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THE POWER EXPANSIONS OF THE SOLUTIONS OF THE FIRST PAINLEVÉ HIERARCHY

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Abstract. In this paper we consider a hierarchy of the first Painlevé equation's higher order analogues. For these equations three types of power expansions, i.e. holomorphic, polar and asymptotic are found. As an example the equation of the fourth order is considered.

Key words: Painlevé equation, P-type equations, power expansions, power geometry

1. Introduction

The first Painlevé equation found by Painlevé and Gambier [7] arises as the result of self-similar reduction of Korteweg-de Vries, Boussinesq, Kadomtsev-Petviashvili equations [4]. It defines new functions in terms of which some solutions of the mentioned and other important PDEs can be expressed. This has aroused the interest in the ODEs of higher order with the Painlevé property which can also define new functions. However, as checking all possible equations with increased order becomes more and more complicated, other approaches should be used.

Building hierarchies of analogues of Painlevé equations is one of the ways to find higher order ordinary differential equations with the Painlevé property. There are several approaches to do this. Historically, the first one is the self-similar reduction of the Korteweg-de Vries hierarchy. In this case ${}_{2n}P_1$ hierarchy occurs [6] (see also [4]):

$$\mathbf{d}^{n+1}(w) + 4z = 0, \tag{1.1}$$

where

$$\begin{aligned} &\mathbf{d}^1(w) = -4w, \\ &\frac{d}{dz}\big(\mathbf{d}^{n+1}(w)\big) = \Big(\frac{d^3}{dz^3} - 8w\frac{d}{dz} - 4w'\Big)\mathbf{d}^n(w), \quad \text{for } n \in \mathbb{N}. \end{aligned} \tag{1.2}$$

For $n=1$, $n=2$ and $n=3$ the following examples (all equations are reduced by -4) can be given:

$$w'' - 6w^2 - z = 0, \quad w^{(4)} - 20ww'' - 10(w')^2 + 40w^3 - z = 0,$$

$$w^{(6)} - 42(w'')^2 - 56w'w''' - 28ww^{(4)} + 280w(w')^2 + 280ww'' - 280w^4 - z = 0.$$

In this paper all local power expansions of the solutions of equations (1.1) are found.

2. The Method Overview

We are looking for power expansions of the solutions of (1.1) in the form

$$w = c_r z^r + \sum_s c_s z^s, \tag{2.1}$$

where r, s are considered to be rational numbers. For this purpose we use power geometry method developed by A. D. Bruno [1] and successfully applied to a similar problem [2] and in recent work [3] to the particular case $n=2$ of the current problem. The following facts will be used.

Let us consider the equation

$$F(w,z) := \sum_k c z^q w^p \frac{d^k w}{dz^k} = 0,$$

where $c = const \in \mathbb{C}$, $q \in \mathbb{Q}$, $p, k \in \mathbb{Z}_{\geq 0}$ and $r > s$ when $z \to \infty$ or $r < s$ when $z \to 0$. Here we hold to the notation, introduced in [2]. Power expansions of the solutions in the neighborhood of $z = z_0 \in \mathbb{C}$ can be obtained by changing the independent variable $z = \tilde{z} + z_0$.

An expression $m = z^q w^p \frac{d^k w}{dz^k}$ is called a differential monomial. Vector exponent of the differential monomial $R(m) \in \mathbb{Q}^2$ is calculated by the rule

$$R(w^p z^q) = (p, q), \quad R(\frac{d^k w}{dz^k}) = (1, -k), \quad R(m_1 m_2) = R(m_1) + R(m_2)$$

for a product of any monomials m_1 and m_2. A differential polynomial $F(w, z)$ is a sum of some differential monomials with coefficients.

The carrier of differential polynomial is a set of vector exponents of its monomials $\mathbf{S}(f) = \{R(m) : \quad m \in F(w, z)\}$. Consider the convex hull of the carrier $\Gamma(F) = Conv(\mathbf{S}(F))$. A boundary of the polygon $\Gamma(F)$ consists of apices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$, which are called (generalized) bounds, where the subscript means a number and the superscript means a dimension of a bound. By extracting from differential polynomial only the monomials with vector exponents on some bound $\Gamma_j^{(d)}$ we get the shortened equation $\hat{F}_j^{(d)}(w, z) = 0$. This equation is easier to solve as it has the solution $\eta_j^{(d)}(z)$ of the form $c_r z^r$ from (2.1), that is used for a change of independent variable $w = \eta_j^{(d)} + u$ in the initial equation. The given algorithm is applied to all the bounds of $\Gamma(F)$ until the equations, we get after the change of variable, satisfy the conditions described in in theorems 2.1, 2.2 and 2.3 in [2], which give the second part of the expansion (2.1) – the term $\sum_s c_s z^s$ and the conditions for its convergence.

3. Power Expansions

Let us denote

$$f_n(w, z) = -(\mathbf{d}^{n+1}(w) + 4z)/4. \tag{3.1}$$

Here we multiply equation (1.1) by $(-\frac{1}{4})$ for convenience.

First, we need to find the carrier of this differential polynomial. After integration by parts the Lenard operator (1.2) has the representation:

$$\mathbf{d}^1(w) = -4w, \quad \mathbf{d}^{n+1}(w) = \frac{d^2}{dz^2}\,\mathbf{d}^n(w) - 8w\,\mathbf{d}^n(w) + 4\int \mathbf{d}^n(w)\,dw. \tag{3.2}$$

Theorem 1. *The carrier of the power sum defined by operator* (1.2) *is given by*

$$\mathbf{S}\big(\mathbf{d}^{n+1}(w)\big) = \{(p, q) = (n + 1 - 2k, k), \quad k = 0, 1, \ldots, n\}. \tag{3.3}$$

Proof. The fact $\mathbf{S}\big(\mathbf{d}^1(w)\big) = \mathbf{S}(-4w) = (1, 0)$ is the basis of mathematical induction. Next we investigate how the points of the carrier move when the order of the operator (3.2) is increased from $\mathbf{d}^n$ to $\mathbf{d}^{n+1}$.

Consider first, how differentiation influences a vector exponent of a differential monomial. It is clear, that differentiation of the monomial $m_0 = w_0^p$

gives $m_0' = p_0 w^{p_0 - 1} w'$, so the vector exponent $(p_0, 0)$ moves by $(0, -1)$ to $(p_0, -1)$. If we assume, that this rule holds for monomials of the form

$$m_k = w_0^p w_1'^p w_2''^p \dots (w^{(k)})^{p_k},$$

then by differentiation of $m_{k+1} = w_0^p w_1'^p w_2''^p \dots (w^{(k+1)})^{p_{k+1}}$ we get the differential polynomial

$$m_{k+1}' = m_k' \, (w^{(k+1)})^{p_{k+1}} + p_{k+1} m_k \, (w^{(k+1)})^{p_{k+1}-1} w^{(k+2)}.$$

The vector exponents of the monomials of this polynomial are shifted by $(0, -1)$. So it is proved, that

$$\mathbf{S}(m_k') = \mathbf{S}(m_k) + (0, -1). \tag{3.4}$$

Here a plus sign means parallel shifting. Hence the first term of (3.2) gives

$$\mathbf{S}\big(\frac{d^2}{dz^2}\,\mathbf{d}^n(w)\big) = \mathbf{S}\big(\mathbf{d}^n(w)\big) + (0, -2).$$

The second term clearly gives $\mathbf{S}(w\,\mathbf{d}^n(w)) = \mathbf{S}(\mathbf{d}^n(w)) + (1, 0)$. Consider the last term. By differentiation we get

$$\Big(\int \mathbf{d}^n(w)\,dw\Big)' = \mathbf{d}^n(w)\,w'.$$

Hence $\mathbf{S}\big((\int \mathbf{d}^n(w)\,dw)'\big) = \mathbf{S}\big(\mathbf{d}^n(w)\big) + (1, -1)$. So due to (3.4) the last term gives $\mathbf{S}\big(\int \mathbf{d}^n(w)\,dw\big) = \mathbf{S}\big(\mathbf{d}^n(w)\big) + (1, 0)$. Combining the carriers found for all the summands of (3.2) we prove the theorem:

$$\mathbf{S}\big(\mathbf{d}^{n+1}(w)\big) = \{\mathbf{S}\big(\mathbf{d}^n(w)\big) + (0, -2)\} \bigcup \{\mathbf{S}\big(\mathbf{d}^n(w)\big) + (1, 0)\}.$$

■

The direct implication of the theorem is that $\mathbf{\Gamma}(f) = Conv(\mathbf{S}(f_n(w, z)))$ is a triangle. Figure 1 presents examples of the carriers of equation (3.1) for $n = 1$, $n = 2$, $n = 3$, $n = 4$ and $n = 5$. It also could be a good illustration to the proof of Theorem 1.

Let us consider the apex $(0, 1)$. The corresponding shortened equation $z = 0$ cannot define a solution. Similar situation arises from considering the apex $(n + 1, 0)$. In this case the shortened equation $w^{n+1} = 0$ leads only to the solution $w = 0$ that cannot be the origin of power expansion.

Theorem 2. *The apex* $(1, -2n)$ *gives* $2n$*–parametric family of power expansions of holomorphic solutions.*

Proof. Theorem 2.3 from [2] shows a way how to find the explicit form of the power expansion. Let us divide the differential polynomial $f_n(w, z)$ from (3.1) into three terms, where $\mathcal{L}(w) = \dfrac{d^{2n}}{dz^{2n}} w$ corresponds to the apex $(1, -2n)$,

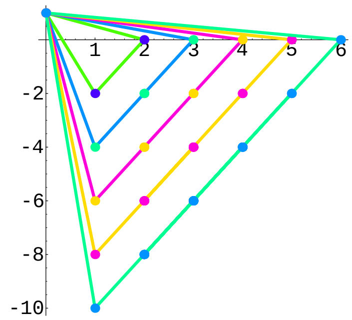

Figure 1. $\mathbf{\Gamma}(f)$ for $n = 1$, $n = 2$, $n = 3$, $n = 4$ and $n = 5$.

$h(z) = -z$ is independent of w, and $g(w,z) = -\dfrac{1}{4}\mathbf{d}^{n+1}(w) - w^{(2n)}$ contains the remaining terms. The characteristic equation

$$\nu(q) = z^{-q+2n}\mathcal{L}z^q = \prod_{k=0}^{2n-1}(q-k) = 0$$

has the roots $q_k = k,\ k = 0, 1, \dots, (2n-1)$. The differences of the coordinates of the carrier's points give the lattice $\mathbb{K} = \{(-k+l; (2n+1)k+2l) : k, l \in \mathbb{Z}\}$ with the base vectors $B_1 = (-1, 2n+1)$ and $B_2 = (1, 2)$. The exponents which take part in the expansion (2.1) are defined by the intersection of the lattice $\mathbb{K}$ and the axis $p = -1$ while $q > 0$. So by substituting $-k+l = -1$ in $(2n+1)k+2l = q$ we get $q = (2n+3)k-2$, where $k > 0$. The power expansion with these exponents is

$$w(z) = \sum_{k=1}^{\infty} c_k z^{(2n+3)k-2} = z^{2n+1}\sum_{l=0}^{\infty} c_l z^{(2n+3)l}. \tag{3.5}$$

As all conditions of Theorem 2.2 from [2] hold, this series converges and defines a holomorphic solution in some circle $|z| < \rho$ near $z = 0$. The coefficients c_k are uniquely defined by equation (3.1) with the fixed order n. So (3.5) is a power expansion of the solution with zero initial conditions.

The roots of the characteristic equation define the exponents in the initial condition $w_0(z) = \sum_{k=0}^{2n-1}\alpha_k z^k$, where α_k are arbitrary constants. A new lattice defined by the set $\mathbf{S}(f_n(w,z)) \bigcup \mathbf{S}(w_0(z))$ has the basis vectors $B_1 = (1, 0)$ and $B_2 = (0, 1)$. Hence any integer numbers greater than $(2n-1)$ can appear as an exponent in the power expansion of such solution

$$w(z) = \sum_{k=0}^{2n-1}\alpha_k z^k + z^{2n}\sum_{k=0}^{\infty} c_k z^k. \tag{3.6}$$

The existence of this solution corresponds the Cauchy theorem. ■

We have analyzed all three apices of $\mathbf{\Gamma}(f)$. Now let us consider the edges. We start with the edge $[(0,1);\,(1,-2n)]$. The corresponding shortened equation is

$$w^{(2n)} - z = 0.$$

It has the solution $w(z) = \dfrac{1}{(2n+1)!}z^{2n+1}$. This is the first member of a power expansion of some solution of (3.1). The direction vector of the edge's normal cone is $(-2n-1,-1)$. As the second coordinate is negative, so the corresponding power expansion contains only integer exponents greater than $(2n+1)$. Therefore according to the Cauchy theorem this expansion coincides with (3.5).

Theorem 3. *The edge* $[(0,1);\,(n+1,0)]$ *gives* $n+1$ *formal series.*

Proof. The shortened equation $\phi_n w^{n+1} - z = 0$, where ϕ_n is a corresponding coefficient, defined by this edge has $n+1$ solutions $\sqrt[n+1]{\frac{z}{\phi_n}}$ corresponding to different branches of the root. Taking into account that the direction vector of the edge's normal cone is $(1,n+1)$ and $(n+1)>0$ we conclude, that the exponents in the power expansion are less than $q<\frac{1}{n+1}$. Now we can change the independent variable in (3.1) following the rule $w(z) = \sqrt[n+1]{\frac{z}{\phi_n}} + u(z)$. Considering the changes in the carrier by the substitution we conclude, that all the points in the segment $[(1,-2n);\,(n+1,0)]$ remain, the point $(0,1)$ disappears (due to the shortened equation), and the point $(0,\frac{1}{n+1}-2n)$ is added due to the term $w^{(2n)}(z) = \Big(\sqrt[n+1]{\frac{z}{\phi_n}} + u(z)\Big)^{(2n)}$. Other new points do not influence the result, i.e. new lattice defined by this carrier has basis vectors $B_1 = (-1,\frac{1}{n+1})$ and $B_2 = (1,2)$. It intersects the axis $p=-1$ at the points with $q = \big(1+(2n+3)k\big)/(n+1)$ where $k<0$. So the power expansion for equation (3.1) is

$$w(z) = \sqrt[n+1]{\frac{z}{\phi_n}} + \sum_{k=-1}^{-\infty} c_k z^{\frac{1}{n+1}+\frac{2n+3}{n+1}k} = \sqrt[n+1]{\frac{z}{\phi_n}} + \sum_{l=0}^{\infty} c_l z^{-2-\frac{2n+3}{n+1}l}, \tag{3.7}$$

where the coefficients c_l are uniquely defined by choosing the branch of the $\sqrt[n+1]{\frac{z}{\phi_n}}$. ■

Note 1. Coefficient ϕ_n in this theorem are found from (3.2). For example, $\phi_1 = -6$, $\phi_2 = 40$, $\phi_3 = -280$ and generally

$$\phi_n = -(-4)^n \frac{(2n+1)!!}{(n+1)!}. \tag{3.8}$$

Unfortunately, expansion (3.7) does not meet the conditions of convergence of Theorem 2.2 from [2]. However one can prove that these series give asymptotic approximation of the solutions. For the first Painlevé equation it was done in [5] (see also [4]). Let us consider the problem for the fourth order equation (i.e. $n=2$ in (3.1)) within the following proposition.

Proposition 1. *For any sector of Ω opening by less than $\pi/3$ with apex at the origin there exists a solution of* (3.1) *for $n = 2$:*

$$w^{(4)} - 20ww'' - 10(w')^2 + 40w^3 - z = 0, \tag{3.9}$$

the asymptotic behavior of which as $z \to \infty$, $z \in \Omega$ is given by asymptotic series (3.7) *for $n = 2$:*

$$w(z) = \left(\frac{1}{40}z\right)^{\frac{1}{3}} + \sum_{k=0}^{\infty} c_k z^{-2-\frac{7}{3}k}. \tag{3.10}$$

Proof. The proof is based on the Wasow theorem [8] (Theorem 12.1, page 75). It's conditions require us to change the variables

$$\xi = \frac{9}{7}z^{7/9}, \quad u_1(\xi) = \frac{w}{\sqrt[3]{z}} - \frac{1}{\sqrt[3]{40}}, \quad u_2 = u_1', \quad u_3 = u_2', \quad u_4 = u_3',$$

then (3.10) becomes $u_1(\xi) = \frac{1}{\sqrt[3]{40}} \sum_{l=1}^{\infty} \tilde{c}_l \xi^{-3l}$, where $\tilde{c}_l = \sqrt[3]{40}\,(7/9)^3 c_{l-1}$ and equation (3.9) becomes

$$\begin{cases} u_1' = & u_2, \\ u_2' = & u_3, \\ u_3' = & u_4, \\ u_4' = & \frac{140}{9}\xi u_1 u_3 + \frac{14\sqrt[3]{25}}{9}\xi u_3 + \frac{80}{49}\xi^{-2}u_3 + \frac{70}{9}\xi u_2^2 + \frac{140}{9}\xi u_1 u_2 \\ & -\frac{160}{49}\xi^{-3}u_2 + \frac{14\sqrt[3]{25}}{9}u_2 - \frac{1960}{81}\xi^2 u_1^3 - \frac{196\sqrt[3]{25}}{27}\xi^2 u_1^2 - \frac{30}{7}\xi^{-1}u_1^2 \\ & -\frac{98\sqrt[3]{5}}{27}\xi^2 u_1 - \frac{6\sqrt[3]{25}}{7}\xi^{-1}u_1 + \frac{6480}{2401}\xi^{-4}u_1 - \frac{3\sqrt[3]{5}}{14}\xi^{-1} - \frac{648\sqrt[3]{25}}{2401}\xi^{-4}. \end{cases}$$

So the Wasow theorem holds with the following parameters $r = 4$, $q = 2$, $b_j = \sqrt[3]{40}\,(7/9)^3 c_{j-1}$, and the eigenvalues of the matrix of the linear part are nonzero and equal to the four different values of $5^{1/12}\sqrt{7}\sqrt[4]{-2/27}$. Consequently, the series (3.10) is asymptotic for some solution of (3.9). ■

One can definitely expect the behaviour of the type described from expansion (3.7) for any n. However as there is no explicit form of equation (3.1), this fact is hard to prove in general.

Theorem 4. *The edge $[(1, -2n); (n+1, 0)]$ gives n families of polar power expansions.*

Proof. The edge defines the shortened equation $\mathbf{d}^{n+1}(w) = 0$. The normal cone for this edge is the ray with the direction vector $(-2, 1)$. Therefore we look for a solution in the form $\eta = bz^{-2}$. The coefficient b can be found from the polynomial equation $\mathbf{d}^{n+1}(bz^{-2}) = 0$.

Lemma 1. *The equation $\mathbf{d}^{n+1}(bz^{-2}) = 0$ has the following roots:*

$$b_m = \frac{m(m+1)}{2}, \quad m = 0, 1, \cdots, n. \tag{3.11}$$

Proof. From the definition (1.2) we obtain $\mathbf{d}^1(bz^{-2}) = -4bz^{-2} = 0$, so $b_0 = 0$. It follows from the structure of operator $\mathbf{d}^{n+1}$ (3.2) that for any b the equality $\mathbf{d}^n(bz^{-2}) = 0$ necessitates $\mathbf{d}^{n+1}(bz^{-2}) = 0$. However, due to the term $-8w\mathbf{d}^n(w)$ in (3.2) the order of $\mathbf{d}^{n+1}(bz^{-2}) = 0$ as an algebraic equation is higher than the order of $\mathbf{d}^n(bz^{-2}) = 0$ and there can be some other roots. Hence let us consider the equation $\dfrac{\mathbf{d}^{n+1}(bz^{-2})}{\mathbf{d}^n(bz^{-2})} = 0$. As $\mathbf{d}^n(bz^{-2}) = c(b)\, z^{-2n}$, then $\dfrac{\frac{d^2}{dz^2}\mathbf{d}^n(bz^{-2})}{\mathbf{d}^n(bz^{-2})} = 2n(2n-1)z^{-2}$ and $\dfrac{\int \mathbf{d}^n(bz^{-2})d(bz^{-2})}{\mathbf{d}^n(bz^{-2})} = \dfrac{b}{n+1}z^{-2}$, so the equation has the following form

$$\frac{\mathbf{d}^{n+1}(bz^{-2})}{\mathbf{d}^n(bz^{-2})} = 2n(2n-1)z^{-2} - 8bz^{-2} + 4\frac{b}{n+1}z^{-2} = 0.$$

We get only one root $b = n(n+1)/2$. ■

To finish the proof of Theorem 4 we need another important fact.

Lemma 2. *Changing the variable* $w = b_m z^{-2} + u$ *in equation* (3.1) *does not affect its carrier (see Fig. 1).*

Proof. Let us consider the result of changing variable for the expression $(w^{(t)})^k$. We get

$$\begin{aligned}\mathbf{S}\big((u^{(t)} + \beta z^{-2-t})^k\big) &= \mathbf{S}\Big(\sum_{l=0}^{k} C_l^k (u^{(t)})^l (z^{-2-t})^{k-l}\Big)\\ &= \{(l, -tl - (2+t)(k-l)) :\ \ l = 0, 1, \dots, k\}\\ &= \{(0, -(2+t)k) + (1,2)l :\ \ l = 0, \dots, k\}.\end{aligned}$$

The differential polynomial $\mathbf{d}^{n+1}(w)$ contains only the monomials of the type $m = C(w^{(t_1)})^{k_1}(w^{(t_2)})^{k_2}\dots(w^{(t_r)})^{k_r}$, where the orders of the derivatives $t_i \le 2n$. As by multiplication of monomials their carriers sum up, we get after changing the variables

$$\mathbf{S}(m) = \{(0, -2n-2) + (1,2)(l_1 + l_2 + \dots + l_r) :\ l_i = 0, 1, \dots, k_i,\ i = 1, 2, \dots, r\}.$$

In case of $(l_1 + l_2 + \dots + l_r) > 0$ we get only the points from the edge $[(1, -2n);\ (n+1, 0)]$. And in case of $(l_1 + l_2 + \dots + l_r) = 0$ we get the point $(0, -2n-2)$, but due to the shortened equation $\mathbf{d}^{n+1}(b\, z^{-2}) = 0$ the coefficient of the summand $C\, z^{-2n-2}$ corresponding to the point $(0, -2n-2)$ equals zero. Hence the point must be excluded from the equation carrier. ■

Now let us finish the proof of Theorem 4. Although the carrier of the equation (3.1) within the change of the variable remains the same, the operator $\mathcal{L}$ is changing (following the proof of Lemma 2) and hence there will be another characteristic equation $\nu(q) = z^{-q+2n}\mathcal{L}z^q$ for every b_m chosen. As all the apices and edges of $\Gamma(f)$ have been considered, we conclude that the next summand in power expansion will have the exponent $2n+1$ due to the direction vector $(-2n-1,-1)$ of the normal cone of the edge $[(0,1);\ (1,-2n)]$. This summand also will have a constantly defined coefficient, because the corresponding shortened equation leads to a first order algebraic equation. So we have n families of polar power expansions:

$$w_m(z) = b_m z^{-2} + \sum_{l=2n+1}^{\infty} c_{m,l}\, z^l, \tag{3.12}$$

where $c_{m,2n+1}$ is constant and some of $c_{m,l}$ are parameters. This series matches the conditions of Theorems 2.2 and 2.3 from [2] therefore they converge in some deleted neighborhood $|z| < \rho$ of $z = 0$. ■

This result can be illustrated with the following example. Let us consider $n = 2$.

Proposition 2. *The equation* ${}_4P_1$ *(defined by* (3.1) *with* $n = 2$*) has one-parametric* (3.13) *and two-parametric* (3.14) *families of polar solutions defined by series* (3.12)*.*

Proof. For $b_1 = 1$ after the change of variable $w = z^{-2} + u$ we have the equation

$$u^{(4)} - 20uu'' - 20\frac{u''}{z^2} - 10(u')^2 + 40\frac{u'}{z^3} + 40u^3 + 120\frac{u^2}{z^2} - z = 0.$$

It defines

$$\mathcal{L} = \frac{d^4}{dz^4} - 20\frac{1}{z^2}\frac{d^2}{dz^2} + 40\frac{1}{z^3}\frac{d}{dz}$$

and the characteristic equation $\nu(q) = q(54 - 9q - 6q^2 + q^3) = 0$ with the roots $q_1 = -3$, $q_2 = 0$, $q_3 = 3$, $q_4 = 6$. The shortened equation corresponding to the edge $[(0,1);\ (1,-4)]$ is

$$u^{(4)} - 20\frac{u''}{z^2} + 40\frac{u'}{z^3} - z = 0$$

and within the substitution $u = \beta z^5$ it gives $\beta = -\frac{1}{80}$. While $2n+1 = 5$ and only $q_4 > 5$, we get one-parametric family of power expansions

$$w_1(z) = z^{-2} - \frac{1}{80}z^5 + \alpha z^6 + \sum_{l=7}^{\infty} c_{1,l} z^l, \tag{3.13}$$

where α is an arbitrary constant, that uniquely defines coefficients $c_{1,l}$.

For $b_2 = 3$ after the change of variable $w = 3z^{-2} + u$ we have the equation:

$$u^{(4)} - 20uu'' - 60\frac{u''}{z^2} - 10(u')^2 + 120\frac{u'}{z^3} + 40u^3 + 360\frac{u^2}{z^2} + 720\frac{u}{z^4} - z = 0$$

It has characteristic equation

$$\nu(q) = -720 - 174q + 49q^2 + 6q^3 - q^4 = 0$$

with the roots $q_1 = -5$, $q_2 = -3$, $q_3 = 6$, $q_4 = 8$. The shortened equation

$$u^{(4)} - 60\frac{u''}{z^2} + 120\frac{u'}{z^3} + 720\frac{u}{z^4} - z = 0$$

by the substitution $u = \beta z^5$ gives $\beta = -\frac{1}{240}$. As this time q_3, $q_4 > 5$ we have the two-parametric family of power expansions:

$$w_2(z) = z^{-2} - \frac{1}{240}z^5 + \alpha_1 z^6 + \alpha_2 z^8 + \sum_{l=9}^{\infty} c_{2,l} z^l \tag{3.14}$$

where again α_1 and α_2 are arbitrary constants, which uniquely define coefficients $c_{2,l}$. The proposition is proved. ■

Thus all the generalized bounds of the carrier of the equation (3.1) (see Fig. 1) have been considered, so the adduced analysis is complete and proves the following result.

Theorem 5. *The equation* (3.1) *has only the following power expansions of solutions:*

1. *$2n$-parametric family of holomorphic solutions* (3.6) *near* $z = 0$,
2. *n families of polar solutions* (3.12) *near* $z = 0$,
3. *$n + 1$ formal power series* (3.7) *near* $z = \infty$.

Let us note, that the power expansions found in the neighborhood of $z = 0$ can be also used while looking for the solutions near an arbitrary point $z_0 \in \mathbb{C}$ within the substitution $z = \tilde{z} + z_0$.

4. Conclusion

The three types power expansions of found for ${}_{2n}P_1$ hierarchy completely coincide with the ones found for the first Painlevé equation. This is another corroboration of the deep analogy of these equations and another argument for the hypothesis, that all the ${}_{2n}P_1$ hierarchy has the Painlevé property.

Another important conclusion is that the power geometry method can be used to find higher order analogues. As follows from Theorem 1 and Fig. 1 scaling a carrier of an equation could uncover the structure of other equations with the Painlevé property.

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