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STABILITY ANALYSIS OF CELLULAR AUTOMATA GENERATED CLUSTERS

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It is more important to "be roughly right than precisely wrong". /A.Einstein/

Abstract. The development of regional urban system still remains one of the main problems during the human race history. There are a lot of problems inside this system like overcrowded cities and decaying countryside. All these situations can be reproduced by modelling them using Cellular Automata (CA) [1, 2, 5]. CA models implement algorithms with simple rules and parameter controls, but the result can be a complex behaviour.

A stability of naturally formed self-organized urban system depends on its critical state parameter τ in the power law $log(f(x)) = -\tau log(x)$. If the system reaches self-organized critical (SOC) state then it remains in it for a long time. The CA model URBACAM (URBAnistic Cellular Automata Model) describes the long-lasting term behaviour and shows that the change in behaviour is sensitive to the urban parameter τ of the power law.

Key words: cellular automata, clustering, urban growth, stability analysis

1. Introduction

Urbanization is a social process whereby cities grow and societies become more urban. Urban system is a good example of clustering process: all kinds of cities are situated in space differently and have different amount of population inside.

Cluster is a set of agents (cells) connected by side or by corner (see Fig. 1). Primitive definition of city is an agglomeration of buildings (agents or cells). Consequently in our model a city is represented as an agglomeration of house cells (H) and we call it cluster.

The model is a rough view of the reality. Even though the system's behaviour has a simple form, it may not be at all easy to construct the model for it. When the behaviour is complex it may take an irreducible amount of computational work to answer any given question about it. However, this is not a sign of model imperfection,

Figure 1. Examples of clusters.

it is merely a fundamental feature of its complex behaviour. With traditional models based on differential equations we try to find the unique model for every problem, consequently the result is unique, too. But we can observe that it is possible for quite different methods to yield essentially the same large–scale behaviour, implying that with different tools there can be many models that have the same consequences, but a different detailed underlying structure.

Systems, which exhibit scale-free power law behaviours, are ubiquitous in physics, biology, geology, sociology and economics. Shortly, a function $f(x)$ produces power law if the independent variable x has an exponent. There are many different systems generating power law behaviour, the SOC system is one among them.

The self-organized criticality manifests that open, dynamical, far from equilibrium systems consisting of many constituents may evolve towards a critical state without any control from outside agents. In the critical state, a small local perturbation may spread to the whole system through domino–like effect and form an "*avalanche*". The spatial and temporal sizes of "avalanches" at the critical state obey a stable scale-free power law. Our model should simulate any real situation just by changing parameters. These parameters for the regional urban system are weights of empty place (E), houses (H), time steps T, initial pattern and grid size. The main goal is to find which parameters are essential in controlling the behaviour of the system. This task requires a lot of calculations, so in this paper we will evaluate only parameters that reflex SOC long-lasting stability state.

The algorithm of calculations is pretty simple: according to the definition the size of clusters is found going from the upper left corner to the bottom right corner of the pattern, from the first house cell around its environment until the cluster is closed and so on. The distribution of cluster sizes should represent the situation that we are modelling. So again, we have to find ruling parameters in our model.

2. Simulating Urban Growing

2.1. Data analysis

Analysing data from real urban situations in Lithuania and some other neighbouring countries (see Fig. 2) we obtained that these systems fit power law with the following urban parameters τ: Lithuania – 0.83, Estonia – 0.94, Sweden – 1, Norway – 0.92.

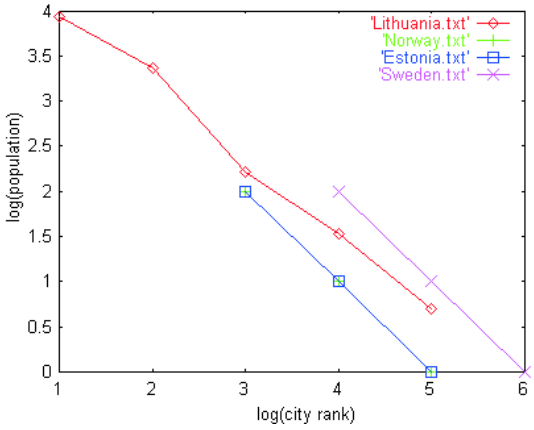

Figure 2. Power law in urban systems: $log(rank\,of\,cities) = -\tau log(population)$.

The main goal of CA modelling and simulation is to get the same urban parameter τ in the model as in practice. We make the main assumption that there are counted conglomerates of houses (clusters) in the model instead of population in the cities: one house in a model is directly proportional to two persons (i.e. the scale is $1:2$).

2.2. Model description

URBAn Cellular Automata Model (URBACAM) is done according to simple rules with an interface of Borland Delphi 6. This notwithstanding cellular automata model can be implemented within different software [1]. Modeling rules are the following [3, 4]:

1. A set of cells' weights

$$W_{n,n} = \begin{bmatrix} w_{11} & \ldots\ldots & w_{1n} \\ \ldots & \ldots\ldots & \ldots \\ \ldots & \ldots\ldots & \ldots \\ w_{n1} & \ldots\ldots & w_{nn} \end{bmatrix},$$

   where n is the dimension of the pattern, w_{ij} are weights of cells representing values of parameters (houses, empty place);
2. $W_{n,n} \Longleftrightarrow W_{n+2m,n+2m}$ is a torus with m rings neighbourhood (in our model $m = 1, 2, 3, 4$);
3. $\tilde{w}_{ij}$ is the weight of cell's neighbourhood

$$\tilde{w}_{kl} = \begin{cases} \sum\limits_{k=i-m}^{i+m} \sum\limits_{l=j-m}^{j+m} w_{kl}, & \text{for neighbourhood of empty cells}, \\ 0, & \text{for neighbourhood of house cells}, \end{cases}$$

   $S = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} \tilde{w}_{ij}$ is the total weight of neighbourhoods of empty places. If $S = 0$ then the process is stopped.

4. Transformation of $2D$ matrix into $1D$ vector:

$$\tilde{W}_{n,n} \Leftrightarrow \tilde{W}_{n^2} = [\tilde{w}_1, \ldots, \tilde{w}_{n^2}],$$

$$\tilde{w}_k = \frac{1}{S}\tilde{w}_{ij} + \begin{cases} \tilde{w}_{k-1}, & \text{if } k \geq 2, \\ 0, & \text{if } k = 1. \end{cases}$$

   Then we have that $0 \leq \tilde{w}_i \leq 1$ and $\tilde{w}_{n^2} = 1$.

5. Transition rule from (E) to (H):

$$p_k = \text{random}(1), \quad k = (i-1)n + j.$$

   If $p_k \in [\tilde{w}_k, \tilde{w}_{k+1}],\ k = 1, \ldots, n^2 - 1$, then cell w_{ij} turns into house cell, where $i = (k-1) \operatorname{div} n + 1,\ j = (k-1) \bmod n + 1$.

6. Recalculate all weights and repeat the procedure.

2.3. Predicting a long-lasting development

Urban model [4] generates clusters that have global systemic characteristic. We expect to get a *log–log* distribution as a stable result of complex system evolution. By setting different parameters we can model any real or imaginable situation, stabilization of which is a long lasting process. But not all real situations are stable, unstable ones increase a gap between small and big clusters, recede from *log–log* distribution. Our task is to define which parameters should be taken and how to vary their values in order to get *log–log* distribution with the gradient τ.

Initially we freeze all parameters except one. During calculations we have changed the value of only one parameter and observed the dynamics of cluster distribution. The analysis proved that only two parameters are essential: E (empty place) and H (houses).

Next we do the following experiment calculations: we set parameters (E, H) and find area where $\tau = 1$. Let's have in mind that in this simplified model we don't have the decay of houses, therefore initial clusters distribution should be constructed in such a way that initial amount of houses is not more than 10% of houses in a final stage. For example, if we are planning to make 1000 steps, initial situation should hold 100 H cells. A sensitive dependence of the solution on initial conditions is associated with a kind of instability in the system. Most of our experiments are done with the blank pattern or the pattern with only one house, which doesn't give us an idea about stability or instability.

3. Modelling Results

We use *URBACAM*, described in Section 2.2., to investigate the sensitivity of the model with respect to initial values of parameters: empty place (see Table 1 and Figure 3*a*), amount of time steps (see Table 2 and Figure 3*c*), and weights of houses (see Table 3 and Figure 3*b*).

It follows from computational results that different distributions of clusters are observed for the different weights of cells. A cluster in our case is a set of weights of houses connected by side or corner. Results aren't hard-and-fast because of random component. We conclude from Fig. 2 that the obtained results hold up very high correlation (~ 0.98) with power law.

Table 1. Dependence of cluster distribution on the weight of empty place.

	Cluster size			
E	>1	>10	>100	>1000
1	129	60	18	4
2	179	75	25	4
3	193	86	24	4
4	246	94	23	5

Table 2. Dependence of cluster distribution on the amount of time steps.

	Cluster size			
Time steps	>1	>10	>100	>1000
14000	257	103	34	1
15000	231	94	27	1
16000	219	89	32	2
17000	163	58	13	3
18000	141	58	15	3
19000	116	40	11	3

Table 3. Dependence of cluster distribution on the weight of houses.

	Cluster size			
H	>1	>10	>100	>1000
795	185	72	21	3
796	223	82	23	2
797	173	85	21	1
798	203	93	31	4
799	200	79	26	3
800	199	83	24	4
801	209	77	18	4
802	210	81	25	2
803	186	83	23	4
804	197	86	22	3
805	208	84	22	4

These and many other calculations enable us to formulate the following hypothesis: when the weights are small, the system is very sensitive to changes. On the other hand, if the weights are large, small changes don't play a big role. But anyway the interval of chosen parameters is not wide. Such intervals vary because parameters depend on each other and because the formation of a cluster is a random process. The reason for discrepancy is that the probabilities for different cells are in fact correlated. In general, such approximations tend to work better for systems described in large number of dimensions, where correlations tend to be less important.

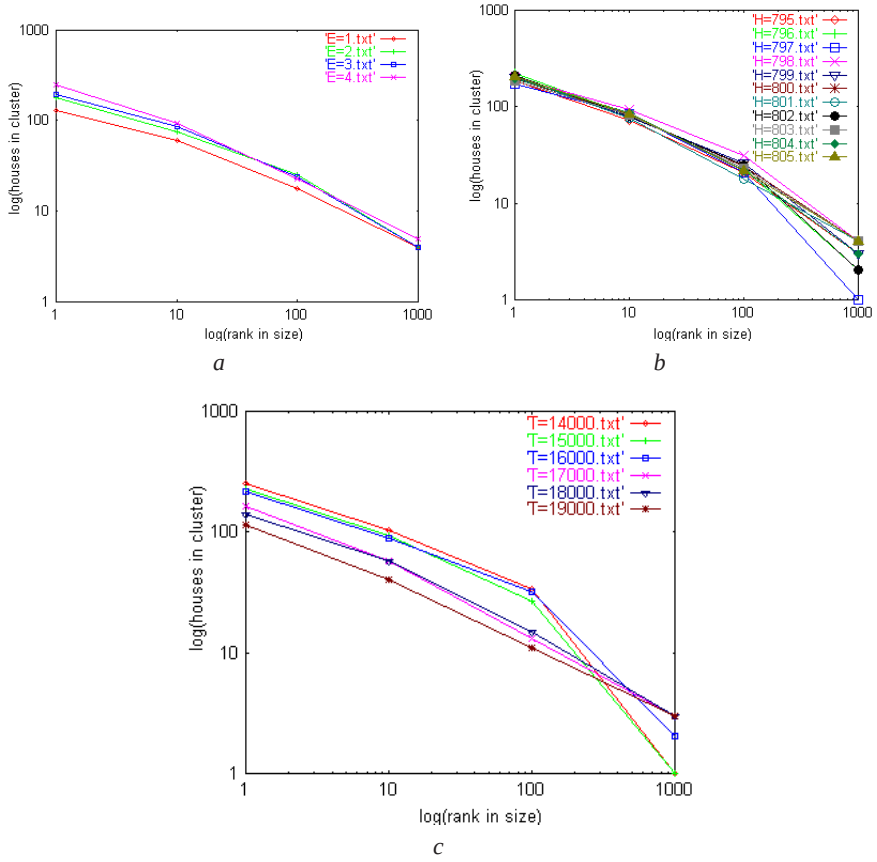

Figure 3. Graphical results of Tables 1–3.

4. Conclusions

In this work we have obtained the main parameters (houses, empty place, time steps) that make system dynamically stable. The initial values of these parameters are the following: weight of houses $(H) = 800$, empty place $(E) = 3$, time steps $[17000 - 18000]$ in a pattern (200×200). With these parameters we have got linearity in *log–log* distribution, but the urban parameter $\tau = 1$ wasn't reached because of not big enough pattern. Actually in our case a grid 200×200 is not enough to form a necessary structure. Bigger grid requires more computer resources and time, so the algorithm should be improved.

In modern science it is usually said that the ultimate test of any model is an agreement of computational results with real system data. But this is often interpreted to mean that if a real data ever disagrees with a model, then the model is wrong. Particularly when the model is simple and the system is complex, however, it is quite a common situation: obtaining the data collection is a complex task rather than the

model that is wrong. We have tended to find still less reliability in the results of complex system [2].

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Ląstelių automatais sugeneruotų klasterių stabilumo analizė

R. Šiugždaitė

Regioninės urbanistinės sistemos vystymasis išlieka viena iš opiausių problemų žmonijos istorijoje. Keletas tokių uždavinių kaip miestų perpildymas, nykstančios kaimo vietovės ir t.t. gali būti nesunkiai modeliuojami naudojant ląstelių automatus (LA). LA metodas ypatingas tuo, kad realizuoja algoritmą paprastų taisyklių bei parametrų valdymo pagalba, tačiau rezultate galima gauti sudėtingą elgseną. Natūraliai susiformavusių urbanistinių sistemų stabilumas priklauso nuo sistemos krizinės savirangos būsenos (KSB) parametro τ. Jei sistema pasiekia KSB, tai ji ilgą laiką išlieka joje. LA modelis URBACAM charakterizuoja ilgalaikę elgseną ir parodo, jog modelyje jos kitimus įtakoja eksponentinio dėsnio urbanistinis parametras τ.