k+1} - u_k) - \lambda_k a_k(u_k - u_{k-1}) = \bar{R}_k,\ k = \overline{1, N-1}, \tag{2.2}$$

$$f(u_N) - \lambda_N a_N(u_N - u_{N-1}) = \bar{R}_N^-, \tag{2.3}$$

where

$$\bar{R}_k = \bar{R}_k^+ + \bar{R}_k^-, \quad \bar{R}_k^+ = I_k^+ + \gamma_{k+1}R_k^+, \quad \bar{R}_k^- = I_k^- + \gamma_k R_k^-,$$

$$I_k = I_k^+ + I_k^-, \quad u_k = u_k(t) = u_k(x_k, t),\ k = \overline{1, N}, \quad u_0 = u_1(0, t),$$

$$R_k^- = \int_{x_{k-1}}^{x_k} \Big(1 - a_k \int_x^{x_k} \frac{d\psi}{p(\psi)}\Big) p(x)\dot{u}_k(x,t)dx$$
$$= a_k \int_{x_{k-1}}^{x_k} \Big(p(x)\dot{u}_k(x,t) \int_{x_{k-1}}^{x} \frac{d\psi}{p(\psi)}\Big)\, dx,$$

$$R_k^+ = \int_{x_k}^{x_{k+1}} \Big(1 - a_{k+1} \int_{x_k}^{x} \frac{d\psi}{p(\psi)}\Big) p(x)\dot{u}_{k+1}(x,t)dx$$
$$= a_{k+1} \int_{x_k}^{x_{k+1}} \Big(p(x)\dot{u}_{k+1}(x,t) \int_{x}^{x_{k+1}} \frac{d\psi}{p(\psi)}\Big)\, dx,$$

$$a_k = \frac{1}{\int_{x_{k-1}}^{x_k} \frac{dx}{p(x)}}, \quad I_k^+ = a_{k+1} \int_{x_k}^{x_{k+1}} \Big(p(x)q_k(x,t) \int_{x}^{x_{k+1}} \frac{d\psi}{p(\psi)}\Big)\, dx,$$

$$I_k^- = \int_{x_{k-1}}^{x_k} \Big(1 - a_k \int_x^{x_k} \frac{d\psi}{p(\psi)}\Big) p(x)q_k(x,t)\ dx$$
$$= a_k \int_{x_{k-1}}^{x_k} \Big(p(x)q_k(x,t) \int_{x_{k-1}}^{x} \frac{d\psi}{p(\psi)}\Big)\, dx.$$

If $p(0) = 0$ (cylindrical and spherical coordinates), then $a_1 = \alpha_0 = 0$ and the difference equation (2.1) is defined as

$$\lambda_1(u_1 - u_0) = R_0^*, \tag{2.4}$$

and the equation (2.2) for $k = 1$ is given as

$$\lambda_2 a_2(u_2 - u_1) = R_1^*, \tag{2.5}$$

where

$$R_0^* = \gamma_1 \int_0^{x_1} p(x)\dot{u}_1(x,t) \int_x^{x_1} \frac{d\psi}{p(\psi)} dx + \int_0^{x_1} p(x)q_1(x,t) \int_x^{x_1} \frac{d\psi}{p(\psi)}\, dx,$$
$$R_1^* = \bar{R}_1^+ + \gamma_1 \int_0^{x_1} p(x)\dot{u}_1(x,t)dx + \int_0^{x_1} p(x)q_1(x,t)\, dx.$$

Summarizing expressions (2.2), (2.3) we obtain equation

$$f(u_N) = Q, \quad Q = \sum_{k=2}^{N-1} \bar{R}_k + \bar{R}_N^- + R_0^* + R_1^*.$$

This equation has a positive root $u_N > 0$, if $\epsilon\sigma T_*^4 + \alpha_L T_L > Q$.

Next we consider only two layers, then $N = 2, x_1 = h_1, x_2 = L = h_1 + h_2$. Then the unknown values are $u_1, u_2 (p(0) = 1, \alpha_0 = \infty)$ or $u_0, u_1, u_2 (p(0) = 0)$.

3. The Cartesian Coordinates

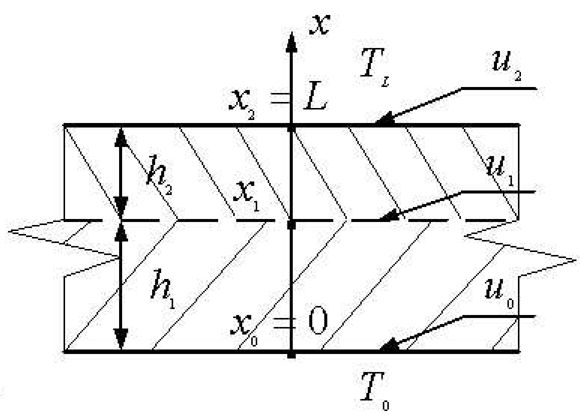

Figure 1. The calculated temperature u_1 in the layer $x = x_1$ in the case of Cartesian coordinates.

In this case (see fig. 1)

$$p(x) = 1, \quad a_k = \frac{1}{h_k}, \quad R_k^- = \frac{1}{h_k} \int_{x_{k-1}}^{x_k} (x - x_{k-1})\dot{u}_k(x,t)\, dx,$$
$$R_k^+ = \frac{1}{h_{k+1}} \int_{x_k}^{x_{k+1}} (x_{k+1} - x)\dot{u}_{k+1}(x,t)\, dx, \quad I_k^- = \frac{1}{h_k} \int_{x_{k-1}}^{x_k} (x - x_{k-1}) q_k\, dx,$$
$$I_k^+ = \frac{1}{h_{k+1}} \int_{x_k}^{x_{k+1}} (x_{k+1} - x) q_{k+1}\, dx,$$

and the finite-difference scheme $(\alpha_0 = \infty)$ is given in the form

$$\begin{cases} \frac{\lambda_2}{h_2}(u_2 - u_1) - \frac{\lambda_1}{h_1}(u_1 - T_0) = \gamma_2 R_1^+ + \gamma_1 R_1^- + I_1 \\ f(u_2) - \frac{\lambda_2}{h_2}(u_2 - u_1) = \gamma_2 R_2^- + I_2^- . \end{cases} \tag{3.1}$$

In the stationary case, if

$$q_k = const, \quad \dot{u}_k = 0, \quad I_1 = 0.5(q_1 h_1 + q_2 h_2), \quad I_2 = 0.5 q_2 h_2,$$

then

$$f(u_2) + D_{12}\big((T_0 - u_2)D_1 - Q_{12}\big) = Q_2, \quad u_1 = \frac{D_2 u_2 + D_1 T_0 - Q_{12}}{D_1 + D_2}, \tag{3.2}$$

where

$$D_i = \frac{\lambda_i}{h_i}, \quad Q_i = 0.5 q_i h_i, \ i = 1, 2, \quad D_{12} = \frac{D_2}{D_1 + D_2}, \quad Q_{12} = Q_1 + Q_2.$$

Therefore the value u_2 can be obtained from the equation $u_2^3 + a = \frac{b}{u_2}$, where

$$a = \frac{\alpha_L + D_{12} D_1}{\epsilon\sigma} > 0, \quad b = \frac{\epsilon\sigma T_*^4 + \alpha_L T_L + D_{12} D_1 T_0 - Q_2 - D_{12} Q_{12}}{\epsilon\sigma}.$$

This equation have a positive root for $b > 0$, e.g., $q_1 \leq 0, q_2 \leq 0$. The exact solution of the stationary problem (1.1)–(1.3) satisfies expressions (3.2).

In the non-stationary case $(\dot{u}_k \neq 0)$, initial-value problems for ODE are used and we compute integrals $R_i^{\pm}$ approximately by the following quadrature formulas

$$J_m = A_1^{(m)} V_2(0) + A_2^{(m)} V_2(1) + A_3^{(m)} V_2'(1) + B_2^{(m)} V_2''(1) + r_m, \ m = 1, 2, \tag{3.3}$$

$$J_3 = A_1^{(3)} V_1(0) + A_2^{(3)} V_1(1) + A_3^{(3)} V_1'(0) + B_1^{(3)} V_1''(0) + r_3, \tag{3.4}$$

where

$$J_1 = \frac{R_1^+}{h_2} = \int_0^1 (1 - \bar{x}) V_2(\bar{x})\, d\bar{x}, \quad J_2 = \frac{R_2^-}{h_2} = \int_0^1 \bar{x} V_2(\bar{x})\, d\bar{x},$$

$$J_3 = \frac{R_1^-}{h_1} = \int_0^1 \bar{x} V_1(\bar{x})\, d\bar{x}, \quad V_1(\bar{x}) = \dot{u}_1(\bar{x} h_1, t), \quad V_2(\bar{x}) = \dot{u}_2(h_1 + \bar{x} h_2, t),$$

$r_m = \frac{h_2^4}{4!} \frac{\partial^4 \dot{u}(\xi_m, t)}{\partial x^4} C_m, \quad \xi_m \in (0, L), \ (m = 1, 2, 3)$ are the errors terms, $A_j^{(m)}, B_j^{(m)}, C_m, \ (j, m = 1, 2, 3)$ are the indefinite coefficients.

Using the power functions $V_j(\bar{x}) = \bar{x}^i, i = \overline{0, 4}, j = 1, 2$ we get the systems of linear algebraic equations for $A_j^{(m)}, B_j^{(m)}$:

$$\begin{cases} g(i, m) = A_1^{(m)} 0^i + A_2^{(m)} + i A_3^{(m)} + i(i-1) B_2^{(m)}, \\ \frac{1}{i+2} = A_1^{(3)} 0^i + A_2^{(3)} + i A_3^{(3)} 0^{i-1} + i(i-1) B_1^{(3)} 0^{i-2}, \end{cases} \tag{3.5}$$

where

$$m=1,2,\;\; 0^0=1,\;\; g(i,1)=\frac{1}{(i+1)(i+2)},\;\; g(i,2)=\frac{1}{i+2}.$$

The solutions of these systems are given as

$$\begin{aligned}
&A_1^{(1)}=\frac{1}{5},\quad A_2^{(1)}=\frac{3}{10},\quad A_3^{(1)}=-\frac{2}{15},\quad B_2^{(1)}=\frac{1}{40},\\
&A_1^{(2)}=\frac{1}{20},\quad A_2^{(2)}=\frac{9}{20},\quad A_3^{(2)}=-\frac{7}{60},\quad B_2^{(2)}=\frac{1}{60},\\
&A_1^{(3)}=\frac{3}{10},\quad A_2^{(3)}=\frac{1}{5},\quad A_3^{(3)}=\frac{2}{15},\quad {}_1^{(3)}=-\frac{1}{40}.
\end{aligned}$$

Constants C_m in the residual r_m are determined from (3.3) – (3.4), for $V_1(\bar{x}) = V_2(\bar{x}) = \bar{x}^4:\ C_1 = C_3 = -\frac{1}{30},\ C_2 = -\frac{1}{60}$.

Using the difference equations (3.1) and the right hand side integrals approximating by (3.3), (3.4) with neglected error terms $r_m, m = 1,2,3$, the approximate numerical solution $u_1(t), u_2(t)$ at every time step $t > 0$ can be found by solving the following stiff system of two nonlinear ODEs of the second order $(\dot{u}_0 = \ddot{u}_0 = 0,\ \ddot{u} = \frac{\partial^2}{\partial t^2},\ \alpha_0 = \infty)$:

$$\begin{cases}\gamma_2 h_2\big[A_1^{(1)}\dot{u}_1 + A_2^{(1)}\dot{u}_2 + \dfrac{h_2}{\lambda_2}\big(A_3^{(1)} f'_u(u_2)\dot{u}_2 + h_2 B_2^{(1)}(\dot{q}_2(L,t)+\gamma_2\ddot{u}_2)\big)\big]\\ +\gamma_1 h_1\big[A_2^{(3)}\dot{u}_1 + \dfrac{h_1^2}{\lambda_1}B_1^{(3)}\dot{q}_1(0,t)\big] + I_1 = \dfrac{\lambda_2}{h_2}(u_2-u_1) - \dfrac{\lambda_1}{h_1}(u_1 - T_0),\end{cases} \tag{3.6}$$

$$\begin{cases}\gamma_2 h_2\big[A_1^{(2)}\dot{u}_1 + A_2^{(2)}\dot{u}_2 + \dfrac{h_2}{\lambda_2}\big(A_3^{(2)} f'_u(u_2)\dot{u}_2\\ \qquad +h_2 B_2^{(2)}(\dot{q}_2(L,t)+\gamma_2\ddot{u}_2)\big)\big] + I_2^- = f(u_2) - \dfrac{\lambda_2}{h_2}(u_2-u_1).\end{cases} \tag{3.7}$$

Here one should take into account that from (1.1), (1.3) it follow that

$$\begin{aligned}
&V_2'(1) = h_2\frac{\partial}{\partial x}\dot{u}_2(L,t) = h_2\frac{\partial}{\partial t}u_2'(L,t) = \frac{h_2}{\lambda_2}\dot{f}(u_2) = \frac{h_2}{\lambda_2}f'_u(u_2)\dot{u}_2,\\
&f'_u(u_2) = -(\alpha_L + 4\epsilon\sigma u_2^3),\ \ V_1'(0) = h_1\frac{\partial}{\partial x}\dot{u}_1(0,t) = h_1\frac{\partial}{\partial t}u_1'(0,t) = h_1\alpha_0\dot{u}_0,\\
&V_2''(1) = h_2^2\frac{\partial^2}{\partial x^2}\dot{u}(L,t) = h_2^2\frac{\partial}{\partial t}u''(L,t) = \frac{h_2^2}{\lambda_2}\frac{\partial}{\partial t}(\gamma_2\dot{u}(L,t) + q(L,t))\\
&\qquad\qquad = \frac{h_2^2}{\lambda_2}(\gamma_2\ddot{u}_2 + \dot{q}_2).
\end{aligned}$$

The initial conditions for ODEs (3.6), (3.7) are the following

$$\begin{cases} u_1(0) = \phi(h_1),\ \ u_2(0) = \phi(L),\\ \dot{u}_1(0) = (\lambda_1\phi''(h_1) - q_1(h_1,0))/\gamma_1,\\ \dot{u}_2(0) = (\lambda_2\phi''(L) - q_2(L,0))/\gamma_2.\end{cases} \tag{3.8}$$

If the derivatives $\ddot{u}$ are not used in (3.3), (3.4) then [3]

$$\begin{cases} R_k^- = \gamma_k h_k\Big(\dfrac{1}{6}\dot{u}_{k-1} + \dfrac{1}{3}\dot{u}_k - \dfrac{h_k^2}{24}\dfrac{\partial^2 \dot{u}(\xi_k^-,t)}{\partial x^2}\Big), \\ R_k^+ = \gamma_{k+1} h_{k+1}\Big(\dfrac{1}{3}\dot{u}_k + \dfrac{1}{6}\dot{u}_{k+1} - \dfrac{h_{k+1}^2}{24}\dfrac{\partial^2 \dot{u}(\xi_k^+,t)}{\partial x^2}\Big), \end{cases} \tag{3.9}$$

where $\xi_k^- \in (x_{k-1}, x_k)$, $\xi_k^+ \in (x_k, x_{k+1})$.

For the equation (1.6), $h_1 = h_2 = h$ the finite difference scheme (3.1) is given in the form

$$\begin{cases} u_2 - 2u_1 + T_0 = h(R_1 + I_1) \\ hf(u_2) - u_2 + u_1 = h(R_2^- + I_2^-), \end{cases} \tag{3.10}$$

where

$$\frac{R_1}{h} = J_1 = \int_0^1 \bar{x}V(\bar{x})\,d\bar{x} + \int_1^2 (2-\bar{x})V(\bar{x})d\bar{x},$$
$$\frac{R_2^-}{h} = J_2 = \int_1^2 (\bar{x}-1)V(\bar{x})\,d\bar{x}, \quad V(\bar{x}) = \dot{u}(\bar{x}h,t), \bar{x} = x/h,$$

$$\begin{cases} J_m = A_1^{(m)}V(0) + A_2^{(m)}V(1) + A_3^{(m)}V(2) + A_4^{(m)}V'(2) \\ \qquad +B_1^{(m)}V''(0) + B_2^{(m)}V''(1) + B_3^{(m)}V''(2) + r_m, \end{cases} \tag{3.11}$$

$r_m = \dfrac{h^7}{7!}\dfrac{\partial^7 \dot{u}(\xi_m,t)}{\partial x^7}C_m$, $\xi_m \in (0,L), m = 1,2$. Using power functions $V(\bar{x}) = \bar{x}^i, i = 0,\dots,6$ in the expressions (3.11) we obtain two systems of 7 linear algebraic equations for determination of $A_j^{(m)}, B_j^{(m)}$

$$g(i,m) = A_1^{(m)}0^i + A_2^{(m)} + A_3^{(m)}2^i + A_4^{(m)}i2^{i-1} \\ + i(i-1)(B_1^{(m)}0^{i-2} + B_2^{(m)} + B_3^{(m)}2^{i-2}), \quad m = 1,2, \tag{3.12}$$

where

$$g(i,1) = \frac{2^{i+2}-2}{(i+1)(i+2)}, \quad g(i,2) = \frac{i2^{i+1}+1}{(i+1)(i+2)}.$$

Constants C_m are determined from (3.11) for $V(\bar{x}) = \bar{x}^n, n = 7:$

$$C_m = g(n,m) - A_2^{(m)} - A_3^{(m)}2^n - A_4^{(m)}n2^{n-1} \\ - n(n-1)(B_2^{(m)} + B_3^{(m)}2^{n-2}). \tag{3.13}$$

The solution of system (3.12) is given by:

$$A_1^{(1)} = A_3^{(1)} = \frac{11}{252}, \quad A_2^{(1)} = \frac{115}{126}, \quad A_4^{(1)} = 0, \quad B_2^{(1)} = \frac{313}{7560},$$
$$B_1^{(1)} = B_3^{(1)} = -\frac{13}{15120}, \quad A_1^{(2)} = A_3^{(2)} = \frac{109}{504}, \quad A_2^{(2)} = \frac{269}{252},$$
$$A_4^{(2)} = 0, \quad B_1^{(2)} = B_3^{(2)} = -\frac{83}{30240}, \quad B_2^{(2)} = \frac{1223}{15120}.$$

We have from (3.13) that $C_1 = 0, C_2 = \frac{32}{9}$. Therefore we get the residuals in the form $r_1 = \dfrac{h^8}{8!}\dfrac{\partial^8 \dot{u}(\xi_1,t)}{\partial x^8}C_1$, and for $n = 1$ we have from (3.13) $C_1 = \dfrac{59}{1890}$.

If $B_3^{(1)} = B_1^{(1)} = 0$, then

$$A_1^{(1)} = A_3^{(1)} = \frac{1}{30},\ A_2^{(1)} = \frac{28}{30},\ B_2^{(1)} = \frac{1}{20},\ r_1 = -\frac{1}{420}\frac{\partial^6 \dot{u}(\xi,t)}{\partial x^6}h^6,\ \xi \in (0,L).$$

The system of ODEs (3.6)–(3.7) is presented in the form

$$h^2(A_2^{(1)}\dot{u}_1 + A_3^{(1)}\dot{u}_2) + h^4\big(B_1^{(1)}\dot{q}_0 + B_2^{(1)}(\dot{q}_1 + \ddot{u}_1) + B_3^{(1)}(\dot{q}_2 + \ddot{u}_2)\big) + I_1 = u_2 - 2u_1 + T_0, \quad (3.14)$$

$$h^2(A_2^{(2)}\dot{u}_1 + A_3^{(2)}\dot{u}_2) + h^4\big(B_1^{(2)}\dot{q}_0 + B_2^{(2)}(\dot{q}_1 + \ddot{u}_1) + B_3^{(2)}(\dot{q}_2 + \ddot{u}_2)\big) + I_2^- = hf(u_2) - u_2 + u_1, \quad (3.15)$$

where $\dot{q}_j = \dot{q}(x_j,t), j = 0,1,2$. If the derivatives are not used in (3.11) then [3]

$$A_1^{(1)} = A_3^{(1)} = \frac{1}{12},\ \ A_2^{(1)} = \frac{10}{12},\ \ r_1 = \frac{h^4}{4!}\frac{\partial^4 \dot{u}(\xi,t)}{\partial x^4}C_1,\ \ C_1 = -\frac{1}{10}.$$

Example 1. If

$$q = 0,\ \alpha_L = \infty,\ \phi(x) = \sin(\frac{\pi x}{L}),\ \dot{u}_2 = \ddot{u}_2 = 0,\ T_0 = T_L = 0,$$

then from the first order ODE (3.14) with $A_2^{(1)} = \frac{5}{6}, B_2^{(1)} = 0, u_1(0) = 1$ it follows that $u_1(t) = \exp(-9.6t/L^2)$. The exact solution of (1.6) is $u(x,t) = \exp(-\pi^2 t/L^2)\sin(\frac{\pi x}{L})$ or $u_1 = u(h,t) = \exp(-\pi^2 t/L)$. Using the approximation

$$u''(h,t) \approx \Lambda u_1 = \frac{1}{h^2}\big(u_0(t) - 2u_1(t) + u_2(t)\big),$$

we get the first order ODE (the method of lines) $\dot{u}_1 = \exp(-8t/L^2)$. The second order ODE (3.14) is given as

$$b_1\ddot{u}_1 + a_1\dot{u}_1 + u_1 = 0\ \ (b_1 = h^4 B_2^{(1)}, a_1 = h^2 A_2^{(1)})$$

with initial conditions $u_1(0) = 1,\ \dot{u}_1(0) = \phi''(h) = -(\pi/L)^2$. It's solution is given by

$$u_1(t) = D_1\exp(\mu_1 t) + D_2\exp(\mu_2 t),$$

where $\mu_{1,2} = -a_1/(2b_1) \pm \sqrt{(a_1/(2b_1))^2 - 1/b_1}$, $D_1 = (\mu_2 + (\pi/L)^2)/(\mu_2 - \mu_1)$, $D_2 = -((\pi/L)^2 + \mu_1)/(\mu_2 - \mu_1)$.

Using the approximation $u''(h,t) \approx \Lambda u_1 - \dfrac{h^2}{12}u^{(4)}(h,t)$, $u^{(4)}(h,t) = \ddot{u}_1(t)$, we have the second order ODE with $b_1 = \dfrac{h^4}{24}$, $a_1 = \dfrac{h^2}{2}$ (the method of lines with a high order approximation).

The results of calculations are presented in Table 1. Here $L = 2$, u_{1*} is the analytical solution, u_{1pp} is a $O(h^8)$ order approximation, u_{1p} is $O(h^6)$ order approximation, u_1 is a usual approximation, u_{1t} is obtained by using the method of lines, u_{1tt} is computed by the method of lines of high approximation.

Table 1. The values $u(1,t)$ at different time t moments.

t	u_{1*}	u_{1pp}	u_{1p}	u_1	u_{1tt}	u_{1t}
.1	.781344	.781340	.78127	.7866	.770	.82
.2	.610498	.610490	.61032	.6188	.607	.67
.5	.291213	.291201	.29094	.3012	.284	.37
.9	.108537	.108529	.10834	.1153	.103	.16

4. The Cylindrical Coordinates

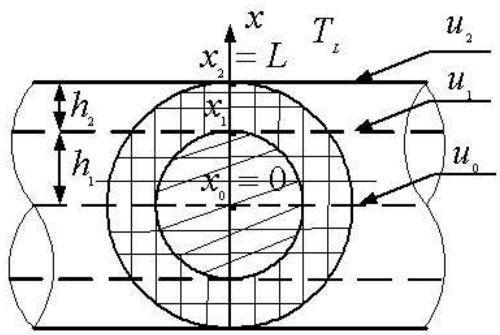

Figure 2. The calculated temperatures u_0, u_1 in layers $x_0 = 0$ and $x = x_1$ in the case of cylindrical coordinates.

In the cylindrical coordinates (see fi g.2) we get the following coeffi cients:

$$p(x) = x, \quad a_k = \frac{1}{\ln \frac{x_k}{x_{k-1}}}, \ k \geq 2, \quad a_1 = 0,$$

$$R_k^- = a_k \int_{x_{k-1}}^{x_k} x \ln \frac{x}{x_{k-1}} \dot{u}_k(x,t)\, dx, \quad R_k^+ = a_{k+1} \int_{x_k}^{x_{k+1}} x \ln \frac{x_{k+1}}{x} \dot{u}_{k+1}(x,t)\, dx,$$

$$I_k^- = a_k \int_{x_{k-1}}^{x_k} x \ln \frac{x}{x_{k-1}} q_k\, dx, \quad I_k^+ = a_{k+1} \int_{x_k}^{x_{k+1}} x \ln \frac{x_{k+1}}{x} q_{k+1}\, dx\,.$$

The fi nite difference scheme $(N = 2, x_1 = h_1, x_2 = L = h_1 + h_2,\ \beta = h_2/h_1)$ is given in the following form

$$\begin{cases} \lambda_1(u_1 - u_0) = \gamma_1 J_2 + I_0, \\ \lambda_2 a_2(u_2 - u_1) = \gamma_2 R_1^+ + I_1^+ \gamma_1 J_1 + I_0^{(1)}, \\ f(u_2) - \lambda_2 a_2(u_2 - u_1) = \gamma_2 R_2^- + I_2^-, \end{cases} \tag{4.1}$$

where

$$J_2 = \int_0^{h_1} x \ln \frac{h_1}{x} \dot{u}_1(x,t)\,dx, \quad I_0 = \int_0^{h_1} x \ln \frac{h_1}{x} q_1(x,t)\,dx,$$
$$a_2 = 1/\ln \frac{L}{h_1}, \quad J_1 = \int_0^{h_1} x \dot{u}_1(x,t)\,dx, \quad I_0^{(1)} = \int_0^{h_1} x q_1(x,t)\,dx.$$

In the stationary case, if $q_k = const, \dot{u}_k = 0$, then

$$I_0 = \frac{q_1 h_1^2}{4}, \quad I_1^+ = q_2\Big(-0.5h_1^2 + 0.25(L^2 - h_1^2)\ln^{-1}\frac{L}{h_1}\Big),$$
$$I_0^{(1)} = \frac{q_1 h_1^2}{2}, \quad I_2^- = q_2\Big(0.5L^2 - 0.25(L^2 - h_1^2)\ln^{-1}\frac{L}{h_1}\Big)$$

and the values u_0, u_1, u_2 can be obtained from the system of equations

$$\begin{cases} \lambda_1(u_1 - u_0) = 0.25 q_1 h_1^2, \\ \lambda_2(u_2 - u_1) = 0.5(q_1 - q_2)h_1^2 \ln \frac{L}{h_1} + 0.25 q_2 (L^2 - h_1^2), \\ f(u_2)\ln \frac{L}{h_1} - \lambda_2(u_2 - u_1) = q_2\big(0.5L^2 \ln \frac{L}{h_1} - 0.25(L^2 - h_1^2)\big). \end{cases} \tag{4.2}$$

The exact solution of the stationary problem (1.1)–(1.3) satisfies expressions (4.2) and $f(u_2) = 0.5(q_2 L^2 + (q_1 - q_2)h_1^2)$. This equation has a positive root if

$$\epsilon\sigma T_*^4 + \alpha_L T_L - 0.5\big(q_2(L^2 - h_1^2) + q_1 h_1^2\big) > 0. \tag{4.3}$$

In the non-stationary case integrals J_1, J_2, R_1^+, R_2^- can be approximated by the following quadrature formulas

$$\frac{J_m}{h_1^2} = A_1^{(m)} V_1(0) + A_2^{(m)} V_1(1) + A_3^{(m)} V_1'(0) + B_1^{(m)} V_1''(0) + r_m, m = 1,2, \tag{4.4}$$
$$\frac{J_m}{h_2^2} = A_1^{(m)} V(0) + A_2^{(m)} V(1) + A_3^{(m)} V'(1) + B_2^{(m)}\Big(V''(1) + \frac{\beta V'(1)}{1+\beta}\Big) + r_m, \tag{4.5}$$

where $r_m = \dfrac{W^{(4)}(\xi_m)}{4!} C_m,\ \xi_m \in (0,1),\ m = 3,4;\ W = V_1, V,$

$$V(\bar{x}) = \dot{u}_2(h_1 + h_2\bar{x}, t), \quad V_1(\bar{x}) = u_1(h_1\bar{x}, t), \quad J_1 = h_1^2 \int_0^1 \bar{x} V_1(\bar{x})\,d\bar{x},$$
$$J_2 = -h_1^2 \int_0^1 \bar{x}\ln(\bar{x}) V_1(\bar{x})\,d\bar{x}, \quad J_3 = h_2^2 \int_0^1 (\beta^{-1} + \bar{x}) V(\bar{x}) \ln(1 + \beta\bar{x})\,d\bar{x},$$
$$J_4 = -h_2^2 \int_0^1 (\beta^{-1} + \bar{x}) V(\bar{x}) \ln\frac{1+\beta\bar{x}}{1+\beta} d\bar{x}, \quad R_2^- = \frac{J_3}{\ln(1+\beta)}, \quad R_1^+ = \frac{J_4}{\ln(1+\beta)}.$$

The unknown coefficients $A_j^{(m)}, B_j^{(m)}$ can be determinated by using $W(\bar{x}) = \bar{x}^i, i = 0,1,2,3$, and solving the system of linear algebraic equations with parameter β :

$$\begin{cases} g(i,m) = A_1^{(m)} 0^i + A_2^{(m)} + iA_3^{(m)} 0^{i-1} + i(i-1)B_2^{(m)} 0^{i-2}\ (m = 1,2), \\ g(i,m) = A_1^{(m)} 0^i + A_2^{(m)} + iA_3^{(m)} + i(i - 1 + \beta/(1+\beta))B_2^{(m)}\ (m = 3,4), \end{cases}$$

where

$$g(i,1) = 1/(i+2), \quad g(i,3) = \int_0^1 (\beta^{-1} + \bar{x}) \ln(1+\beta\bar{x})\bar{x}^i \, d\bar{x},$$
$$g(i,2) = 1/(i+2)^2, \quad g(i,4) = -\int_0^1 (\beta^{-1} + \bar{x}) \ln \frac{1+\beta\bar{x}}{1+\beta}\bar{x}^i \, d\bar{x}.$$

As an example, if $\beta = 0.25$, then we get

$$A_1^{(3)} = 0.057368822, \quad A_2^{(3)} = 0.5789255389, \quad A_3^{(3)} = -0.1486490492,$$
$$B_2^{(3)} = 0.02021926484, \quad C_3 = -0.01752470,$$
$$A_1^{(4)} = 0.1505753322, \quad A_2^{(4)} = 0.2528510778, \quad A_3^{(4)} = -0.1135916341,$$
$$B_2^{(4)} = 0.0202143207, \quad C_4 = -0.03082307.$$

The other coeffi cients are given by

$$A_1^{(1)} = \frac{3}{10}, \ A_2^{(1)} = \frac{1}{5}, \ A_3^{(1)} = \frac{2}{15}, \ B_1^{(1)} = \frac{1}{40}, \ C_1 = -\frac{1}{30},$$
$$A_1^{(2)} = \frac{21}{100}, \ A_2^{(2)} = \frac{1}{25}, \ A_3^{(2)} = \frac{16}{225}, \ B_1^{(2)} = \frac{9}{800}, \ C_2 = -\frac{11}{900}.$$

The following stiff system of three ODEs of the second order is obtained for fi nding $u_0(t), u_1(t), u_2(t)$:

$$\begin{cases} \gamma_1 h_1^2 \big[A_1^{(2)}\dot{u}_0 + A_2^{(2)}\dot{u}_1 + \dfrac{h_1^2}{2\lambda_1}B_1^{(2)}(\dot{q}_1(0,t) + \gamma_1\ddot{u}_0)\big] + I_0 = \lambda_1(u_1 - u_0), \\ \dfrac{\gamma_2 h_2^2}{\ln\frac{L}{h_1}}\Big[A_1^{(4)}\dot{u}_1 + A_2^{(4)}\dot{u}_2 + \dfrac{h_2}{\lambda_2 L}A_3^{(4)} f'_u(u_2)\dot{u}_2 + \dfrac{h_2^2}{\lambda_2}B_2^{(4)}\big(\dot{q}_2(L,t) + \gamma_2\ddot{u}_2\big)\Big] \\ \qquad + \gamma_1 h_1^2\big[A_1^{(1)}\dot{u}_0 + A_2^{(1)}\dot{u}_1 + \dfrac{h_1^2}{2\lambda_1}B_1^{(1)}(\dot{q}_1(0,t) + \gamma_1\ddot{u}_0)\big] + I_1^+ + I_0^{(1)} \\ \qquad\qquad = \lambda_2/\ln\frac{L}{h_1 q}\big(u_2 - u_1\big), \\ \dfrac{\gamma_2 h_2^2}{\ln\frac{L}{h_1}}\Big[A_1^{(3)}\dot{u}_1 + A_2^{(3)}\dot{u}_2 + \dfrac{h_2}{\lambda_2 L}A_3^{(3)} f'_u(u_2)\dot{u}_2 \dfrac{h_2^2}{\lambda_2}B_2^{(3)}\big(\dot{q}_2(L,t) + \gamma_2\ddot{u}_2\big)\Big] \\ \qquad + I_2^- = f(u_2) - \lambda_2/\ln\frac{L}{h_1}(u_2 - u_1). \end{cases}$$

The initial conditions are

$$u_0(0) = \phi(0), \ \ u_1(0) = \phi(h_1), \ \ u_2(0) = \phi(L),$$
$$\dot{u}_0(0) = (2\lambda_1\phi''(0) - q_1(0,0))/\gamma_1,$$
$$\dot{u}_2(0) = (\lambda_2(\phi''(L) + L^{-1}\phi'(L)) - q_2(L,0))/\gamma_2.$$

For the equation (1.6) $(h_1 = h_2 = h = L/2, \alpha_L = \infty)$ the fi nite difference scheme (4.1) is defi ned as

$$u_1 - u_0 = J_2 + I_0, \quad T_L - u_1 = J_5 + I_1^+ + I_0^{(1)}, \tag{4.6}$$

where $J_5 = \ln 2 J_1 + J_0^*,\ J_0^* = \int_h^L x \ln \frac{L}{x} \dot{u}(x,t)\,dx,\ V(\bar{x}) = \dot{u}(\bar{x}h, t)$,

$$\frac{J_5}{h^2} = \ln 2 \int_0^2 \bar{x} V(\bar{x})\,d\bar{x} - \int_1^2 \bar{x} \ln \bar{x} V(\bar{x})\,d\bar{x} = A_1^{(5)} V(0) + A_2^{(5)} V(1)$$
$$+ A_3^{(5)} V(2) + B_1^{(5)} V''(0) + \frac{B_2^{(5)} (\bar{x} V'(\bar{x}))'}{\bar{x}}\Big|_{\bar{x}=1} + \frac{V^{(5)}(\xi)}{5!} C_5,\ \xi \in (0,1).$$

If $V(\bar{x}) = \bar{x}^i, i = \overline{0,5}$, then we get the following results

$$A_1^{(5)} = \frac{98}{2475},\ \ A_2^{(5)} = \frac{6653}{9900},\ \ A_3^{(5)} = \frac{19}{495},$$
$$B_1^{(5)} = -\frac{53}{26400},\ \ B_2^{(5)} = \frac{287}{9900},\ \ C_5 = -\frac{4028}{121275}.$$

The following system of two ODEs of the second order is obtained for finding $u_0,\ u_1$:

$$\begin{cases} h^2(A_1^{(2)}\dot{u}_0 + A_2^{(2)}\dot{u}_1) + 0.5h^4 B_1^{(2)}(\dot{q}_0 + \ddot{u}_0) + I_0 = u_1 - u_0, \\ h^2(A_1^{(5)}\dot{u}_0 + A_2^{(5)}\dot{u}_1 + A_3^{(5)}\dot{u}_2) + h^4\big(0.5B_1^{(5)}(\dot{q}_0 + \ddot{u}_0) + B_2^{(5)}(\dot{q}_1 + \ddot{u}_1)\big) \\ \qquad\qquad + I_1^+ + I_0^{(1)} = T_L - u_1. \end{cases}$$

Example 2. [5] If $q = 0,\ T_L = 0,\ \dot{u}_2 = 0,\ \phi(x) = J_0(\mu x/L)$ is the Bessel's function of first kind, $\mu = 2.404825558$ the first positive root of equation $J_0(\mu) = 0$, then the exact solution is $u(x,t) = \exp(-(\mu/L)^2 t) J_0(\mu x/L)$ with

$$u_0(t) = \exp\Big(-\Big(\frac{\mu}{L}\Big)^2 t\Big), u_1(t) = J_0\Big(\frac{\mu}{2}\Big) \exp\Big(-\Big(\frac{\mu}{L}\Big)^2 t\Big), J_0\Big(\frac{\mu}{2}\Big) = 0.6699297389.$$

The results of calculations are presented in Table 2, where $L = 2$, u_{0*}, u_{1*} are the exact values, u_{0p}, u_{1p} approximate values.

Table 2. The values $u(0,t), u(1,t)$ in the time t.

t	u_{0*}	u_{0p}	u_{1*}	u_{1p}
.1	.86538	.86540	.57974	.57972
.2	.74889	.74891	.50171	.50165
.5	.48534	.48530	.32515	.32505
.9	.27220	.27212	.18236	.18226

Example 3. [1] For heat transfer in cylindrical wire-metal (copper) conductor $(x \in [0, h_1])$ with insulation $(x \in (h_1, L])$ the term q_1 describes Joule's heat generation

and it can be written as [1] $q_1 = -\sigma_0^{-1}\frac{I^2}{S^2}$, where $\sigma_0[\frac{1}{\Omega.m}]$ is the electric conductivity of metal, $I[A]$ is the electric current, $S[m^2]$ is the cross-section of the wire $(S = \pi h_1^2)$. The numerical results are obtained for the following values of parameters:

$$h_1 = 4\,[mm],\;\; L = 5\,[mm],\;\; \beta = 0.25,\;\; I = 173\,[A],\;\; \alpha_L = 1\,[\frac{W}{m^3K}],$$
$$\epsilon = 0.5,\;\; \sigma_0^{-1} = 1.7\cdot 10^{-8}\,[\Omega m],\;\; c_1 = 410\,[\frac{J}{kgK}],\;\; c_2 = 840\,[\frac{J}{kgK}],$$
$$\rho_1 = 8960\,[\frac{kg}{m^3}],\;\; \rho_2 = 500[\frac{kg}{m^3}],\;\; q_1 = -0.217\cdot 10^7\,[\frac{W}{m^3}],\;\; q_2 = 0,$$
$$\lambda_1 = 400\,[\frac{W}{mK}],\;\; \lambda_2 = 0.2\,[\frac{W}{mK}],\;\; T_* = T_L = 293\,[K].$$

We have calculated stationary and non-stationary solutions in two cases:
1) with radiation $u_0 = 316.78[K]$, $u_1 = 316.76[K]$, $u_2 = 297.42[K]$,
2) without radiation $u_0 = 329.69[K]$, $u_1 = 329.67[K]$, $u_2 = 310.33[K]$.

In the non-stationary case the results are presented in tables 3 (with radiation) and 4 (without radiation). We can see the effect of radiation.

Table 3. The values $u_0 = u_1, u_2, \dot{u}_0, \dot{u}_2$ in time t with radiation.

t	u_0	$\dot{u}_0$	u_2	$\dot{u}_2$
1	293.6	0.560	293.1	0.094
10	298.1	0.450	293.9	0.086
50	309.7	0.171	296.1	0.032
100	314.7	0.051	297.0	0.009
150	316.2	0.015	297.3	0.003
200	316.6	0.004	297.4	0.001

Table 4. The values $u_0 = u_1, u_2, \dot{u}_0, \dot{u}_2$ in time t without radiation.

t	u_0	$\dot{u}_0$	u_2	$\dot{u}_2$
1	293.6	0.560	293.1	0.193
10	298.3	0.490	295.3	0.240
50	312.8	0.260	302.3	0.132
100	322.0	0.120	306.6	0.061
150	326.1	0.055	308.6	0.026
200	328.0	0.025	309.5	0.012

5. The Spherical Coordinates

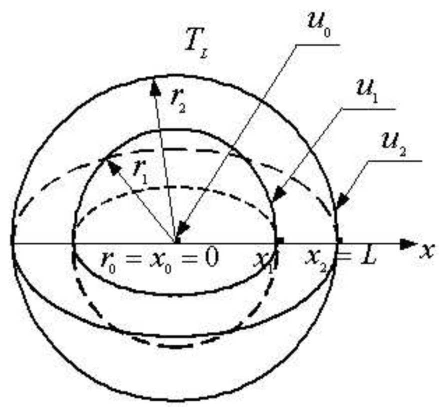

Figure 3. The calculated temperatures u_0 and u_1 in the points $x = 0$ and $x = x_1$ in the case of spherical coordinates.

In the spherical coordinates (see Fig. 3) we get the following coefficients

$$p(x) = x^2, \quad R_k^- = \frac{a_k}{x_{k-1}} \int_{xk-1}^{x_k} x(x - x_{k-1})\dot{u}_k(x,t)\,dx,$$
$$a_k = \Big(\frac{1}{x_{k-1}} - \frac{1}{x_k}\Big)^{-1}, \quad R_k^+ = \frac{a_{k+1}}{x_{k+1}} \int_{x_k}^{x_{k+1}} x(x_{k+1} - x)\dot{u}_{k+1}(x,t)\,dx.$$

The finite-difference scheme $(N = 2)$ is given by (4.1), where

$$J_2 = h_1^{-1} \int_0^{h_1} x(h_1 - x)\dot{u}_1(x,t)\,dx, \quad I_0 = h_1^{-1} \int_0^{h_1} x(h_1 - x)q_1\,dx,$$
$$J_1 = \int_0^{h_1} x^2\dot{u}_1(x,t)\,dx, \quad I_0^{(1)} = \int_0^{h_1} x^2 q_1\,dx, \quad a_2 = \frac{L}{\beta},\ \beta = \frac{h_2}{h_1}.$$

In the stationary case $(q_k = const)$ we get:

$$I_0 = \frac{q_1 h_1^2}{6}, \quad I_0^{(1)} = \frac{q_1 h_1^3}{3},$$
$$I_2^- = \frac{q_2 L}{6\beta}\big(2\beta L^2 + h_1^2 - L^2\big), \quad I_1^+ = \frac{q_2}{6\beta}\big(L^3 - h_1^3 - 3\beta h_1^3\big).$$

From (4.1) it follows that

$$\begin{cases} \lambda_1(u_1 - u_0) = \dfrac{q_1 h_1^2}{6}, \\ \dfrac{\lambda_2\beta}{L}\big(u_2 - u_1\big) = \dfrac{q_1 h_1^3}{3} + \dfrac{q_2}{6\beta}\big(L^3 - h_1^3 - 3\beta h_1^3\big), \\ f(u_2) - \dfrac{\lambda_2\beta}{L}\big(u_2 - u_1\big) = \dfrac{q_2 L}{6\beta}\big(2\beta L^2 + h_1^2 - L^2\big). \end{cases} \tag{5.1}$$

The exact solution of the problem (1.1)–(1.3) satisfies (5.1) and

$$f(u_2) - \frac{q_2 L^3}{3} = \frac{h_1^3}{3}\big(q_1 - q_2\big).$$

This equation has a positive root if

$$\epsilon\sigma T_*^4 + \alpha_L T_L - \frac{1}{3}\big(q_2(L^3 - h_1^3) + q_1 h_1^3\big) > 0.$$

In the non-stationary case we have the following quadrature formulas ($x = \bar{x}h_1, x = h_1 + \bar{x}h_2$):

$$\frac{J_m}{h_1^3} = A_1^{(m)} V_1(0) + A_2^{(m)} V_1(1) + A_3^{(m)} V_1'(0) + B_1^{(m)} V_1''(0) + r_m,\ m = 1,2,$$

$$\frac{J_m}{h_2^2 L} = A_1^{(m)} V(0) + A_2^{(m)} V(1) + A_3^{(m)} V'(1) + B_2^{(m)}\Big(V''(1) + \frac{2\beta V'(1)}{1+\beta}\Big) + r_m,$$

where $r_m = \dfrac{W^{(4)}(\xi_m)}{4!} C_m, \xi_m \in (0,1), m = 3,4;\ W = V, V_1, m = 1,2,3,4,$

$$J_1 = h_1^3 \int_0^1 \bar{x}^2 V_1(\bar{x})\, d\bar{x},\ J_2 = h_1^3 \int_0^1 \bar{x}(1-\bar{x}) V_1(\bar{x})\, d\bar{x},\ J_3 = \beta R_2^-,\ J_4 = \beta R_1^+,$$

$$J_3 = h_2^2 L \int_0^1 (\beta\bar{x} + 1) V(\bar{x})\, d\bar{x},\quad J_4 = \frac{h_2^2 L}{1+\beta}\int_0^1 (\beta\bar{x}+1)(1-\bar{x}) V(\bar{x})\, d\bar{x}.$$

The undefinite coefficients are given by:

$$A_1^{(1)} = A_2^{(1)} = \frac{1}{6},\ A_3^{(1)} = \frac{1}{12}, B_1^{(1)} = \frac{1}{60},\ C_1 = -\frac{1}{42},\ A_1^{(2)} = \frac{2}{15},$$

$$A_2^{(2)} = \frac{1}{30},\ A_3^{(2)} = \frac{1}{20},\ B_1^{(2)} = \frac{1}{120},\ C_2 = -\frac{1}{105},\ A_1^{(3)} = \frac{\beta+3}{60},$$

$$A_2^{(3)} = \frac{19\beta+27}{60},\ A_3^{(3)} = -\frac{5\beta^2 + 13\beta + 7}{60(1+\beta)},\ B_2^{(3)} = \frac{\beta+2}{120},\ C_3 = -\frac{\beta+2}{120},$$

$$A_1^{(4)} = \frac{\beta+6}{30(1+\beta)},\ A_2^{(4)} = \frac{4\beta+9}{30(\beta+1)},\ A_3^{(4)} = -\frac{2\beta^2 + 7\beta + 4}{30(1+\beta)^2},$$

$$B_2^{(4)} = \frac{\beta+3}{120(1+\beta)},\quad C_4 = -\frac{2\beta+7}{210(1+\beta)}.$$

Therefore we obtain the following system of ODEs:

$$\begin{cases} \gamma_1 h_1^2 \big[A_1^{(2)}\dot{u}_0 + A_2^{(2)}\dot{u}_1 + + \dfrac{h_1^2 B_1^{(2)}}{3\lambda_1}\big(\dot{q}_1(0,t)+\gamma_1\ddot{u}_0\big)\big] + I_0 = \lambda_1(u_1-u_0), \\ \dfrac{\gamma_2 h_2^2 L}{\beta}\Big[A_1^{(4)}\dot{u}_1 + A_2^{(4)}\dot{u}_2 + \dfrac{h_2}{\lambda_2 L^2}A_3^{(4)} f_u'(u_2)\dot{u}_2 + \dfrac{h_2^2}{\lambda_2}B_2^{(4)}\big(\dot{q}_2(L,t)+\gamma_2\ddot{u}_2\big)\Big] \\ \qquad + \gamma_1 h_1^3 \Big[A_1^{(1)}\dot{u}_0 + A_2^{(1)}\dot{u}_1 + \dfrac{h_1^2}{3\lambda_1}B_1^{(1)}\big(\dot{q}_1(0,t)+\gamma_1\ddot{u}_0\big)\Big] \\ \qquad + I_1^+ + I_0^{(1)} = \dfrac{\lambda_2 L}{\beta}\big(u_2-u_1\big), \\ \dfrac{\gamma_2 h_2^2 L}{\beta}\Big[A_1^{(3)}\dot{u}_1 + A_2^{(3)}\dot{u}_2 + \dfrac{h_2}{\lambda_2 L^2}A_3^{(3)} f_u'(u_2)\dot{u}_2 + \dfrac{h_2^2}{\lambda_2}B_2^{(3)}\big(\dot{q}_2(L,t)+\gamma_2\ddot{u}_2\big)\Big] \\ \qquad + I_2^- = f(u_2) - \dfrac{\lambda_2 L}{\beta}\big(u_2-u_1\big). \end{cases}$$

The initial conditions are given by

$$\begin{cases} u_0(0) = \phi(0), \;\; u_1(0) = \phi(h_1), \;\; u_2(0) = \phi(L), \\ \dot{u}_0(0) = (3\lambda_1\phi''(0) - q_1(0,0))/\gamma_1, \\ \dot{u}_2(0) = \big(\lambda_2(\phi''(L) + 2L^{-1}\phi'(L)) - q_2(L,0)\big)/\gamma_2. \end{cases}$$

For the equation (1.6) $(h_1 = h_2 = h = L/2, \alpha_L = \infty, \dot{u}_2 = \ddot{u}_2 = 0)$ we have the finite difference scheme in the form

$$u_1 - u_0 = J_2 + I_0, \quad T_L - u_1 = J_5 + I_1^+ + I_0^{(1)}, \tag{5.2}$$

where

$$\begin{aligned} & J_5 = \frac{J_1 + J_0^*}{L}, \;\; J_0^* = \int_h^L x(L-x)\dot{u}(x,t)\,dx, \\ & \frac{J_5}{h^2} = \frac{1}{2}\Big(\int_0^1 \bar{x}^2 V(\bar{x})\,d\bar{x} + \int_1^2 \bar{x}(2-\bar{x})V(\bar{x})\,d\bar{x}\Big) = A_1^{(5)}V(0) + A_2^{(5)}V(1) \\ & + A_3^{(5)}V(2) + B_1^{(5)}V''(0) + \frac{B_2^{(5)}(\bar{x}^2V'(\bar{x}))'}{\bar{x}^2}\Big|_{\bar{x}=1} + \frac{B_3^{(5)}(\bar{x}^2V'(\bar{x}))'}{\bar{x}^2}\Big|_{\bar{x}=2} \\ & \qquad + \frac{V^{(6)}(\xi)}{6!}C_5, \;\; \xi \in (0,2), \quad V(\bar{x}) = \dot{u}(\bar{x}h,t) = \bar{x}^i, \; i = 0,\dots,6, \\ & A_1^{(5)} = \frac{1}{1008}, \;\; A_2^{(5)} = \frac{115}{252}, \;\; A_3^{(5)} = \frac{43}{1008}, \;\; B_1^{(5)} = \frac{1}{5040}, \\ & B_2^{(5)} = \frac{107}{5040}, \;\; B_3^{(5)} = -\frac{1}{1260}, \;\; C_5 = -\frac{1}{315}. \end{aligned}$$

If we add in expression J_5/h^2 the term $A_4^{(5)}V'(0)$ with error term $\frac{V^{(7)}(\xi)}{7!}C_5$, then we have the following values of coefficients:

$$\begin{aligned} & A_1^{(5)} = B_1^{(5)} = 0, \;\; A_2^{(5)} = \tfrac{115}{252}, \;\; A_3^{(5)} = \frac{11}{252}, \;\; B_2^{(5)} = \frac{313}{15120}, \\ & B_3^{(5)} = -B_2^{(5)}, \;\; A_4^{(5)} = -B_2^{(5)}, \;\; C_5 = -\frac{59}{3780}. \end{aligned} \tag{5.3}$$

$$h^2(A_1^{(2)}\dot{u}_0 + A_2^{(2)}\dot{u}_1) + h^4\big(B_1^{(2)}\frac{\dot{q}_0 + \ddot{u}_u}{3} + B_2^{(2)}(\dot{q}_1 + \ddot{u}_1)\big) + I_0 = u_1 - u_0, \quad (5.4)$$

$$h^2(A_1^{(5)}\dot{u}_0 + A_2^{(5)}\dot{u}_1) + h^4\big(B_1^{(5)}\frac{\dot{q}_0 + \ddot{u}_0}{3} + B_2^{(5)}(\dot{q}_1 + \ddot{u}_1) + B_3^{(5)}\dot{q}_2\big) + I_1^+ + I_0^{(1)} = T_L - u_1. \quad (5.5)$$

Example 4. If $q = 0$, $T_L = 0$, $\phi(x) = x^{-1}\sin(\pi x/L)$, then the exact solution is

$$u(x,t) = \frac{1}{x}\exp(-(\pi/L)^2 t)\sin(\pi x/L),$$
$$u_0(t) = \frac{\pi}{L}\exp(-(\pi/L)^2 t), \quad u_1(t) = \frac{2}{L}\exp(-(\pi/L)^2 t).$$

We have solved the system (5.4)–(5.5) with initial conditions

$$u_0(0) = \frac{\pi}{L}, \ \ u_1(0) = \frac{2}{L}, \ \ \dot{u}_1(0) = -\frac{2\pi^2}{L^3}, \ \ \dot{u}_0 = -\Big(\frac{\pi}{L}\Big)^3, \ \ L = 2.$$

The results are presented in Table 5.

Table 5. The values $u(0,t), u(1,t)$ in the time t.

t	u_{0*}	u_{0p}	u_{1*}	u_{1p}
.1	1.22733	1.22741	0.781344	0.781339
.2	0.95897	0.95907	0.610498	0.610488
.5	0.45744	0.45749	0.291213	0.291198
.9	0.17049	0.17050	0.108537	0.108527

6. Conclusions

1. The proposed method allows us to reduce 1D heat transfer problem in Cartesian, cylindrical and spherical coordinates to the system of the ordinary differential equations of the second order.

2. The described methods make it possible to fi nd the distribution of temperature in the case of different layers with the heat source in between the layers and on layers borders.

3. In different coordinates it is possible to enlarge the accuracy of the given algorithm, when second order derivatives are used instead of fi rst order derivatives.

4. Such formulations have a big practical importance as compared to Cartesian coordinates, e.g. for analysis of heat transfer in cylindrical wire-metal (coper) conductor with insulation.

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Kai kurių inžinerinių, dvisluoksnėje srityje, šilumos laidumo uždavinių tikslumo didinimas

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Šiame straipsnyje yra nagrinėjami paprasti dvisluoksnės srities šilumos laidumo problemos modeliavimo algoritmai, keičiant diferencialines lygtis dalinėmis išvestinėmis į paprastas diferencialines lygtis. Parodoma, kad didesnio tikslumo pasiekimui, vietoje pirmos eilės paprastų diferencialinių lygčių pradinio uždavinio nagrinėjamos antros eilės diferencialinės lygtys. Ši procedūra leidžia gauti paprastą inžinerinį dvisluoksnės srities šilumos laidumo lygties sprendinį stačiakampėje, cilindrinėje (su ašių simetrija) ir sferinėje (su spinduline simetrija) koordinačių sistemoje. Tiksli baigtinių skirtumų schema buvo sudaryta stacionariam atvejui.