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Boundedness in a Biofilm-Chemotaxis Model in Evolving Porous Media

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Abstract. The article concerns with a model discribing the growth of a biofilm made by chemotactical bacteria within a saturated porous media and affects the flow through the pores. The underlying model describing this process on the macro-scale is derived in [21]. Therein also solvability in a weak sense and boundedness of solutions with high regularity is investigated. This current paper verifies the existence of bounded weak solutions in case of less regularity assumptions on the data.

Keywords: evolving porous media, biological fluid mechanics, chemotaxis, boundedness, parabolic PDE.

AMS Subject Classification: 76S05, 74F10, 76Zxx, 35K55, 35Q92, 92C17.

1 Introduction

In porous media, biofilms, which attach on the surface of the solid matrix, occupy pore space. Since microbial biofilms can significantly affect the hydrodynamics (porosity, permeability, diffusivity, etc.) of a porous medium, cf. [7], they also influence the flow as well as the mass transport of dissolved substances within the pores. Thus, such microbes can be used in building bio-barriers that restrict the flow of ground water, e.g. to control the propagation of contaminants. Besides, in filter systems they lead to an unwanted decrease in efficiency (biofouling). Further examples of the beneficial applicability of biofilms in porous media are subsurface remediation, carbon sequestration, and enhanced oil recovery.

In terms of non-rigid porous media, the literature offers only a small number of analytical results. Upscaled diffusion–precipitation equations with effective coefficients coupled with a level set equation are considered in [22]. An effective model describing biofilm growth in porous media is already derived in [20], but neglects completely chemotactical effects.

Some bacteria even are able to move chemotactically, thereby is chemotaxis the ability of organisms to direct their movements according to certain chemicals in their environment. The significant role of chemotaxis of bacteria in porous media has been studied by numerous researchers, see e.g. [6, 10, 27]. Indeed, the averaged bacterial velocity caused by chemotaxis is of a similar scale as the averaged velocity of groundwater flow in fine-pored soil: $\sim 1m/\text{day}$. The chemotactical movement is controlled in such a gradient in the direction of higher concentrations of attractive substances:

$$\nabla \cdot (b\chi(n)\nabla n)\,,$$

where b is the concentration of bacteria, n is the concentration of attractive substances and $\chi(n) := \frac{\chi_0 K}{(K+n)^2}$ denotes the chemotactic sensitivity for some constants $\chi_0, K > 0$. Owing to this high order nonlinear term, chemotaxis leads to mathematical difficulties which have been investigated by several authors, see e.g. [1, 3, 4, 13, 32]. For example, it is well known that classical solutions to the so called Keller-Segel system

$$\begin{cases} \partial_t b - \Delta b + \nabla \cdot (b\nabla n) = 0, & \text{in } \Omega \times (0,T), \\ \partial_t n - \Delta n = -n + b, & \text{in } \Omega \times (0,T) \end{cases}$$

remain bounded when either the dimension is 2 or the total mass $\int_\Omega b_0$ of the initial data is sufficiently small. In contrast, finite in time blow-up does occur for large classes of initial data if one of these both conditions is not satisfied, e.g. cf. [15, 30]. Hence, the boundedness of solutions to chemotactical systems is of particluar interest. A possible way to overcome this difficulty is to add a self-diffusive motility flux of the bacteria, i.e. the diffusion coefficient is assumed to be bounded from below by $c_b b^{\bar{m}}$ for some $c_b, \bar{m} > 1$, cf. [5]. Such a regularizing effect can be justified since the random mobility of the bacteria may increase for large concentrations. In [12, 25] such a chemotaxis system is coupled with the Stokes equations and describes the motion of swimming bacteria in an incompressible fluid. It turns out that in the two-dimensional case marginal self-diffusivity $\bar{m} > 0$ suffices to entail a bounded solution to this chemotaxis-Stokes model. However, to ensure the existence of a bounded global in time weak solution to the three-dimensional model [2] determines a lower bound for $\bar{m}$ which is improved recently in [31] to $\bar{m} > 7/6$. In [26] this condition is weakened to $\bar{m} > \frac{8}{7}$ to obtain global-in-time weak solutions still bounded within each finite time interval $(0,T)$. In this case, the solution may become unbounded in the limit $T \to \infty$.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with smooth boundary $\partial\Omega$ and be filled with a saturated porous medium. In [21] the growth of a biofilm produced by chemotactical bacteria within this porous medium is considered and an upscaled model for this evolution is formally derived. This model describes the behavior of the flow, the biofilm, and the transport of mobile bacteria as well as nutrients in $\Omega_T := \Omega \times (0,T)$:

Darcy's law:

$$\begin{aligned} q &= -\mathbb{K}(\theta)\nabla p, \qquad (1.1a) \\ \nabla \cdot q &= -\partial_t \theta. \end{aligned}$$

Transport equations:

$$\partial_t(\theta b) = \nabla \cdot \big(\mathbb{D}_b(\theta, b)\nabla b - \theta b \chi(n)\nabla n - qb\big) + \theta(R_{\mathrm{mon},l}(b,n) - b) + R_I(b,n,\theta), \tag{1.1b}$$

$$\partial_t(\theta_s n) = \nabla \cdot \big(\mathbb{D}_n(\theta)\nabla n - qn\big) - \theta R_{\mathrm{mon},l}(b,n) - (\theta_s - \theta)\, R_{\mathrm{mon},b}(n). \tag{1.1c}$$

Change of porosity:

$$\partial_t \theta \quad = \quad R_b(n,\theta) + R_I(b,n,\theta) \tag{1.1d}$$

with homogeneous Dirichlet boundary conditions, i.e.

$$b = c = n = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad q \cdot \nu = 0 \quad \text{on } \partial\Omega\,. \tag{1.1e}$$

The unknowns of the above equations (1.1) are b, n, θ, v and p denoting the concentration of the bacteria and the nutritious substances, the porosity as well as the velocity and the pressure of the fluid, respectively. The hydrodynamic parameters namely the permeability $\mathbb{K}$ of the porous medium, the diffusivity with respect to the bacterial transport $\mathbb{D}_b$ and to the transport of the nutrients $\mathbb{D}_n$ depend on the porosity θ. Due to the assumed self-diffusivity of the microbes, the diffusivity $\mathbb{D}_b$ is additionally depending on b. The change of porosity is caused by inner reactions R_b (bacterial reproduction and mortality) as well as by de-/attachment reactions R_I along the bioflim-fluid interface at the pore-scale. Furthermore, $R_{\mathrm{mon},l}$ and $R_{\mathrm{mon},b}$ denote the bacterial reproduction within the fluid and the biofilm, respectively. The function χ describes the chemotactical sensitivity. The clean surface porosity $\theta_s \in (0,1)$ is the sum of volume fractions of the fluid and the biomass.

The solvability in a weak sense (cf. Definition 1) is shown in [21] in the two-dimensional case if slight self-diffusivity is provided. With more regularity assumptions on the initial data unique existence global in time or at least up to a possible clogging phenomenon is proven. Let us remark at this point that (post-)clogging phenomena are not considered. In [21] the proof for boundedness of the solution requires sufficient regularity assumptions on the initial data, cf. (2.5) below. These assumptions are necessary to control the nonlinear chemotactical term. In more detail, an estimate

$$\sup_{t\in(0,T)} \|\nabla n(t)\|_\gamma < C(\gamma) \quad \text{for all } \gamma < \infty \tag{1.2}$$

with an appropriate upper bound $C(\gamma)$ is needed to apply an iteration procedure of Moser type and hence obtain boundedness of b.

This current article verifies bounded weak solutions in case of less regularity assumptions on the data. For this we use entropy-type estimate and proceed mainly as in [25]. Together with a semigroup approach, this method enables us to establish an estimate of type (1.2). Contrary to [25], including the porosity requires an adaption of the applied estimates. Furthermore, we extend this method to the three-dimensional case. Under rather weak regularity assumptions on the initial data and a self-diffusivity of order $m > \frac{2}{9}(d-2)$ we finally prove the existence of a bounded weak solution up to a possible clogging phenomenon. At this point we emphasize that the following investigations and appropriate results may be obtained for general models describing chemotactical movements in evolving microstructures. That means the porous media is not restricted to change only due to biofilm growth but also because of heterogenous reactions (dissolution of minerals [28], precipitation of electrical charged particles [19], etc.).

Beside the intrinsic mathematical interest the result of this current article is also of significance for real applications. The underlying biofilm-chemotaxis model is not able to describe the behavior of a specific microorganism in detail. Nevertheless, it presents the phenomenological interactions and importance of several physical variables. In this sense, also this paper provides a qualitative statement of corresponding solutions which is a first step for understanding the investigated process.

This paper is organized as follows. In Sect. 2 we present the main results, Theorem 1 and 2.3 on boundedness of strong solutions and on the existence of bounded weak solutions, respectively. Next, in Sect. 3 we establish several auxiliary lemmata. In particular, we obtain the useful estimate for ∇n of type (1.2), see Lemma 4 below. Sect. 4 and 5 concerns with the proofs of Theorem 1 and 2.3. Finally, the article ends with a brief conclusion.

2 Main result

Let us introduce the following function spaces

$$H^1_* := L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)),$$
$$H^2_* := L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) \cap H^1_*,$$
$$\mathcal{Y} := H^1(0,T;L^2(\Omega)), \quad \mathcal{Z} := L^2(0,T;H^1_{\mathrm{div},0}(\Omega)) \times L^2(0,T;L^2(\Omega)/\mathbb{R}) \quad \text{with}$$
$$H^1_{\mathrm{div},0}(\Omega) := \{\phi \in L^2(\Omega)^2 \,|\, \nabla \cdot \phi \in L^2(\Omega) \text{ and } \phi \cdot \nu = 0 \text{ on } \partial\Omega\}.$$

We denote the essential solution space by $\mathbb{X}_0 := H^1_* \times H^2_* \times \mathcal{Y} \times \mathcal{Z}$.

In the following the norm of the Banach space $L^p(\Omega)$, $p \in [1,\infty]$, is denoted by $\|\,.\,\|_p$. All the rest of occuring norms below are denoted intuitively. Throughout this article C describes positive constants, where the value may differ from one occasion to another. The indices in the constant $C(\cdot,...,\cdot)$ indicate the dependence of parameters.

Before presenting the main results we define the reaction functions occuring in (1.1). Let us introduce the notation $[a]_+ := \max\{0,a\}$ for all $a \in \mathbb{R}$, the averaged macroscopic de-/attachment rates $R_{\substack{\mathrm{det}\\ \mathrm{att}}}$ as well as the following terms

$$R_I(b,n,\theta) := |\Gamma_I(\theta)|\,(R_{\mathrm{det}} - R_{\mathrm{att}})\,, \quad R_b(n,\theta) := [\theta_s - \theta]_+\,(1 - R_{\mathrm{mon},b}(n))\,,$$

which separate the processes leading to a change of the biofilm at the interface R_I from inner de-/increase R_b. Here $|\Gamma_I|$ denotes the specific surface of the biofilm-fluid interface. Furthermore, we assume bacterial reproduction of Monod-type

$$R_{\mathrm{mon},l}(b,n) := \mu_l \frac{n}{1+|n|} b\,, \quad R_{\mathrm{mon},b}(n) := \mu_b \frac{n}{1+|n|}$$

and the chemotactic coefficient

$$\chi(n) := \frac{\chi}{(1+|n|)^2}\,.$$

These definitions are motivated in [20, 21]. However, the following analytical results hold more generally if the functions above are at least assumed to be bounded and sufficiently smooth.

In the following we list several integrability and continuity assumptions on the parameters $\mathbb{K}, \mathbb{D}_b, \mathbb{D}_n$ as well as on the reaction functions. Let the interface reaction rate R_I be bounded, sufficiently smooth and a Lipschitz continuous map $R_I : \tilde{\mathbb{X}} \to L^\infty(\Omega_T)$, depending on $(b, n, \theta) \in \tilde{\mathbb{X}} := L^2(0, T; L^2(\Omega))^2 \times \mathcal{Y}$, i.e. for some constant $L > 0$ there holds the estimate

$$\begin{aligned}\|R_I(b_1, n_1, \theta_1) - R_I(b_2, n_2, \theta_2)\|_{L^\infty(\Omega_T)} \\ \le L\|(b_1 - b_2, n_1 - n_2, \theta_1 - \theta_2)\|_{\tilde{\mathbb{X}}}\end{aligned} \tag{2.1a}$$

for all $(b_i, n_i, \theta_i) \in \tilde{\mathbb{X}}$, $i = 1, 2$. Also the Jacobian $\mathcal{D}R_I : \tilde{\mathbb{X}} \to L^\infty(\Omega_T)^3$ with respect to (b, n, θ) is assumed to be bounded and Lipschitz continuous in the sense of (2.1a). Similar properties hold for the nonnegative functions $R_b, R_{\text{mon},l}/b, R_{\text{mon},b}$ and χ. To ensure $\theta \le \theta_s$ and also nonnegativity of the concentrations b and n an additional assumption on R_I is needed, cf. [21]:

$$R_I(\theta) \le 0 \quad \text{for } \theta \ge \theta_s \quad \text{and} \quad R_I(b) \ge 0 \quad \text{for } b < 0\,, \tag{2.1b}$$

that means in particular with respect to our underlying model that in case of a clear surface attachment of bacteria is the dominant interface reaction.

The effective permeability $\mathbb{K}$, the effective diffusivity parameters $\mathbb{D}_b$ as well as $\mathbb{D}_n$ are assumed to be symmetric, continuous and uniformly positive definite with respect to θ within every interval $[\theta_{\text{clog}} + \delta, \theta_s]$, $\delta > 0$, cf. [20], where $\theta_s \in (0, 1)$ and $\theta_{\text{clog}} := \inf\{\theta \in [0, \theta_s) \,|\, \partial Y \cap \Gamma_I(\theta) = \emptyset, x \in \Omega\} \in [0, 1)$. Furthermore, we assume these maps depending on θ (and the derivatives) also to be Lipschitz continuous with respect to their corresponding (co)domains:

$$\mathbb{K}^{-1}, \mathbb{D}_b, \mathbb{D}_n : \mathcal{Y}^\delta \to L^\infty(\Omega_T)^{(d,d)}\,, \tag{2.2a}$$

where we define $\mathcal{Y}^\delta := \{\theta \in \mathcal{Y} \,|\, \theta(x, t) \in [\theta_{\text{clog}} + \delta, \theta_s] \text{ a. e. } (x, t) \in \Omega_T\}$ for $\delta > 0$. Moreover, beside the "standard" diffusivity caused by the fluid we additionally assume self-diffusivity of the bacteria, i.e. $\mathbb{D}_b$ does not only depend on θ but also on b in such a way that for some $c_b, C_b > 0$ and $M \ge m \in (0, 2)$

$$c_b(|b|^m + 1) \le \mathbb{D}_b(b, \theta) \le C_b(|b|^M + 1) \tag{2.2b}$$

and the Jacobian of $\mathbb{D}_b$ with respect to the argument (b, θ) can be estimated by

$$\mathcal{D}\mathbb{D}_b(b, \theta) \le C_b(|b|^M + 1)\,. \tag{2.2c}$$

Definition 1. A tuple $(b, n, \theta, q, p) \in \mathbb{X}_0$ of functions is called a *weak solution* to the upscaled model (1.1) if for all test functions $(\varphi_1, \varphi_2, \varphi_3) \in H^1_0(\Omega) \times$

$H^1_{\mathrm{div},0}(\Omega) \times L^2(\Omega)$ and a. e. $t \in (0,T)$ there holds

$$\int_\Omega \mathbb{K}^{-1}(\theta) q \cdot \varphi_2 = -\int_\Omega p \nabla \cdot \varphi_2 \,, \quad \int_\Omega (\nabla \cdot q)\varphi_3 = -\int_\Omega (\partial_t \theta)\varphi_3 \,, \tag{2.3a}$$

$$\begin{aligned}\langle \partial_t(\theta b), \varphi_1 \rangle_{H^{-1},H^1_0} = &-\int_\Omega \big(\mathbb{D}_b(\theta)\nabla b - \theta b \chi(n) \nabla n - qb\big)\nabla \varphi_1 \\ &+ \int_\Omega \big(\theta R_{\mathrm{mon},l}(b,n) - \theta b + R_I(b,n,\theta)\big)\varphi_1 \,,\end{aligned} \tag{2.3b}$$

$$\begin{aligned}\int_\Omega \theta_s (\partial_t n)\varphi_1 = &-\int_\Omega (\mathbb{D}_n(\theta)\nabla n - qn\big)\nabla \varphi_1 \\ &- \int_\Omega \big(\theta R_{\mathrm{mon},l}(b,n) + (\theta_s - \theta)\, R_{\mathrm{mon},b}(n)\big)\varphi_1 \,,\end{aligned} \tag{2.3c}$$

$$\int_\Omega (\partial_t \theta)\varphi_3 = \int_\Omega \big(R_b(n) + R_I(b,n,\theta)\big)\varphi_3 \tag{2.3d}$$

and if (b,n,θ) takes the initial value $(b_0,n_0,\theta_0) \in L^2(\Omega)^3$ in the sense

$$|\langle b(t) - b_0, \phi \rangle_{L^2,L^2}| + |\langle n(t) - n_0, \phi \rangle_{L^2,L^2}| + |\langle \theta(t) - \theta_0, \phi \rangle_{L^2,L^2}| = 0 \tag{2.3e}$$

for all $\phi \in L^2(\Omega)$ if $t \searrow 0$.

We say a weak solution $(b,n,\theta,q,p) \in \mathbb{X}_0$ is *strong*, if the triple (b,n,θ) even belongs to the space

$$H^2_* \times \big(L^\infty(0,T;H^2(\Omega)) \cap H^1(0,T;H^1(\Omega))\big) \times L^\infty(0,T;W^{1,\kappa}(\Omega))$$

for some $\kappa > d$. The additional assumption on θ guarantees the continuity of the θ-depending parameters. Let us remark that a strong solution satisfies not only (2.3c) but also

$$\begin{aligned}\int_\Omega \theta_s (\partial_t n)\varphi_3 = &\int_\Omega \nabla \cdot \big(\mathbb{D}_n(\theta)\nabla n - qn\big)\varphi_3 \\ &- \int_\Omega \big(\theta R_{\mathrm{mon},l}(b,n) + (\theta_s - \theta)\, R_{\mathrm{mon},b}(n)\big)\varphi_3\end{aligned} \tag{2.4}$$

for all $\varphi_3 \in L^2(\Omega)$ and a. e. $t \in (0,T)$.

In case of $d = 2$, [21] ensures boundedness of the solution for initial data with high regularity

$$b_0 \in L^\infty(\Omega) \cap H^1_0(\Omega)\,, \quad n_0 \in H^1_0(\Omega) \cap H^2(\Omega) \ \text{ and } \ \theta_0 \in W^{1,\rho}(\Omega) \tag{2.5}$$

for some $\rho > 2 + \sqrt{2}$. On the other hand, for $d = 3$ sufficiently smooth initial data lead also to a strong solution, but boundedness can not be verified with the methods applied in [21]. Therein the inequality (1.2) is obtained via the Solobev embedding $W^{1,d}(\Omega) \hookrightarrow L^\gamma(\Omega)$ for all $\gamma < \infty$. But the standard parabolic L^2-theory enables to estimate $\sup_{t\in(0,T)} \|\nabla n(t)\|_{H^1}$. However, strong L^p-theory of parabolic equations lead to bounded (strong) solutions for smooth initial data if $d = 3$.

The purpose of this work is to prove for $d = 2$ the existence of bounded global in time weak solutions to (1.1) corresponding to the initial data

$$b_0 \in L^\infty(\Omega) \cap H^{\frac{1}{3}}(\Omega) =: \mathcal{X}\,, \quad n_0 \in W_0^{1,\infty}(\Omega) \quad \text{and} \quad \theta_0 \in W^{1,\infty}(\Omega) \tag{2.6}$$

if clogging does not occur during the evolution. Otherwise we show the existence of a bounded weak solution to (1.1) associating with (2.6) up to such a clogging phenomenon. Actually, in order to reduce the regularity of b and n we assume more integrability on θ_0.

Beside the case $d = 2$, we deal with 3 dimensions. In this case we assume the diffusivity parameter $\mathbb{D}_n$ to be independent of θ, i.e. $\mathbb{D}_n$ is a constant tensor. But this models equality of the diffusivity in the fluid D_l and in the biofilm D_b. Experimental measurements show that D_b/D_l ranges from $0,2$ to at least $0,8$ when water represents the fluid, cf. [23]. In [33] a scheme for numerical calculation of this ratio is provided. However, the simplification of a constant diffusivity tensor $\mathbb{D}_n$, i.e. $D_b/D_l = 1$, is less of practical importance, but rather of mathematical interest. In this case, the transport of the nutrients in pores with different biomass only differs due to advection, since the diffusion coincides. As already mentioned, in the three-dimensional case we do not obtain boundedness of strong solutions via the methods of [21].

An estimate of the L^∞-norms for strong solutions is established for $d = 2$ in [21], where the upper bound depend on rather strong norms of the initial data. In order to transfer the property of boundedness from strong to weak solutions, the L^∞-norms should be estimated from above uniformly by some $C(\|b_0\|_{\mathcal{X}}, \|n_0\|_{W_0^{1,\infty}}, \|\theta_0\|_{W^{1,\infty}}, T) > 0$. However, the following result yields such a wanted estimate for strong solutions.

Theorem 1 [Boundedness of the strong solutions]. *Let (b, n, θ, q, p) be a strong solution to (1.1) in $\Omega \times (0, T_{\max})$, $T_{\max} \in (0, \infty]$, corresponding to nonnegative initial data (b_0, n_0, θ_0) of type (2.6) and $m > \frac{2}{9}(d-2)$. Then for all $T < T_{\max}$ there holds*

$$\sup_{t\in(0,T)} \|b(t)\|_\infty < C(\|b_0\|_{\mathcal{X}}, \|n_0\|_{W_0^{1,\infty}}, \|\theta_0\|_{W^{1,\infty}}, T)\,.$$

Proceeding as in [25] we approximate the initial data in (2.6) with a sequence of smooth initial data inducing bounded strong solutions. Owing to the previous theorem, the L^∞-norms of the approximative solutions may be estimated from above uniformly by an appropriate constant. Then a compactness argument yields the existence of a bounded weak solution (up to a possible clogging phenomenon) corresponding to the initial data (2.6).

Theorem 2 [Existence of bounded weak solutions]. *Let $m > \frac{2}{9}(d-2)$ and (b_0, n_0, θ_0) be a nonnegative initial data satisfying (2.6) with $\theta_0(x) \in (\theta_{\text{clog}} + \delta, \theta_s(x))$ for a. e. $x \in \Omega$. Moreover, let (2.1) and (2.2) be satisfied. Then there exists a weak solution $(b, n, \theta, q, p) \in \mathbb{X}_0$ solving equations (2.3) either global in time (i.e. $T_{\max} := \infty$) or there exists an instant $T_{\max} > 0$ such that $\lim_{t\to T_{\max}} \|\theta^{-1}(t)\|_\infty = \theta_{\text{clog}}^{-1}$. In particular, the solution satisfies $b(x,t)$, $n(x,t) \geq 0$, $\theta(x,t) \in \big(\theta_{\text{clog}}, \theta_s(x)\big]$ for a. e. $(x,t) \in \Omega_{T_{\max}}$ and*

$$\sup_{t\in(0,T)} \|b(t)\|_\infty + \sup_{t\in(0,T)} \|n(t)\|_\infty < \infty \qquad \textit{for all } T < T_{\max}\,.$$

3 A gradient estimate for n

First of all let us briefly introduce the following useful Lemmata. Boundedness of n, θ and q is obtained easily:

Lemma 1. *Let* (b, n, θ, q, p) *be a nonnegative weak solution to* (2.3). *Then* n *and* θ *are bounded, i.e.*

$$\sup_{t\in(0,T)} \|n(t)\|_\infty + \sup_{t\in(0,T)} \|\theta(t)\|_\infty < \infty \qquad \text{for all } T < T_{\max}\,.$$

Proof. For $\gamma \geq 2$ we test $n^{\gamma-1}$ to (2.3c) and obtain with (2.3a)

$$\frac{1}{\gamma}\frac{d}{dt}\int_\Omega \theta_s n^\gamma + \frac{1}{2}(\gamma-1)\|n^{\frac{\gamma-2}{2}}|\nabla n|\|_2^2 \\ \leq (\gamma-1)\int_\Omega q n^{\gamma-1}\nabla n = -\frac{\gamma-1}{\gamma}\int_\Omega \partial_t\theta n^\gamma\,.$$

Gronwall's Lemma yields

$$\sup_{t\in(0,T)} \|n(t)\|_\infty = \sup_{t\in(0,T)}\left(\lim_{\gamma\to\infty}\|n(t)\|_\gamma\right) \leq \frac{2}{\delta}\|n_0\|_\infty \exp\left(\|\partial_t\theta\|_\infty T\right)\,.$$

The boundedness of θ follows directly from the boundedness of R_b and R_I. □

Lemma 2. *Let* (b, n, θ, q, p) *be a nonnegative strong solution to* (2.3). *Then* q *is bounded, i.e.* $\sup_{t\in(0,T)}\|q(t)\|_\infty < \infty$ *for all* $T < T_{\max}$.

Proof. We consider the elliptic equation $-\nabla\cdot(\mathbb{K}(\theta(t))\nabla\hat{p}(t)) = -\partial_t\theta(t)$ in Ω for a. e. $t \in (0,T)$. Owing to the continuity of $\mathbb{K}(\theta(t)) \in W^{1,\kappa}(\Omega) \hookrightarrow C(\bar{\Omega})$ there exists a unique strong solution $\hat{p}(t) \in W^{2,\kappa}(\Omega)$ satisfying $\|\hat{p}(t)\|_{W^{2,\kappa}} \leq C(\rho,\mathbb{K})\|\partial_t\theta\|_\infty$, cf. [8]. Defining $\hat{q} := -\mathbb{K}(\theta)\nabla\hat{p} \in W^{1,\rho}(\Omega)$ the tuple $(\hat{q},\hat{p}) \in \mathcal{Z}$ solves the equations (2.3a) and $\sup_t\|\hat{q}(t)\|_\infty \leq C(\mathbb{K})\sup_t\|\partial_t\theta(t)\|_\infty < \infty$. Due to the uniqueness of solutions to (2.3a) in $\mathcal{Z}$ there holds $(\hat{q},\hat{p}) = (q,p)$. □

In the following we establish a gradient estimate for n of type (1.2) if there holds $\sup_{t\in(0,T)}\|b(t)\|_\gamma \leq C(\gamma)$. For $d = 3$ we apply [11, Lemma 1] or [9, Lemma 4.3] since $\mathbb{D}_n$ is independent of θ. Instead of these articles a strong solution n satisfies in the two-dimensional case the non-autonomous abstract Cauchy problem

$$\begin{cases} \partial_t n(t) &= A(t)n(t) + f(t), \quad \text{for } t \in (0,T), \\ n(0) &= n_0 \end{cases}$$

with the operator $A(t) := \nabla\cdot(\mathbb{D}_n(\theta)(t)\nabla) - \rho\mathrm{Id} : D(A(t)) \to L^2(\Omega)$ and the inhomogeneity $f := \rho n - \nabla\cdot(qn) - (\theta R_{\mathrm{mon},l} + (\theta_s - \theta)R_{\mathrm{mon},b})$. The constant $\rho > 0$ is chosen sufficiently large such that the spectrum of $A(t)$ lies in $\{z \in \mathbb{C}\,|\,\mathrm{Re}(z) < 0\}$. The domain $D := D(A(t)) = H^2(\Omega)\cap H_0^1(\Omega)$ is dense in $L^2(\Omega)$ and independent of t. Since $\mathbb{D}_n$ is assumed to be symmetric and uniformly positive definite, the densely defined, closed operator $A(t)$ is self-adjoint and dissipative. Furthermore, this operator is sectorial for each $t \in [0,T)$.

Owing to the assumption $\theta_0 \in W^{1,\infty}(\Omega)$ and $H^1(\Omega) \hookrightarrow L^\gamma(\Omega)$ for $d=2$ and $\gamma < \infty$ the gradient of $\theta(t)$ can be estimated in each L^γ-norm, $\gamma < \infty$:

$$\|\nabla\theta(t)\|_\gamma \leq C(\gamma)\|\nabla\theta(t)\|_{H^1} \leq C(\gamma,\theta_0,R_I)\|b(t)\|_{H^2} + \|n(t)\|_{H^2}\,.$$

Therefore, the operator $A_\gamma(t) := \nabla\cdot(\mathbb{D}_n(\theta)(t)\nabla) : D(A_\gamma(t)) \to L^\gamma(\Omega)$, $\gamma < \infty$, with $D(A_\gamma(t)) = W^{2,\gamma}(\Omega) \cap W_0^{1,\gamma}(\Omega)$ is well-defined and is nothing but the restriction of $A(t)$ on $D(A_\gamma(t))$.

Instead of a representation formula for n via semigroups as in the autonomous case, we may represent n via a *evolution system* $(U(t,s))_{t\geq s}$, see [17, § 5, Def. 5.3]:

Definition 2 [Evolution system]. A two-parameter family of bounded linear operators $U(t,s)$, $0 \leq s \leq t < T$, on a Banach space X is called an *evolution system* if the following conditions are satisfied:

(i) $U(s,s) = \mathrm{Id}_X$ and $U(t,r)U(r,s) = U(t,s)$ for $0 \leq s \leq r \leq t < T$.

(ii) The map $(t,s) \mapsto U(t,s)$ is strongly continuous for $0 \leq s \leq t < T$.

Moreover, the following property of A is necessary to obtain a useful representation of n: there exists a constant $\alpha \in (\frac{1}{2},1]$ such that

$$\|(A(t)-A(s))\,A(\tau)^{-1}\|_{L^\gamma\to L^\gamma} \lesssim |t-s|^\alpha \qquad \text{for } s,t,\tau \in [0,T)\,. \tag{3.1}$$

In fact, assuming (3.1) there exists a unique evolution system $(U(t,s))_{t\geq s}$ such that

$$n(t) = U(t,0)n_0 + \int_0^t U(t,\tau)f(\tau)\,d\tau\,, \tag{3.2}$$

cf. [17, § 5, Theor. 6.1].

For each fixed $t \in (0,T)$, the operator $-A(t)$ generates a C_0-semigroup $S_t(s)$, $s \geq 0$, on the Banach space $L^\gamma(\Omega)$ the evolution system $U(t,s)$ of A reads as follows, cf. [17, § 5, (6.3)]:

$$U(t,s) = S_s(t-s) + \int_s^t S_\tau(t-\tau)R(\tau,s)\,d\tau$$

with $R(t,s) = \sum_{m=1}^\infty R_m(t,s)$ and recursively defined summands

$$R_1(t,s) := (A(s)-A(t))\,S_s(t-s)\,,\ R_{m+1}(t,s) := \int_s^t R_1(t,\tau)R_m(\tau,s)\,d\tau\,.$$

However, for our purpose the inequality

$$\|R(t,s)\|_{L^\gamma\to L^\gamma} \leq C(\gamma)(t-s)^{\alpha-1} \tag{3.3}$$

suffices, cf. [17, § 5, (6.26)].

The next result is a standard estimate and holds in a much more general setting. Since we will use it only once to obtain the gradient estimate, it is stated in this special way:

Lemma 3. *Let $s, t, \tau \in [0, T)$, $\beta \in (0, 1)$ and $\gamma < \infty$. Then there holds the inequality*

$$\|A(t)^{\beta} S_s(\tau)\|_{L^\gamma \to L^\gamma} \leq C(\gamma)\tau^{-\beta}.$$

Proof. The proof for the autonomous case, see [17, § 2, Theor. 6.13], holds for $A(s)^{\beta} S_s(\tau)$, but can easily be extended by applying the bounded operator $A(t)A(s)^{-1} : L^\gamma(\Omega) \to L^\gamma(\Omega)$. □

With the help of the previous Lemma, we are now able to prove an appropriate gradient estimate for n:

Lemma 4. *Let (b, n, θ, q, p) be a strong solution to problem*(2.3) *with* $\sup_t \|b(t)\|_\gamma \leq C(\gamma)$, $\gamma < \infty$. *Then the gradient of n satisfies*

$$\sup_{t \in (0,T)} \|\nabla n(t)\|_\gamma \leq C(\gamma) \quad \textit{for all } T < \infty.$$

Proof. Defining $f_1 := -q\nabla n$ and $f_2 := f - f_1$ we have with Lemma 1

$$\|f_2(t)\|_\gamma \leq C(\gamma) \left((\rho + \|(\partial_t \theta)(t)\|_\infty) \|n(t)\|_\infty + \|b(t)\|_\gamma + 1 \right) \leq C(\gamma). \tag{3.4}$$

The first summand N_1 of the representation formula of n

$$\begin{aligned} n(t) &= S_0(t)n_0 + \left(\int_0^t S_\tau(t-\tau)R(\tau, 0)\, d\tau\ n_0 + \int_0^t U(t,\tau) f(\tau)\, d\tau \right) \\ &=: N_1(t) + N_2(t), \end{aligned} \tag{3.5}$$

which comes from (3.2), is bounded in $W^{1,\gamma}(\Omega)$, i.e. $\|S_0(t)n_0\|_{W^{1,\gamma}} \leq C(\gamma, T)$. By applying $A(t)^\beta$ for $\beta \in (0, \alpha)$ on the remaining function N_2 we obtain with Lemma 3 and (3.3)

$$\begin{aligned} \|A(t)^\beta N_2(t)\|_\gamma &\leq \int_0^t \|A(t)^\beta S_\tau(t-\tau) R(\tau, 0) n_0\|_\gamma \, d\tau \\ &\quad + \int_0^t \|A(t)^\beta S_s(t-s) f(s)\|_\gamma \, ds + \int_0^t \int_s^t \|A(t)^\beta S_\tau(t-\tau) R(\tau, s) f(s)\|_\gamma \, d\tau ds \\ &\leq C(\gamma) \left[\int_0^t (t-\tau)^{-\beta} \tau^{\alpha-1} \, d\tau \ \|n_0\|_\gamma \right. \\ &\quad \left. + \int_0^t (t-s)^{-\beta} \|f(s)\|_\gamma \, ds + \int_0^t \int_s^t (t-\tau)^{-\beta} (\tau - s)^{\alpha-1} \|f(s)\|_\gamma \, d\tau ds \right]. \end{aligned}$$

Now we use (3.4) and have

$$\|A(t)^\beta N_2(t)\|_\gamma \leq C(\gamma) \Bigg[\left(\frac{1}{\alpha} + \frac{1}{1-\beta} \ \left(\frac{T}{2}\right)^{\alpha-\beta} \right) \|n_0\|_\gamma$$

$$+\left(\frac{1}{1-\beta}T^{1-\beta}+\left(\frac{1}{\alpha}+\frac{1}{1-\beta}\right)\frac{1}{1+\alpha-\beta}\left(\frac{T}{2}\right)^{1+\alpha-\beta}\right)\sup_{t\in(0,T)}\|f_2(t)\|_\gamma$$
$$+\int_0^t\left((t-s)^{-\beta}+\left(\frac{1}{\alpha}+\frac{1}{1-\beta}\right)\left(\frac{t-s}{2}\right)^{\alpha-\beta}\right)\|f_1(s)\|_\gamma\,ds\Bigg]$$
$$\leq C(\gamma)\Bigg[T^{\alpha-\beta}\|u_0\|_{W^{1,\infty}}+T^{2+\alpha-2\beta}\,(C(\gamma)+1)$$
$$+\sup_{t\in(0,T)}\|q(t)\|_\infty\int_0^t(t-s)^\alpha\|\nabla n(s)\|_\gamma\,ds\Bigg]. \tag{3.6}$$

Real interpolation implies for $\beta\in(\frac{1}{2},\alpha)$

$$D(A_\gamma(t)^\beta)\subset\big(L^\gamma(\Omega),D(A_\gamma(t))\big)_{\beta,\infty}\subset\big(L^\gamma(\Omega),D(A_\gamma(t))\big)_{\frac{1}{2},\gamma}\subseteq W^{1,\gamma}(\Omega)\,.$$

cf. [14, Prop. 1.4, 4.7]. Finally, we estimate $\|A(t)^\beta N_2(t)\|_\gamma$ from below by $\|\nabla N_2(t)\|_\gamma$, combine (3.5) with (3.6) and apply Gronwall's Lemma to obtain

$$\sup_{t\in(0,T)}\|\nabla n(t)\|_\gamma\leq C(\gamma,T)\exp\left(C(T)\sup_{t\in(0,T)}\|q(t)\|_\infty\right)$$

as was to be shown. □

We conclude this section with an elementary inequality, which can be verified directly by applying twice Young's inequality.

Lemma 5. *Let $\alpha_i>0$, $i=1,2,3$, with $\sum_{i=1}^3\alpha_i<1$. Then for all $a_i\geq 0$, $i=1,2,3$, the estimate*

$$a_1^{\alpha_1}a_2^{\alpha_2}a_3^{\alpha_3}\leq\alpha_1a_1+Ca_2^\lambda+\alpha_3a_3$$

holds, where $C=C(\alpha_1,\alpha_3):=1-\alpha_1-\alpha_3>0$ and $\lambda=\lambda(\alpha_1,\alpha_2,\alpha_3):=\frac{\alpha_2}{1-\alpha_1-\alpha_3}<1$.

4 Proof of Theorem 1

First of all let us briefly note that the estimate (4.6) and hence (4.1) below are not necessary for the three-dimensional case since we assumed independence of $\mathbb{D}_n$ of θ. However, in the two-dimensional case the result holds for any $b_0\in\mathcal{X}_\rho$, $\rho>0$, by choosing κ sufficiently close to 2. In more detail, the initial data $b_0\in\mathcal{X}$ entails $\nabla b\in L^2(0,T;H^{\frac{1}{3}}(\Omega))\hookrightarrow L^2(0,T;L^3(\Omega))$.

Proof of Theorem 1*:* We assume (b,n,θ,q,p) to be a strong solution to (2.3). Testing the ODE

$$\partial_t(\partial_{x_i}\theta)=\mathcal{D}R_b(\tilde n,\tilde\theta)\partial_{x_i}(\tilde n,\tilde\theta)+\mathcal{D}R_I(\tilde b,\tilde n,\tilde\theta)\partial_{x_i}(\tilde b,\tilde n,\tilde\theta)$$

for the spatial derivative $\partial_{x_i}\theta$, $i=1,...,d$, which corresponds to (1.1d) with $(\partial_{x_i}\theta)^2$ we have

$$\frac{d}{dt}\|\nabla\theta\|_3^3\leq C(R_b,R_I)\left(\|\nabla\theta\|_3+\|\nabla b\|_3+\|\nabla n\|_3\right)\|\nabla\theta\|_3^2\,. \tag{4.1}$$

Let $\gamma, r > 0$ be numbers satisfying $\gamma > \max\left\{6, m+\frac{2}{d}, C_0(b_0, n_0, \theta_0, T)\right\}$ and $r \in \left(\frac{d}{2}, r_{\sup}\right)$ with $r_{\sup} := \frac{1}{\gamma-m}\left[(m+\gamma)\frac{d}{d-2}(1-\frac{1}{\gamma})+1\right]$ (note for $d = 2$: $r \in (1,\infty)$). By testing $b^{\gamma-1}$ to (2.3b) we obtain

$$\frac{1}{\gamma}\frac{d}{dt}\int_\Omega b^\gamma + (\gamma-1)\int_\Omega \left(|\nabla b^{\frac{m+\gamma}{2}}|^2 + |\nabla b^{\frac{\gamma}{2}}|^2\right) \leq (\gamma-1)\int_\Omega b^{-m+\gamma}|\nabla n|^2 + C(R_I)\int_\Omega (|b|+1)\,|b|^{\gamma-1}.$$

We estimate the first summand on the right-hand side by Hölder's inequality $\left(\frac{1}{r}+\frac{1}{r'}=1\right)$:

$$\int_\Omega b^{-m+\gamma}|\nabla n|^2 \leq \|b^{\frac{m+\gamma}{2}}\|_{\frac{2(-m+\gamma)r}{m+\gamma}}^{\frac{2(-m+\gamma)}{m+\gamma}} \|\nabla n\|_{2r'}^2 .$$

Furthermore, the Gagliardo-Nirenberg inequality [16] yields

$$\|b^{\frac{m+\gamma}{2}}\|_{\frac{2(-m+\gamma)r}{m+\gamma}}^{\frac{2(-m+\gamma)}{m+\gamma}} \leq C(\gamma)\Big(\|\nabla b^{\frac{m+\gamma}{2}}\|_2^{\frac{2(-m+\gamma)}{m+\gamma}a}\,\|b^{\frac{m+\gamma}{2}}\|_s^{\frac{2(-m+\gamma)}{m+\gamma}(1-a)} + \|b^{\frac{m+\gamma}{2}}\|_s^{\frac{2(-m+\gamma)}{m+\gamma}}\Big), \tag{4.2}$$

where $a = 1-\frac{1}{(-m+\gamma)r} \in (0,1)$ and $\frac{1}{s} = \frac{m+\gamma}{2} - \frac{d-2}{2d}\left[(-m+\gamma)r-1\right] \in (0, \frac{m+\gamma}{2}]$, and also with Lemma 1

$$\|\nabla n\|_{2r'}^2 \leq C\left(\|\Delta n\|_2^{\frac{2}{4-d}(\frac{d}{r}-d+2)}\|n\|_\infty^{\frac{2}{4-d}(2-\frac{d}{r})} + \|n\|_\infty^2\right) \leq C\left(\|\Delta n\|_2^{\frac{2}{4-d}(\frac{d}{r}-d+2)}+1\right). \tag{4.3}$$

Owing to $\frac{1}{\gamma} < \frac{2}{m+\gamma}\frac{1}{s}$ (this holds, since $r < r_{\sup}$) we may estimate $\|b^{\frac{m+\gamma}{2}}\|_s^{\frac{2}{m+\gamma}} = \|b\|_{\frac{m+\gamma}{2}s} \leq C\|b\|_\gamma$ and hence with (4.2)

$$\|b^{\frac{m+\gamma}{2}}\|_{\frac{2(-m+\gamma)r}{m+\gamma}}^{\frac{2(-m+\gamma)}{m+\gamma}} \leq C(\gamma)\Big(\|\nabla b^{\frac{m+\gamma}{2}}\|_2^{2\frac{-m+\gamma-\frac{1}{r}}{m+\gamma}} \cdot \|b\|_\gamma^{\frac{1}{r}} + \|b\|_\gamma^{-m+\gamma}\Big). \tag{4.4}$$

Now let us test (2.4) with $-\Delta n \in L^2(\Omega)$:

$$\frac{d}{dt}\int_\Omega |\nabla n|^2 + \int_\Omega |\Delta n|^2 \leq C\int_\Omega \left[(|\nabla\theta|+|q|)|\nabla n| + |b| + 1 + |\partial_t\theta||n|\right]^2. \tag{4.5}$$

We use Lemma 2 and the estimate for $d = 2$

$$\int_\Omega |\nabla\theta|^2|\nabla n|^2 \leq C\|\nabla\theta\|_3^2\|\nabla n\|_6^2 \leq C\|\nabla\theta\|_3^2\left(\|\Delta n\|_2^{2\cdot\frac{2}{3}}+1\right) \tag{4.6}$$

to obtain

$$\frac{d}{dt}\int_\Omega |\nabla n|^2 + \int_\Omega |\Delta n|^2 \leq C\left(\|\nabla\theta\|_3^6 + \|b\|_2^2 + 1\right) \leq C(\gamma)\left(\|\nabla\theta\|_3^6 + \|b\|_\gamma^\gamma + 1\right).$$

Finally, this leads with (4.3) and (4.4) to

$$\frac{d}{dt}\left(\frac{1}{\gamma}\|b\|_\gamma^\gamma + \|\nabla n\|_2^2\right) + (\gamma-1)\left(\|\nabla b^{\frac{m+\gamma}{2}}\|_2^2 + \|\nabla b^{\frac{\gamma}{2}}\|_2^2\right) + \|\Delta n\|_2^2$$
$$\leq C(\gamma)(\gamma-1)\left\{\|\nabla b^{\frac{m+\gamma}{2}}\|_2^{2\frac{-m+\gamma-\frac{1}{r}}{m+\gamma}} \cdot \|b\|_\gamma^{\frac{1}{r}} + \|b\|_\gamma^{-m+\gamma}\right\}$$
$$\times\left\{\|\Delta n\|_2^{\frac{2}{4-d}\left(\frac{d}{r}-d+2\right)} + 1\right\} + C(\gamma, R_I)\left((\|b\|_\gamma + 1)\|b\|_\gamma^{\gamma-1} + \|\nabla\theta\|_3^6 + 1\right).$$

Adapting the idea of [25, Lem. 2.6] we set $\alpha_1(\xi) := \frac{-m+\gamma-\xi}{m+\gamma}$, $\alpha_2(\xi) := \frac{\xi}{\gamma}$ and $\alpha_3(\xi) := \frac{1}{4-d}(d\xi - d + 2)$ for $\xi \geq 0$. Then for $\xi_0 := \begin{cases} 0 & \text{, if } d = 2, \\ \frac{1}{3} & \text{, if } d = 3 \end{cases}$ there holds

$$0 < \sum_{i=1}^{3} \alpha_i(\xi_0) = \frac{-m+\gamma-\xi_0}{m+\gamma} + \frac{\xi_0}{\gamma} + 0 < 1.$$

Since the functions α_i are continuous, there exists for sufficiently large γ a real number $r \in \left(\frac{d}{2}, \frac{d}{d-2}\left(1-\frac{1}{\gamma}\right)\right)$ with $\frac{1}{r}$ arbitrary close to ξ_0 such that $\sum_{i=1}^3 \alpha_i < 1$, where $\alpha_i := \alpha_i\left(\frac{1}{r}\right)$, and $1 - \frac{m}{\gamma} + \alpha_3 < 1$ (note for $d = 3$ and $m > \frac{2}{9}$ there exists a number $r < r_{\sup}$ such that $\frac{d}{r} < (4-d)\frac{m}{\gamma} + d - 2$). Applying Lemma 3.6 leads to

$$\frac{d}{dt}\left(\frac{1}{\gamma}\|b\|_\gamma^\gamma + \|\nabla n\|_2^2\right) + \|\nabla b^{\frac{m+\gamma}{2}}\|_2^2 + \|\nabla b^{\frac{\gamma}{2}}\|_2^2 + \|\Delta n\|_2^2$$
$$\leq (C(\gamma)(\gamma-1))^{\lambda\gamma r}\|b\|_\gamma^{\lambda\gamma} + C(\gamma, R_I)\|b\|_\gamma^\gamma + \|\nabla\theta\|_3^6 + 1, \tag{4.7}$$

where $\lambda := \frac{\alpha_2}{1-\alpha_1-\alpha_3} < 1$. Combining (4.7) with (4.1) yields

$$\frac{d}{dt}\left(\frac{1}{\gamma}\|b\|_\gamma^\gamma + \|\nabla n\|_2^2 + \|\nabla\theta\|_3^6\right) \leq (C(\gamma)(\gamma-1))^{\lambda\gamma r}\|b\|_\gamma^{\lambda\gamma}$$
$$+ C(\gamma, R_b, R_I)\left(\|b\|_\gamma^\gamma + \|\nabla\theta\|_3^6 + (\|\nabla b\|_3 + \|\nabla n\|_3)^2\|\nabla\theta\|_3^4 + 1\right).$$

The norms $\|\nabla b\|_{L^2(L^3)}$ and $\|\nabla n\|_{L^2(L^3)}$ can be estimated by an appropriate constant $C_0(b_0, n_0, \theta_0, T) := C(\|b_0\|_{\mathcal{X}}, \|n_0\|_{W^{1,\infty}}, \|\theta_0\|_{W^{1,\infty}}, T) > 0$. Thus Gronwall's Lemma [24, Prop. 3.4] together with the restriction on γ leads to

$$\sup_{t\in(0,T)}\left(\frac{1}{\gamma}\|b(t)\|_\gamma^\gamma + \|\nabla n(t)\|_2^2 + \|\nabla\theta(t)\|_3^6\right) \lesssim \left[\frac{1}{\gamma}\|b_0\|_\gamma^\gamma + \|\nabla n_0\|_2^2\right.$$
$$\left. + \|\nabla\theta_0\|_3^6 + 1 + \left((C(\gamma)(\gamma-1))^{\frac{\lambda}{\alpha_2}}T\right)^{\frac{1}{1-\lambda}} + \left(\frac{1}{3}\gamma\right)^3\right]\exp(\gamma T)$$
$$\lesssim \left[\frac{1}{\gamma}\|b_0\|_\gamma^\gamma + \|\nabla n_0\|_2^2 + \|\nabla\theta_0\|_3^6 + 1 + \tilde{C}(\gamma)\right]\exp(\gamma T),$$

where $\tilde{C}(\gamma) := \left((C(\gamma)(\gamma-1))^{\frac{\lambda}{\alpha_2}}T\right)^{\frac{1}{1-\lambda}} + \left(\frac{1}{3}\gamma\right)^3$ and hence with $\gamma^{\frac{1}{\gamma}} \lesssim 1$

$$\sup_{t\in(0,T)}\|b(t)\|_\gamma \lesssim C(T)\left[\|b_0\|_\gamma + \|\nabla n_0\|_2^{\frac{2}{\gamma}} + \|\nabla\theta_0\|_3^{\frac{6}{\gamma}} + 1 + \tilde{C}(\gamma)^{\frac{1}{\gamma}}\right].$$

We assume that r even is chosen in that way such that $\alpha_1 + \tilde{\alpha}_2 + \alpha_3 < 1$ is satisfied, where $\tilde{\alpha}_2 := \frac{m+\frac{1}{r}}{\gamma} \in (\alpha_2, 1-\alpha_1-\alpha_3)$. Thus, $\lambda < \frac{1}{mr+1}$ and hence $\frac{1}{1-\lambda} < 1 + \frac{2}{dm}$. Moreover, there holds $\frac{\lambda}{\alpha_2} \cdot \frac{1}{\gamma} < \frac{1}{\tilde{\alpha}_2\gamma} < \frac{1}{m}$. However, since $\gamma > 6$ we obtain with $\frac{1}{m} \cdot (1 + \frac{2}{dm}) < \frac{1}{\tilde{m}}$, $\tilde{m} := \{\frac{m^2}{m+1}, \frac{3m}{5}\}$

$$\sup_{t\in(0,T)} \|b(t)\|_\gamma \lesssim C(T)\left[\|b_0\|_\gamma + \|\nabla n_0\|_2 + \|\nabla\theta_0\|_3 + 1 + \tilde{C}(\gamma)^{\frac{1}{\gamma}}\right]$$
$$\lesssim C(T)\left[\|b_0\|_\infty + \|\nabla n_0\|_2 + \|\nabla\theta_0\|_3 + 1 + (C(\gamma)(\gamma-1))^{\frac{1}{\tilde{m}}}\right].$$

Finally, the assumptions of Lemma 4 are satisfied. That means we have

$$\sup_{t\in(0,T)} \|\nabla n(t)\|_\gamma^\gamma \lesssim C(\|b_0\|_{\mathcal{X}}, \|n_0\|_{W_0^{1,\infty}}, \|\theta_0\|_{W^{1,\kappa}}, T). \tag{4.8}$$

To prove boundedness of b an iteration procedure of Moser type may be applied, cf. [21, Theor. 4.6]. However, the assertion

$$\sup_{t\in(0,T)} \|b(t)\|_\infty < C(\|b_0\|_\infty, \|n_0\|_{W_0^{1,\infty}}, \|\theta_0\|_{W^{1,\kappa}}, T) \qquad \text{for all } T < T_{\max}$$

is verified. □

At this point the author likes to mention that in (4.2) the application of the Gagliardo-Nirenberg inequality is not standard, since $\frac{1}{s} \sim \gamma$ and hence $s < 1$. Therefore, if a function $f : \Omega \to \mathbb{R}$ is measurable we define $\|f\|_s^s := \int_\Omega |f|^s$ even for all $s > 0$. The Gagliardo-Nirenberg preserves validity in such a case, cf. [29, Lem. 3.2].

5 Proof of Theorem 2

Let $(b_0, n_0, \theta_0) \in L^\infty(\Omega) \times W_0^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$ and a sequence of smooth initial data $(b_{0,k}, n_{0,k})_{k\in\mathbb{N}} \in C_0^\infty(\bar{\Omega})^2$ is assumed to satisfy $b_{0,k} \overset{*}{\rightharpoonup} b_0$ in $\mathcal{X}$ and $n_{0,k} \overset{*}{\rightharpoonup} n_0$ in $W_0^{1,\infty}(\Omega)$. Then for all $k \in \mathbb{N}$ there exists a unique strong solution $(b_k, n_k, \theta_k, q_k, p_k)$ to (1.1) with initial data $(b_{0,k}, n_{0,k}, \theta_0)$, cf. [21, Theor. 4.4]. This solution either exists globally in time or up to a possible clogging phenomenon, cf. [21, Theor. 4.5]. However, the time interval of existence corresponding to solution $(b_k, n_k, \theta_k, q_k, p_k)$ is denoted by $(0, T_{\max,k})$ with $T_{\max,k} \le \infty$. Thereby θ_k satisfies

$$\lim_{t\nearrow T_{\max,k}} \|\theta_k^{-1}(t)\|_\infty = \theta_{\mathrm{clog}}^{-1} \qquad \text{or} \qquad T_{\max,k} = \infty. \tag{5.1}$$

We define $T_{\max} := \liminf_{k\to\infty} T_{\max,k} > 0$. The case $T_{\max} = 0$ would contradict the assumption (2.3e) on θ_k and $\theta_0 > \theta_{\mathrm{clog}} + \delta$.

Now let us fix $k \in \mathbb{N}$. Owing to the uniform boundedness principle and (2.6) there exists some $K > 0$ such that $\|b_{0,k}\|_{\mathcal{X}} + \|n_{0,k}\|_{W_0^{1,\infty}} < K$. Then Theorem 1, Lemma 1 and (4.8) yield the following estimates for all $T < T_{\max,k}$ and all $\gamma > \max\left\{6, m + \frac{2}{d}, C(\|b_0\|_{\mathcal{X}}, \|n_0\|_{W^{1,\infty}}, \|\theta_0\|_{W^{1,\infty}}, T)\right\}$:

$$\sup_{t\in(0,T)} \|b_k(t)\|_\infty + \sup_{t\in(0,T)} \|n_k(t)\|_\infty < C(T), \quad \sup_{t\in(0,T)} \|\nabla n_k(t)\|_\gamma < C(\gamma, T). \tag{5.2}$$

Note that the constants of the above estimates do not depend on k. Inserting (5.2) into (4.7) and (4.5) yields

$$\|\nabla b_k^{\frac{m+\gamma}{2}}\|_{L^2(\Omega_T)}\leq C(\gamma,T)\,, \qquad \|\Delta n_k\|_{L^2(\Omega_T)}\leq C(T)\,. \tag{5.3}$$

Due to (2.3a) there holds

$$\frac{1}{2\alpha+2}\frac{d}{dt}\int_\Omega \theta_k|b_k|^{2\alpha+2} = \langle\theta_k(\partial_t b_k), |b_k|^{2\alpha}b_k\rangle_{H^{-1},H_0^1} + \int_\Omega |b_k|^{2\alpha}b_k q_k\nabla b_k$$

for $\alpha>0$. Thus, applying the test function $|b_k|^{2\alpha}b_k$ to (2.3b) we obtain with the self-diffusivity (2.2b) and $a=2\alpha+2$ the inequality

$$\begin{aligned}&\frac{\delta}{a}\frac{d}{dt}\|b_k^a\|_1+\frac{1}{2}(a-1)\|(|b_k|^{\frac{m}{2}}+1)|b_k|^\alpha\nabla b_k\|_2^2\\ &\qquad\leq C\Big((a-1)\|\nabla n_k\|_{2\frac{a}{m}}^2\|b_k\|_a^{a-m}+(\|b_k\|_a+1)\,\|b_k\|_a^{a-1}\Big)\end{aligned}$$

and with (5.2)

$$\|(|b_k|^{\frac{m}{2}}+1)|b_k|^\alpha\nabla b_k\|_2\leq C(a,T)\,. \tag{5.4}$$

Also it is required to prove uniform boundedness of $\partial_t b_k$ in $L^2(H^{-1})$-norm:

$$\begin{aligned}\|\theta_k\partial_t b_k\|_{L^2(H^{-1})}^2 &= \int_0^T\Big(\sup_{\|\varphi\|_{H_0^1}=1}|\langle\theta_k\partial_t b_k,\varphi\rangle_{H^{-1},H_0^1}|\Big)^2\\ &\leq C\int_0^T\big(\|\nabla b_k\|_2^2+\|b_k\|_\infty\|\nabla n_k\|_2+\|q_k\|_2^2\|b_k\|_\infty^2+\|b_k\|_2^2+1\big)\leq C(T).\end{aligned} \tag{5.5}$$

Similarly we obtain $\|\partial_t n_k\|_{L^2(\Omega_T)}\leq C(T)$. Let us remark that $\|q_k\|_2\leq C\|\partial_t\theta_k\|_2 \leq C$, cf. [18, Theor. 7.4.1]. Owing to (5.3)–(5.5) together with the embedding $L^1(\Omega)\hookrightarrow H^{-2}(\Omega)$, the compactness lemma of Aubin-Lions implies strong convergence of subsequences $(b_j)_j$ and $(n_j)_j$ in $L^2(\Omega_T)$ to b and n, respectively. In particular, there are subsequences again denoted by $(b_j)_j$ and $(n_j)_j$ converging almost everywhere in Ω_T to b and n. The estimates (5.2)–(5.4) entail weak* convergence of subsequences $(b_j)_j$, $(n_j)_j$ in $L^\infty(0,T;L^\infty(\Omega))$ as well as weak convergence of $(\nabla b_j)_j$ and $(\Delta n_j)_j$ in $L^2(\Omega_T)$. This also ensures non negativity of b and n.

The boundedness of R_b and R_I imply uniform boundedness of $(\partial_t\theta_k)$ in $L^\infty(0,T;L^\infty(\Omega))$ as well as of $(\nabla\theta_k)_k$ in $L^2(\Omega_T)$ and hence strong convergence of $(\theta_j)_j$ to θ in $L^2(\Omega_T)$. Therefore, this limit satisfies (2.3d) and for a. e. $t\in(0,T)$ the estimate

$$\theta_{\text{clog}}<\liminf_{j\to\infty}\big(\|\theta_j^{-1}(t)\|_\infty\big)^{-1}\leq\|\theta(t)\|_\infty\leq\limsup_{j\to\infty}\|\theta_j(t)\|_\infty\leq\theta_s\,.$$

Moreover, the assumptions (2.1a) on R_b and R_I entail strong L^∞-convergence of $(\theta_j(t))_j$ for a. e. $t\in(0,T)$. Thus the limit θ inherits a similar property to (5.1) from the sequence $(\theta_j)_j$. Finally, the proof of Theorem 4.3 is complete. □

Since pores may clog over time, in general we cannot expect global-in-time solutions to our underlying problem. At least in the case of a rigid porous medium, i.e. the porosity does not change in time: $\partial_t \theta(t) = 0$, we obtain a bounded global-in-time weak solution. The same holds, if the reaction terms are modified such that already sufficiently close to θ_{clog} the pores can not contract further.

Conclusions

In this article, we considered a model derived in [21] which describes the growth of a biofilm produced by chemotactical bacteria within a saturated porous medium, as well as its interplay with fluid flow. Beside self-diffusivity of the bacteria, in [21] high regularity assumptions are needed to obtain (in case of no clogging) existence of a bounded global in time solution. In the current paper we improved this result by adapting the main ideas of [25], applying entropy-type estimates and a semigroup approach to control the norms $\sup_{t\in(0,T)} \|\nabla n(t)\|_\gamma$ for all $\gamma < \infty$ in an appropriate way. Beside the underlying model, the main difference to [25] is the dependence of the coefficients on θ and the extension to three dimensions. In particular, the θ-dependence leads in Section 3 to a nonautonomous Cauchy problem. However, we are able to prove the existence of a bounded weak solution up to a possible clogging phenomenon, if (2.6) and $m > \frac{2}{9}(d-2)$ is satisfied. Comparing this result with [26] or [31] suggest that in the three-dimensional case restriction $m > \frac{2}{9}$ can be relaxed. Therefore, future work should comprise investigation also in this direction. A drawback is the simplification that for $d = 3$ the diffusivity parameter $\mathbb{D}_n$ does not depend on θ. This makes the model less applicable and should be avoided in future work.

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