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# Global Existence and Asymptotic Behavior of Solutions for the Cauchy Problem of a Dissipative Boussinesq-Type Equation

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**Abstract.** We consider the Cauchy problem for a Boussinesq-type equation modeling bidirectional surface waves in a convecting fluid. Under small condition on the initial value, the existence and asymptotic behavior of global solutions in some time weighted spaces are established by the contraction mapping principle.

**Keywords:** dissipative Boussinesq equation, asymptotic behavior, Sobolev spaces.

**AMS Subject Classification:** 35B30; 35Q55; 35Q72.

## 1 Introduction

The Kuramoto-Sivashinsky (KS) equation

$$u_t + \gamma u_{xxxx} + \alpha u_{xx} + uu_x = 0,\ u = u(x,t),\ (x,t) \in \mathbb{R} \times \mathbb{R}^+$$

is a well-known model of one-dimensional turbulence derived in various physical contexts such as chemical-reaction waves, propagation of combustion fronts in gases, surface waves in a film of a viscous liquid flowing along an inclined plane, patterns in thermal convection, rapid solidification (see e.g. [14, 19]), where $\alpha$ and $\gamma$ are constant coefficients accounting for the long-wave instability (gain) and short-wave dissipation, respectively. By combining the dispersive effects of the KdV equation and the dissipative effects of the KS equation, the Kuramoto-Sivashinsky-Korteweg-de Vries (KS-KdV) equation

$$u_t + u_{xxx} + \gamma u_{xxxx} + \alpha u_{xx} + uu_x = 0,\ u = u(x,t),\ (x,t) \in \mathbb{R} \times \mathbb{R}^+$$

appears; which was first introduced by Benney [1]. This equation finds various applications in the study of unstable drift waves in plasmas [5], fluid flow along an inclined plane [1, 16] convection in fluids with a free surface [7] the Eckhaus instability of traveling waves [9], in solar dynamo wave [12], hydrodynamics and other fields [4].

The derivation of this equation in the physical situations mentioned above involves the assumption of unidirectional waves. The assumption of unidirectional waves for surface waves was removed in [10,13] and a modified Boussinesq system of equations was derived. One of these type of equation is the following dissipative Boussinesq equation:

$$u_{tt} - \Delta u + \Delta^2 u + \alpha \Delta u_t + \gamma \Delta^2 u_t = \Delta(\beta f(u_t) + g(u)). \tag{1.1}$$

Here $u = u(x,t)$ is the unknown function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ , $t > 0$ and $\beta > 0$ and $\alpha \in \mathbb{R}$ are constants. The operator $\Delta$ is defined to be $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ and $\Delta^2 = \Delta\Delta$. The term $u_t$ represents a frictional function dissipation, and the nonlinear term $f(v)$ and $g(v)$ are smooth functions of $v$ under considerations and satisfies $f(v) = O(|v|^2)$ and $g(v) = O(|v|^2)$ for $v \to 0$. Equation (1.1) arises in the study of the stability of one-dimensional periodic patterns in systems with Galilean invariance and also the oscillations of elastic beams [3]. Ignoring the dissipation, (1.1) turns into the classical Boussinesq equation (see [2])

$$u_{tt} - \Delta u \pm \Delta^2 u = \Delta(u^2),\ u = u(x,t),\ (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

appeared not only in the study of the dynamics of thin inviscid layers with free surface but also in the study of the nonlinear string, the shape-memory alloys, the propagation of waves in elastic rods and in the continuum limit of lattice dynamics or coupled electrical circuit. When $\gamma = \beta = 0$, the existence, uniqueness and long-time asymptotic of solutions to the Cauchy problem and the initial boundary value problem of equation (1.1) has been studied by several authors, see for instance [6, 11, 17, 18] and references therein.

In this paper we study the asymptotic behavior of solutions of the Cauchy problem associated to (1.1) with the initial values

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{1.2}$$

The article is organized as follows. In Section 2 we obtain the solution formula of (1.1) and study the decay property of the solution operators appearing in the solution formula. Then, in Section 3, we discuss the linear problem and show the decay estimates of the solutions in $L^1$. We prove global existence and asymptotic behavior of solutions for the Cauchy problem (1.1) and (1.2) in $L^2$ in Section 4. Throughout this paper we assume $\gamma = 1 \le -\alpha$.

**Notations.** Function $\hat{f}$ denotes the Fourier transform of $f(x)$, defined as

$$\hat{f}(\xi) = F(f)(\xi) = \int_{\mathbb{R}^n} f(x) \mathrm{e}^{\mathrm{i}x\cdot\xi} \mathrm{d}x.$$

We denote its inverse transform by $F^{-1}$. If a function $f \in L^r = L^r(\mathbb{R}^n)$, its usual norm is written as $\|f\|_{L^r}$. The usual Sobolev space of order $s$ is defined by $W^{s,p} = W^{s,p}(\mathbb{R}^n) = (I - \Delta)^{s/2} L^p$ with the norm $\|f\|_{W^{s,p}} = \|(I - \Delta)^{s/2} f\|_{L^p}$.

The corresponding homogeneous Sobolev space of order $s$ is defined by $\dot{W}^{s,p} = \dot{W}^{s,p}(\mathbb{R}^n) = (-\Delta)^{-s/2}L^p$ with the norm $\|f\|_{\dot{W}^{s,p}} = \|(-\Delta)^{s/2}\|_{L^p}$. Note that $(I-\Delta)^{s/2}$ and $(-\Delta)^{s/2}$ are the Fourier multipliers defined by $(I-\Delta)^{s/2}f = F^{-1}((1+|\xi|^2)^{s/2}\hat{f}(\xi))$ and $(-\Delta)^{s/2}f = F^{-1}(|\xi|^s\hat{f}(\xi))$. For $p=2$, we write $H^s = H^s(\mathbb{R}^n) = W^{s,2}$ and $\dot{H}^s = \dot{W}^{s,2}$. For a nonnegative integer $k$, $\partial_x^k$ denotes the totality of all the $k$th-order derivatives with respect to $x \in \mathbb{R}^n$. Also, for an interval $I$ and a Banach space $X$, $C^k(I;X)$ denotes the space of $k$-times continuously differential functions on $I$ with values in $X$.

Convention: Throughout the paper, we write sometimes function $f(x,t)$ by $f(t)$, where $t$ is time variable.

## 2 Decay property of the linear part

The aim of this section is to derive the solution formula for the problem (1.1) and (1.2). First of all, we investigate the linear equation of (1.1).

$$u_{tt} - \Delta u + \Delta^2 u + \alpha\Delta u_t + \Delta^2 u_t = 0 \tag{2.1}$$

with the initial data (1.2). By applying the Fourier transform to (2.1) we have

$$\hat{u}_{tt} + (|\xi|^4 - \alpha|\xi|^2)\hat{u}_t + (|\xi|^2 + |\xi|^4)\hat{u} = 0. \tag{2.2}$$

The corresponding initial values are given as

$$\hat{u}(\xi,0) = \hat{u}_0(\xi), \qquad \hat{u}_t(\xi,0) = \hat{u}_1(\xi). \tag{2.3}$$

The characteristic equation of (2.2) is

$$\lambda^2 + (|\xi|^4 - \alpha|\xi|^2)\lambda + (|\xi|^4 + |\xi|^2) = 0.$$

Let $\lambda = \lambda_\pm(\xi)$ be the corresponding eigenvalues, i.e

$$\lambda_\pm(\xi) = \frac{(\alpha|\xi|^2 - |\xi|^4) \pm \sqrt{(|\xi|^4 - \alpha|\xi|^2)^2 - 4(|\xi|^2 + |\xi|^4)}}{2}. \tag{2.4}$$

The solution to the problem (2.2) and (2.3) is given in the form

$$\hat{u}(\xi,t) = \hat{G}(\xi,t)\hat{u}_1(\xi) + \hat{H}(\xi,t)\hat{u}_0(\xi), \tag{2.5}$$

where

$$\hat{G}(\xi,t) = (e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t})/(\lambda_+(\xi) - \lambda_-(\xi)), \tag{2.6}$$

$$\hat{H}(\xi,t) = (\lambda_+(\xi)e^{\lambda_-(\xi)t}\lambda_-(\xi)e^{\lambda_+(\xi)t})/(\lambda_+(\xi) - \lambda_-(\xi)). \tag{2.7}$$

Let

$$G(x,t) = F^{-1}[\hat{G}(\xi,t)](x), \quad H(x,t) = F^{-1}[\hat{H}(\xi,t)](x). \tag{2.8}$$

With applying $F^{-1}$ to (2.5), we obtain

$$u(x,t) = G(\cdot,t) * u_1 + H(\cdot,t) * u_0, \tag{2.9}$$

where $*$ is the convolution in $x$. By our assumptions on the nonlinear terms of (1.1), we obtain the following solution formula to (1.1) and (1.2)

$$u(x,t) = G(\cdot,t) * u_1 + H(\cdot,t) * u_0 + \int_0^t G(\cdot,t-\tau) * \Delta(Z(u,u_t)(\tau)))d\tau, \quad (2.10)$$

where $Z(u,u_t)(t) = f(u(t)) + \beta g(u_t(t))$. This integral formula is derived by using the Duhamel principle. One can refer to [8, 15] to see how the Duhamel principle is applied for various nonlinear problems. Now we study the decay property of the linear equation (1.1). Our aim is to prove the following decay estimates of the solution operators $G(t)$ and $H(t)$ appearing in (2.9).

**Lemma 1.** *The solution of* (2.2) *and* (2.3) *satisfies*

$$\begin{aligned} &|\xi|^2(1+|\xi|^2)|\hat{u}(\xi,t)|^2 + |\hat{u}_t(\xi,t)|^2 \\ &\quad \le Ce^{-c\omega(\xi)t}(|\xi|^2(1+|\xi|^2)|\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2) \end{aligned} \quad (2.11)$$

*for* $\xi \in \mathbb{R}^n$ *and* $t \ge 0$, *where* $\omega(\xi) = |\xi|^2/(1+|\xi|^2)$.

*Proof.* By multiplying (2.2) by $\bar{\hat{u}}_t$ and taking the real part, we deduce that

$$\frac{1}{2}\frac{d}{dt}(|\hat{u}_t|^2 + (|\xi|^2 + |\xi|^4)|\hat{u}|^2) + (|\xi|^4 - \alpha|\xi|^2)|\hat{u}_t|^2 = 0. \quad (2.12)$$

Multiplying (2.2) by $\bar{\hat{u}}$ and take the real part yields

$$\frac{1}{2}\frac{d}{dt}((|\xi|^4 - \alpha|\xi|^2)|\hat{u}|^2 + 2Re(\hat{u}_t.\bar{\hat{u}})) + (|\xi|^2 + |\xi|^4|)|\hat{u}|^2 - |\hat{u}_t|^2 = 0. \quad (2.13)$$

Multiplying both sides of (2.12) and (2.13) by $(1+|\xi|^2)$ and $|\xi|^2$ respectively, summing up the products yields

$$\frac{d}{dt}E + F = 0, \quad (2.14)$$

where

$$\begin{aligned} E &= (1+|\xi|^2)|\hat{u_t}|^2 + \{(1+|\xi|^2)(|\xi|^2+|\xi|^4) + |\xi|^2(|\xi|^4 - \alpha|\xi|^2)\}|\hat{u}|^2 + 2|\xi|^2\Re(\hat{u_t}\bar{\hat{u}}), \\ F &= \{2(1+|\xi|^2)(|\xi|^4 - \alpha|\xi|^2) - 2|\xi|^2\}\ |\hat{u_t}|^2 + 2|\xi|^2(|\xi|^2 + |\xi|^4)|\hat{u}|^2, \end{aligned}$$

where $\Re(\hat{u_t}\bar{\hat{u}})$ is the real part of $\hat{u_t}\bar{\hat{u}}$. It is easy to see that

$$C(1+|\xi|^2)E_0 \le E \le C(1+|\xi|^2)E_0, \quad (2.15)$$

where

$$E_0 = |\hat{u_t}|^2 + |\xi|^2(1+|\xi|^2)|u|^2.$$

Noting that $F \ge |\xi|^2 E_0$ and with (2.15), we obtain

$$F \ge c\,\omega(\xi)E, \quad (2.16)$$

where $\omega(\xi) = |\xi|^2/(1+|\xi|^2)$. Using (2.14) and (2.16), we get

$$\frac{d}{dt}E + c\,\omega(\xi)E \le 0.$$

Thus

$$E(\xi,t) \le e^{-c\ \omega(\xi)t}E(\xi,0),$$

which together with (2.15) proves the desired estimate (2.11). □

**Lemma 2.** *Assume that $\hat G(\xi,t)$ and $\hat H(\xi,t)$ are fundamental solutions of* (2.1) *in the Fourier space, which are given explicitly in* (2.6) *and* (2.7)*. Then we have the pointwise estimates*

$$|\xi|^2(1+|\xi|^2)|\hat G(\xi,t)|^2 + |\hat G_t(\xi,t)|^2 \le Ce^{-c\omega(\xi)t}, \tag{2.17}$$
$$|\xi|^2(1+|\xi|^2)|\hat H(\xi,t)|^2 + |\hat H_t(\xi,t)|^2 \le C|\xi|^2(1+|\xi|^2)e^{-c\omega(\xi)t}, \tag{2.18}$$

*for $\xi \in \mathbb{R}^n$ and $t \ge 0$, where $\omega(\xi) = \frac{|\xi|^2}{1+|\xi|^2}$.*

*Proof.* If $\hat u_0(\xi) = 0$, then from (2.5) we get

$$\hat u(\xi,t) = \hat G(\xi,t)\hat u_1(\xi), \quad \hat u_t(\xi,t) = \hat G_t(\xi,t)\hat u_1(\xi).$$

Substituting the equalities into (2.11) with $\hat u_0(\xi) = 0$ we obtain (2.17). In what follows, we consider $\hat u_1(\xi) = 0$. We have from (2.5) that

$$\hat u(\xi,t) = \hat H(\xi,t)\hat u_0(\xi), \quad \hat u_t(\xi,t) = \hat H_t(\xi,t)\hat u_0(\xi).$$

Substituting the equalities into (2.11) with $\hat u_1(\xi) = 0$, we obtain (2.18), which together with (2.17), we have completed the proof of the lemma. □

**Lemma 3.** *Let $l, k, j$ be nonnegative integers and assume that $1 \le p \le 2$. Then we have*

$$\begin{aligned}\|\partial_x^k G(t) * \phi\|_{L^2} &\le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k-j}{2}}\ \|\partial_x^j\phi\|_{\dot W^{-1,p}}\\ &\quad + Ce^{-ct}\|\partial_x^{k+l-2}\phi\|_{L^2},\end{aligned} \tag{2.19}$$
$$\begin{aligned}\|\partial_x^k H(t) * \psi\|_{L^2} &\le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k-j}{2}}\ \|\partial_x^j\psi\|_{L^p}\\ &\quad + Ce^{-ct}\|\partial_x^{k+l}\phi\|_{L^2},\end{aligned} \tag{2.20}$$

*for $0 \le j \le k$, where $k+l-2 \ge 0$ in* (2.19)*. Similarly, we have*

$$\begin{aligned}\|\partial_x^k G_t(t) * \phi\|_{L^2} &\le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1-j}{2}}\ \|\partial_x^j\phi\|_{\dot W^{-1,p}}\\ &\quad + Ce^{-ct}\|\partial_x^{k+l}\phi\|_{L^2},\end{aligned} \tag{2.21}$$
$$\begin{aligned}\|\partial_x^k H_t(t) * \psi\|_{L^2} &\le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1-j}{2}}\ \|\partial_x^j\psi\|_{L^p}\\ &\quad + Ce^{-ct}\|\partial_x^{k+l+2}\phi\|_{L^2},\end{aligned} \tag{2.22}$$

*for $0 \le j \le k+1$.*

*Proof.* We only give a proof of (2.19). We apply the Plancherel theorem and use the pointwise estimate for $\hat{G}$ in (2.17). This gives

$$\begin{aligned}
\|\partial_x^k G_t(t) * \phi\|_{L^2}^2 &= \int_{\mathbb{R}^n} |\xi|^{2k} |\hat{G}(\xi,t)|^2 \|\hat{\phi}(\xi)|^2 d\xi \\
&= \int_{|\xi|\le 1} |\xi|^{2k} |\hat{G}(\xi,t)|^2 \hat{\phi}(\xi)|^2 d\xi + \int_{|\xi|\ge 1} |\xi|^{2k} |\hat{G}(\xi,t)|^2 \hat{\phi}(\xi)|^2 d\xi \\
&\le \int_{|\xi|\le 1} |\xi|^{2k-2} e^{-c|\xi|^2 t} |\hat{\phi}(\xi)|^2 d\xi \\
&\quad + C\int_{|\xi|\ge 1} e^{-c\omega(\xi)t} |\xi|^{2k} (|\xi|^2(1+|\xi|^2))^{-1} |\hat{\phi}(\xi)|^2 d\xi \\
&\le C\| \, |\xi|^{j-1}\hat{\phi}(\xi) \, \|_{L^{\acute{p}}}^2 \Big(\int_{|\xi|\le 1} |\xi|^{2(k-j)q} e^{-cq|\xi|^2 t} d\xi\Big)^{\frac{1}{q}} \\
&\quad + Ce^{-ct} \int_{|\xi|\ge 1} |\xi|^{2k-4} |\hat{\phi}(\xi)|^2 d\xi \\
&\le C\| \, |\xi|^{j-1}\hat{\phi}(\xi) \, \|_{L^{\acute{p}}}^2 \big(\| \, |\xi|^{2(k-j)} e^{-c|\xi|^2 t} \|_{L^q} \\
&\quad + Ce^{-ct} \int_{|\xi|\ge 1} |\xi|^{2(k+l-2)} |\hat{\phi}(\xi)|^2 d\xi,
\end{aligned}$$

where we used Hölder inequality with $\frac{1}{q} + \frac{2}{\acute{p}} = 1$, $\frac{1}{p} + \frac{1}{\acute{p}} = 1$. With a straight computation, we obtain

$$\||\xi|^{2(k-j)} e^{-c|\xi|^2 t}\|_{L^q(|\xi|\le 1)} \le C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-(k-j)}.$$

It follows from the Hausdorff-Young inequality that

$$\||\xi|^{j-1}\hat{\phi}(\xi)\|_{L^{\acute{p}}} \le \|\partial_x^j \phi\|_{\dot{W}^{-1,p}}.$$

Combining the above three inequalities yields (2.19). Similarly, we can prove (2.20)–(2.22). Thus the lemma is proved. □

Immediately we have from previous lemma the following corollary.

*Corollary 1.* Let $1 \le p \le 2$,and let $k$, $j$ and $l$ be nonnegative integers. Also, assume that $G(x,t)$ and $H(x,t)$ be the fundamental solution of (2.1) which are given in (2.6) and (2.7), respectively. Then we have

$$\|\partial_x^k G(t) * \Delta g\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1-j}{2}} \|\partial_x^j g\|_{L^p} + Ce^{-ct}\|\partial_x^{k+l} g\|_{L^2}, \tag{2.23}$$

for $0 \le k \le j+1$. It also for $0 \le k \le j+2$ holds that

$$\|\partial_x^k G_t(t) * \Delta g\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+2-j}{2}} \|\partial_x^j g\|_{L^p} + Ce^{-ct}\|\partial_x^{k+l+2} g\|_{L^2}. \tag{2.24}$$

## 3 Global existence and asymptotic behavior of solutions for $L^1$

The aim of this section is to prove the existence and asymptotic behavior of solutions to (1.1) and (1.2) with $L^1$ data. We first state the following lemma, which comes from [20].

**Lemma 4.** *Assume that $f = f(v)$ is smooth function, where $v = (v_1, \dots, v_n)$ is a vector function. Suppose that $f(v) = O(|v|^{1+\theta})(\theta \geq 1$ is an integer) when $|v| \leq v_0$. Then, for the integer $m \geq 0$, if $v, w \in W^{m,q}(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$ and $\|v\|_{L^\infty} \leq v_0$,$\|w\|_{L^\infty} \leq v_0$, then $f(v) - f(w) \in W^{m,r}(\mathbb{R}^n)$. Furthermore, the following inequalities hold:*

$$\|\partial_x^m f(v)\|_{L^r} \leq C\|v\|_{L^p}\|\partial_x^m v\|_{L^q}\|v\|_{L^\infty}^{\theta-1}, \tag{3.1}$$

$$\begin{aligned}\|\partial_x^m (f(v) - f(w))\|_{L^r} \leq C\{(\|\partial_x^m v\|_{L^q})\|v - w\|_{L^p} \\ + (\|v\|_{L^p} + \|w\|_{L^p}\|\partial_x^m (v - w)\|_{L^q})\}(\|v\|_{L^\infty} + \|w\|_{L^\infty})^{\theta-1},\end{aligned} \tag{3.2}$$

*where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $1 \leq p, q, r \leq +\infty$.*

Based on the decay estimates of solutions to the linear problem (2.1), we define the following solution space:

$$X = \{u \in C([0, \infty); H^{s+2}(\mathbb{R}^n)) \bigcap C^1([0, \infty); H^s(\mathbb{R}^n)) : \|u\|_X < \infty\},$$

where

$$\|u\|_X = \sup_{t\geq 0} \left\{ \sum_{k\leq s+2} (1 + t)^{\frac{n}{4}+\frac{k}{2}} \|\partial_x^k u(t)\|_{L^2} + \sum_{k\leq s} (1 + t)^{\frac{n}{4}+\frac{k}{2}} \|\partial_x^k u_t(t)\|_{L^2} \right\}.$$

For $R > 0$, we define $X_R = \{u \in X : \|u\|_X \leq R\}$.

**Theorem 1.** *Let $n \geq 1$, $s \geq \max\{0, [\frac{n}{2}] - 1\}$ and suppose that functions $u_0 \in H^{s+2}(\mathbb{R}^n) \bigcap L^1(\mathbb{R}^n)$, $u_1 \in H^s(\mathbb{R}^n) \bigcap \dot{W}^{-1,1}(\mathbb{R}^n)$ and functions $f(v), g(v)$ are smooth and satisfy $f(v) = O(v^2)$, $g(v) = O(v^2)$ for $v \to 0$. Put*

$$E_0 := \|u_0\|_{L^1} + \|u_1\|_{\dot{W}^{-1,1}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}.$$

*If $E_0$ is suitably small, the Cauchy problem (1.1) and (1.2) has a unique global solution $u(x, t)$ satisfying*

$$X = u \in C([0, \infty); H^{s+2}(\mathbb{R}^n)) \bigcap C^1([0, \infty); H^s(\mathbb{R}^n)).$$

*Also, the solution satisfies the decay estimates*

$$\|\partial_x^k u(t)\|_{L^2} \leq CE_0(1 + t)^{-\frac{n}{4}-\frac{k}{2}}, \quad \|\partial_x^l u_t(t)\|_{L^2} \leq CE_0(1 + t)^{-\frac{n}{4}-\frac{l+1}{2}}$$

*for $0 \leq k \leq s + 2$ and $0 \leq l \leq s$.*

*Proof.* The Gagliardo-Nirenberg inequality gives

$$\|u(t)\|_{L^\infty} \le C\|\partial_x^{s_0} u\|_{L^2}^{\theta}\|u\|_{L^2}^{1-\theta} \le C(1+t)^{-\frac{n}{2}}\|u\|_X,$$

where $s_0 = \frac{n}{2}+1$, $\theta = \frac{n}{2s_0}$; i.e. $s \ge [\frac{n}{2}]-1$. We define (see (2.10))

$$\Phi(u) = G(t)*u_1 + H(t)*u_0 + \int_0^t G(t-\tau)*\Delta(Z(u,u_t)(\tau)))d\tau,$$

where $Z(u,u_t)(t)) = f(u(t)) - \beta g(u_t(t))$. We apply $\partial_x^k$ to $\Phi$ and take the $L^2$ norm. We obtain

$$\begin{aligned}\|\partial_x^k\Phi(u)\|_{L^2} &\le \|\partial_x^k G(t)*u_1\|_{L^2} + \|\partial_x^k H(t)*u_0\|_{L^2}\\ &+ C\int_0^t \|\partial_x^k G(t-\tau)*\Delta(Z(u,u_t)(\tau)))\|_{L^2}d\tau := I_1+I_2+J. \qquad (3.3)\end{aligned}$$

First, we estimate $I_1$. We apply (2.19) with $p=1$, $j=0$, $l=0$ and get

$$I_1 \le C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\|u_1\|_{\dot W^{-1,1}} + Ce^{-ct}\|\partial_x^{(k-2)_+}u_1\|_{L^2} \le CE_0(1+t)^{-\frac{n}{4}-\frac{k}{2}},$$

where $(k-2)_+ = \max\{k-2, 0\}$.

For the term $I_2$, we apply (2.20) with $p=1, j=0$ and $l=0$. This yields

$$I_2 \le C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2} \le CE_0(1+t)^{-\frac{n}{4}-\frac{k}{2}}.$$

Next, we estimate $J$. Let

$$\begin{aligned} J &= \int_0^t G(t-\tau)*\Delta(Z(u,u_t)(\tau)))d\tau\\ &= \int_0^{t/2} G(t-\tau)*\Delta(Z(u,u_t)(\tau)))d\tau\\ &\quad + \int_{t/2}^t G(t-\tau)*\Delta(Z(u,u_t)(\tau)))d\tau =: J_1+J_2.\end{aligned}$$

For the term $J_1$, using (2.23) with $p=1, j=0$ and $l=0$, we have

$$\begin{aligned} J_1 &\le C\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}}\|Z(u,u_t)(\tau))\|_{L^1}d\tau\\ &+ C\int_0^{t/2} e^{-c(t-\tau)}\|\partial_x^k(Z(u,u_t)(\tau)))\|_{L^2}d\tau =: J_{11}+J_{12}.\end{aligned}$$

Note that by lemma (3.1) we have

$$\begin{aligned} \|f(u)\|_{L^1} &\le C\|u\|_{L^2}^2 \le CR^2(1+\tau)^{-\frac{n}{2}},\\ |g(u_t)\|_{L^1} &\le C\|u_t\|_{L^2}^2 \le CR^2(1+\tau)^{-\frac{n}{2}}.\end{aligned}$$

Therefore we have

$$\begin{aligned} J_{11} &\le CR^2\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}}(1+\tau)^{-\frac{n}{2}}d\tau\\ &\le CR^2(1+t)^{-\frac{n}{4}-\frac{k+1}{2}}\int_0^{t/2}(1+\tau)^{\frac{n}{2}}d\tau\\ &\le CR^2(1+t)^{-\frac{n}{4}-\frac{k}{2}}\eta(t),\end{aligned}$$

where

$$\eta(t) = \begin{cases} 1, & n = 1, \\ (1+t)^{-\frac{1}{2}} \ln(2+t), & n = 2, \\ (1+t)^{-\frac{1}{2}}, & n \geq 3. \end{cases} \tag{3.4}$$

We use (3.1) and obtain

$$\|\partial_x^k (Z(u, u_t)(\tau)))\|_{L^2} \leq CR^2 (1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac{n}{2}}. \tag{3.5}$$

Consequently, we get

$$J_{12} \leq CR^2 \int_0^{t/2} e^{-c(t-\tau)} (1+\tau)^{-\frac{n}{4}-\frac{k}{2}-\frac{n}{2}} d\tau \leq CR^2 e^{-ct}.$$

Finally, we estimate the term $J_2$ on the time interval $[t/2, t]$. Applying (2.23) with $p = 2$, $j = k$, $l = 0$ and using (3.5), we can estimate term $J_2$ as

$$\begin{aligned} J_2 &\leq C \int_{t/2}^{t} (1+t-\tau)^{-\frac{1}{2}} \|\partial_x^k (Z(u, u_t)(\tau)))\|_{L^2} d\tau \\ &\quad + C \int_{t/2}^{t} e^{-c(t-\tau)} \|\partial_x^k (Z(u, u_t)(\tau)))\|_{L^2} \\ &\leq CR^2 (1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac{n-1}{2}}. \end{aligned} \tag{3.6}$$

Thus we have shown that

$$J \leq CR^2 (1+t)^{-\frac{n}{4}-\frac{k}{2}} \eta(t).$$

Substituting all these estimates into (3.3), we have

$$(1+t)^{\frac{n}{4}+\frac{k}{2}} \|\partial_x^k \Phi(u)\| \leq CE_0 + CR^2, \tag{3.7}$$

for $0 \leq k \leq s+2$. It follows from that (3.3)

$$\begin{aligned} \Phi(u)_t &= G_t(t) * u_1 + H_t(t) * u_0 \\ &\quad + \int_0^t G_t(t-\tau) * \Delta(Z(u, u_t)(\tau)))\|_{L^2} d\tau. \end{aligned} \tag{3.8}$$

We use $\partial_x^k$ to $\Phi(u)_t$ and take $L^2$ norm. This yields

$$\begin{aligned} \|\partial_x^k \Phi(u)_t\|_{L^2} &\leq \|\partial_x^k G_t(t) * u_1\|_{L^2} + \|\partial_x^k H_t(t) * u_0\|_{L^2} \\ &\quad + C \int_0^t \|\partial_x^k G_t(t-\tau) * \Delta(Z(u, u_t)(\tau)\|_{L^2} d\tau =: \acute{I}_1 + \acute{I}_2 + \acute{J}, \end{aligned} \tag{3.9}$$

for $0 \leq k \leq s$. For the term $\acute{I}_1$, we apply (2.21) with $p = 1$, $j = 0$ and $l = 0$ and obtain

$$\acute{I}_1 \leq C(1+t)^{-\frac{n}{4}-\frac{k+1}{2}} \|u_1\|_{\dot{W}^{-1,1}} + Ce^{-ct} \|\partial_x^k u_1\|_{L^2} \leq CE_0 (1+t)^{-\frac{n}{4}-\frac{k+1}{2}}.$$

Also, for the term $\acute{I}_2$, we apply (2.22) with $p = 1$, $j = 0$ and $l = 0$ and get

$$\acute{I}_2 \leq C(1+t)^{-\frac{n}{4}-\frac{k+1}{2}} \|u_0\|_{L^1} + Ce^{-ct} \|\partial_x^{k+2} u_0\|_{L^2} \leq CE_0 (1+t)^{-\frac{n}{4}-\frac{k+1}{2}}.$$

To estimate the nonlinear term $\acute{J}$, we divide as $\acute{J} = \acute{J}_1 + \acute{J}_2$, where $\acute{J}_1$ and $\acute{J}_2$ correspond to the time intervals $[0, t/2]$ and $[t/2, t]$, respectively. By applying (2.24) with $p = 1$, $j = 0$ and $l = 0$, we have

$$\acute{J}_1 \le C \int_0^{t/2} (1 + t - \tau)^{-\frac{n}{4} - \frac{k+2}{2}} \|Z(u, u_t)(\tau)\|_{L^1} d\tau \\ + C \int_0^{t/2} e^{-c(t-\tau)} \|\partial_x^{k+2}(Z(u, u_t)(\tau)\|_{L^2} d\tau =: \acute{J}_{11} + \acute{J}_{12}.$$

By (3.1), we obtain

$$\|Z(u, u_t)(\tau)\|_{L^1} \le CR^2(1 + \tau)^{-\frac{n}{2}}.$$

Therefore we get

$$\begin{aligned} \acute{J}_{11} &\le CR^2 \int_0^{t/2} (1 + t - \tau)^{-\frac{n}{4} - \frac{k+2}{2}} (1 + \tau)^{-\frac{n}{2}} d\tau \\ &\le CR^2(1 + t)^{-\frac{n}{4} - \frac{k+1}{2}} \eta(t). \end{aligned}$$

Similarly as before,we can estimate $\acute{J}_{12}$ and obtain $\acute{J}_{12} \le CR^2 e^{-ct}$. Finally, we estimate the term $\acute{J}_2$ by using (2.24) with $p = 2$, $j = k + 2$, $l = 0$ and get

$$\begin{aligned} \acute{J}_2 &\le C \int_{t/2}^{t} \|\partial_x^{k+2}(Z(u, u_t)((\tau))\|_{L^2} d\tau \\ &\quad + C \int_{t/2}^{t} -\frac{n}{4} - \frac{k+1}{2} - \frac{n-1}{2} e^{-c(t-\tau)} \|\partial_x^{k+2}(Z(u, u_t)(\tau))\|_{L^2} \\ &\le CR^2 \int_{t/2}^{t} (1 + \tau)^{-\frac{n}{4} - \frac{k+1}{2} - \frac{n}{2}} d\tau \le CR^2(1 + t)^{-\frac{n}{4} - \frac{k+1}{2} - \frac{n-1}{2}}. \end{aligned}$$

Consequently we have that

$$\acute{J} \le CR^2(1 + t)^{-\frac{n}{4} - \frac{k+1}{2}} \eta(t).$$

The above inequality implies

$$(1 + t)^{\frac{n}{4} + \frac{k+1}{2}} \|\partial_x^k \Phi(u)_t\|_{L^2} \le CE_0 + CR^2. \tag{3.10}$$

Combining (3.10) and (3.7) and taking $E_0$ and $R$ suitably small, we obtain $\|\Phi(u)\|_X \le R$. For $u, \tilde{u} \in X_R$, (3.3) gives

$$\begin{aligned} \|\partial_x^k(\Phi(u) - \Phi(\tilde{u}))\|_{L^2} &= \int_0^t \|\partial_x^k G(t - \tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t)) \\ &\quad - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau = \int_0^{t/2} \|\partial_x^k G(t - \tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t)) \\ &\quad - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau + \int_{t/2}^{t} \|\partial_x^k G(t - \tau) * \Delta(f(u) - f(\tilde{u}) \\ &\quad - \beta(g(u_t) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau =: J_1 + J_2. \end{aligned}$$

For the term $J_1$, we apply (2.23) with $p=1$, $j=0$ and $l=0$, we arrive at

$$\begin{aligned} J_1 &\le C\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}}\|(f(u)-f(\tilde u)(\tau)-\beta(g(u_t)-g(\tilde u))(\tau)\|_{L^1}d\tau \\ &+C\int_0^{t/2} e^{-c(t-\tau)}\|\partial_x^k(f(u)-f(\tilde u)-\beta(g(u_t-g(u_t)))(\tau)\|_{L^2}d\tau \\ &=: J_{11}+J_{12}. \end{aligned}$$

By (3.2), we can estimate $J_{11}$ as

$$\begin{aligned} J_{11} &\le CR\|u-\tilde u\|_X\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}}(1+\tau)^{-\frac{n}{2}}d\tau \\ &\le CR\|u-\tilde u\|_X(1+t)^{-\frac{n}{4}-\frac{k}{2}}(1+t)^{-\frac{n}{4}-\frac{k}{2}}\eta(t), \end{aligned}$$

where $\eta$ be defined in (3.4). It follows from the Gagliardo-Nirenberg inequality and (3.2) that

$$\begin{aligned} J_{12} &\le \int_0^{t/2} e^{-c(t-\tau)}\Big[(\|\partial_x^k u\|_{L^2}+\|\partial_x^k\tilde u\|_{L^2})\|u-\tilde u\|_{L^\infty} \\ &+(\|u\|_{L^\infty}+\|\tilde u\|_{L^\infty})\|\partial_x^k(u-\tilde u)\|_{L^2}+(\|\partial_x^k u_t\|_{L^2}+\|\partial_x^k\tilde u_t\|_{L^2}) \\ &\times\|u_t-\tilde u_t\|_{L^\infty}+(\|u_t\|_{L^\infty}+\|\tilde u_t\|_{L^\infty})\|\partial_x^k(u_t-\tilde u_t)\|_{L^2}\Big]d\tau \\ &\le CR\int_0^{t/2} e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{4}-\frac{k}{2}-\frac{n}{2}}\|u-\tilde u\|_X d\tau \\ &\le CR\|u-\tilde u\|_X e^{-ct}. \end{aligned}$$

Finally, we estimate term $J_2$ on the time $[t/2,t]$. Applying (2.23) with $p=2$, $j=k$, $l=0$ and using (3.2), we obtain

$$\begin{aligned} J_2 &\le C\int_{t/2}^t(1+t-\tau)^{-\frac{1}{2}}\|\partial_x^k(f(u)-f(\tilde u)-\beta(g(u_t)-g(\tilde u_t))(\tau))\|_{L^2}d\tau \\ &+C\int_{t/2}^t e^{-c(t-\tau)}\|\partial_x^k(f(u)-f(\tilde u)-\beta(g(u_t)-g(\tilde u_t))(\tau))\|_{L^2} \\ &\le CR\|u-\tilde u\|_X\int_{t/2}^t(1+t-\tau)^{-\frac{1}{2}}(1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac{n}{2}}d\tau \\ &\le CR\|u-\tilde u\|_X(1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac{n-1}{2}}, \end{aligned}$$

which implies

$$(1+t)^{\frac{n}{4}+\frac{k}{2}}\|\partial_x^k\Phi(u)-\Phi(\tilde u)\|_{L^2}\le CR\|u-\tilde u\|_X.$$

Similarly, for $0\le k\le s$ and $u,\tilde u\in X$ from (3.2) and (2.24), we deduce that

$$\begin{aligned} &\|\partial_x^k(\Phi(u)-\Phi(\tilde u)_t\|_{L^2} \\ &= \int_0^t\|\partial_x^k G_t(t-\tau)*\Delta(f(u)-f(\tilde u)-\beta(g(u_t)-g(\tilde u_t))(\tau)\|_{L^2}d\tau \end{aligned}$$

$$\begin{aligned} &= \int_0^{t/2} \|\partial_x^k G_t(t-\tau) * \Delta(f(u)-f(\tilde{u})-\beta(g(u_t)-g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\ &\quad + \int_{t/2}^{t} \|\partial_x^k G_t(t-\tau) * \Delta(f(u)-f(\tilde{u})-\beta(g(u_t)-g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\ &=: \acute{J}_1 + \acute{J}_2. \end{aligned}$$

For the term $J_1$, we use (2.24) with $p=1$, $j=0$ and $l=2$. We have

$$\begin{aligned} \acute{J}_1 &\le C\int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} \|f(u)-f(\tilde{u})(\tau)-\beta(g(u_t)-g(\tilde{u}_t))(\tau)\|_{L^1} d\tau \\ &\quad + C\int_0^{t/2} e^{-c(t-\tau)} \|\partial_x^{k+2}(f(u)-f(\tilde{u}_t)-\beta(g(u_t)-g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\ &=: \acute{J}_{11} + \acute{J}_{12}. \end{aligned}$$

By (3.2), we have

$$\begin{aligned} \acute{J}_{11} &\le \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} (\|u\|_{L^2}+\|\tilde{u}_{L^2})\|u-\tilde{u}\|_{L^2} \\ &\quad +(\|u_t\|_{L^2}+\|\tilde{u}_t\|_{L^2})(\|u_t-\tilde{u}_t\|_{L^2}) d\tau \\ &\le CR\|u-\tilde{u}\|_X (1+t)^{-\frac{n}{4}-\frac{k+1}{2}} \int_0^{t/2} (1+t-\tau)^{-\frac{n}{2}} d\tau \\ &\le CR|u-\tilde{u}\|_X (1+t)^{-\frac{n}{4}-\frac{k+1}{2}} \eta(t), \end{aligned}$$

where $\eta$ be defined in (3.4). Also, the term $\acute{J}_{12}$ is estimated similarly as before and we can estimate the term $\acute{J}_{12}$ as

$$\acute{J}_{12} \le CR\|u-\tilde{u}\|_X e^{-ct}.$$

By applying (2.24) with $p=2$, $j=k+2$, $l=0$ and (3.1), we obtain

$$\begin{aligned} \acute{J}_2 &\le C\int_{t/2}^{t} \|\partial_x^{k+2}(f(u)-f(\tilde{u})-\beta(g(u_t)-g(\tilde{u}_t)(\tau))\|_{L^2} d\tau \\ &\quad + C\int_{t/2}^{t} e^{-c(t-\tau)} \|\partial_x^{k+2}(f(u)-f(\tilde{\tilde{u}})-\beta(g(u_t)-g(\tilde{u}_t)(\tau))\|_{L^2} d\tau \\ &\le \int_{t/2}^{t} (\|\partial_x^{k+2}u\|_{L^2}+\|\partial_x^{k+2}\tilde{u}\|_{L^2})\|u-\tilde{u}\|_{L^\infty} \\ &\quad +(\|u\|_{L^\infty}+\|\tilde{u}\|_{L^\infty})\|\partial_x^{k+2}(u-\tilde{u})\|_{L^2} \\ &\quad +((\|\partial_x^{k+2}u_t\|_{L^2}+\|\partial_x^{k+2}\tilde{u}_t\|_{L^2})\|u-\tilde{u}_t\|_{L^\infty} \\ &\quad +(\|u_t\|_{L^\infty}+\|\tilde{u}_t\|_{L^\infty})\|\partial_x^{k+2}(u-\tilde{u}_t)\|_{L^2}) d\tau \\ &\le CR\|u-\tilde{u}\|_X \int_{t/2}^{t} (1+\tau)^{-\frac{n}{4}-\frac{k+2}{2}-\frac{n}{2}} d\tau \\ &\le CR\|u-\tilde{u}\|_X (1+t)^{-\frac{n}{4}-\frac{k+1}{2}-\frac{n-1}{2}} \\ &\le CR\|u-\tilde{u}\|_X (1+t)^{-\frac{n}{4}-\frac{k+1}{2}}. \end{aligned}$$

Substituting all these estimates together with the previous estimate and taking $R$ suitably small, yields

$$\|\Phi(u) - \Phi(\tilde{u})\|_X \leq \frac{1}{2}\|u - \tilde{u}\|_X. \tag{3.11}$$

From (3.11), we deduce that $\Phi$ is strictly contracting mapping. Then there exists a fixed point $u \in X_R$ of the mapping $\Phi$, which is a solution to (1.1)–(1.2).The proof of the theorem is now complete. □

The proof of the previous theorem shows that when $n \geq 2$, the solution $u$ to the integral equation (2.10) is asymptotic to the linear solution $u_L(t)$ given by the formula $u_L(t) = G(t) * u_1 + H(t) * u_0$ as $t \to \infty$. This result is stated as follows.

**Lemma 5.** *Let $n \geq 2$ and assume the same conditions of Theorem 1. Then the solution $u$ of problem* (1.1)–(1.2) *which is constructed in Theorem 1, can be approximated by the solution $u_L$ to the linearized problem* (2.1), (2.2) *as $t \to \infty$. More precisely, we have*

$$\|\partial_x^k(u - u_L)(t)\|_{L^2} \leq CE_0^2(1+t)^{-\frac{n}{4}-\frac{k}{2}}\eta(t),$$
$$\|\partial_x^k(u - u_L)_t(t)\|_{L^2} \leq CE_0^2(1+t)^{-\frac{n}{4}-\frac{k+1}{2}}\eta(t),$$

*for $0 \leq k \leq s+2$ and $0 \leq k \leq s$, respectively, where $u_L(t) := G(t)*u_1+H(t)*u_0$ is the linear solution and $\eta(t)$ is defined in* (3.4).

## 4 Decay estimates of solutions for $L^2$

In the previous section, we have proved global existence and asymptotic behavior of solutions to the Cauchy problem (1.1)–(1.2) with $L^1$ data.

In this section, we prove a similar decay estimate of solution with $L^2$ data for $n \geq 2$. Based on the decay estimates of solutions to the linear problem (2.1)–(2.2), we define the following solution space:

$$X = \{u \in C([0,\infty); H^{s+2}(\mathbb{R}^n)) \bigcap C^1([0,\infty); H^s(\mathbb{R}^n)) : \|u\|_X < \infty\},$$

where

$$\|u\|_X = \sup_{t\geq 0}\left\{\sum_{k\leq s+2}(1+t)^{\frac{k}{2}}\|\partial_x^k u(t)\|_{L^2} + \sum_{k\leq s}(1+t)^{\frac{k}{2}}\|\partial_x^k u_t(t)\|_{L^2}\right\}.$$

For $R > 0$, we define

$$X_R = \{u \in X : \|u\|_X \leq R\}.$$

Note that from the Gagliardo-Nirenberg inequality for $u \in X_R$, we have

$$\|u(t)\|_{L^\infty} \leq C(1+t)^{-\frac{n}{4}}.$$

**Theorem 2.** *Suppose that* $u_0 \in H^{s+2}, u_1 \in H^s(\mathbb{R}^n)\bigcap \dot{W}^{-1,2}(\mathbb{R}^n)$*, such that* $n \geq 1, s \geq max\{0, [\frac{n}{2}]-1\}$*, and the functions* $f(v)$ *and* $g(v)$ *are smooth and satisfies* $f(v) = O(v^2), g(v) = O(v^2)$ *for* $v \to 0$*. Let*

$$E_1 := \|u_0\|_{L^2} + \|u_1\|_{\dot{W}^{-1,2}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}.$$

*If* $E_0$ *is suitably small, the Cauchy problem* (1.1) *and* (1.2) *has a unique global solution* $u(x,t)$ *satisfying*

$$X = u \in C([0,\infty); H^{s+2}(\mathbb{R}^n))\bigcap C^1([0,\infty); H^s(\mathbb{R}^n)).$$

*The solution* $u$ *also satisfies the decay estimate*

$$\|\partial_x^k u(t)\|_{L^2} \leq CE_1(1+t)^{-\frac{k}{2}}, \quad \|\partial_x^h u_t(t)\|_{L^2} \leq CE_1(1+t)^{-\frac{h+1}{2}}, \tag{4.1}$$

*for* $0 \leq k \leq s+2$ *and* $0 \leq h \leq s$*.*

*Proof.* Let the mapping $\Phi$ be defined in (3.3). Applying $\partial_x^k$ to $\Phi$ and take $L^2$ norm. We have

$$\begin{aligned}\|\partial_x^k \Phi(u)\|_{L^2} &\leq \|\partial_x^k G(t) * u_1\|_{L^2} + \|\partial_x^k H(t) * u_0\|_{L^2} \\ &\quad + C\int_0^t \|\partial_x^k G(t-\tau) * \Delta(Z(u,u_t)(\tau)\|_{L^2} d\tau \ := I_1 + I_2 + J.\end{aligned}$$

We use (2.19) with $p = 2$ and $j = l = 0$ and get

$$I_1 \leq C(1+t)^{-\frac{k}{2}}\|u_1\|_{\dot{W}^{-1,2}} + Ce^{-ct}\|\partial_x^{(k-2)_+} u_1\|_{L^2} \leq CE_1(1+t)^{-\frac{k}{2}},$$

where $(k-2)_+ = \max\{k-2, 0\}$. By applying (2.20) with $p = 2$ and $j = l = 0$, we get

$$I_2 \leq C(1+t)^{-\frac{k}{2}}\|u_0\|_{L^2} + Ce^{-ct}\|\partial_x^k u_0\|_{H^{s+2}} \leq CE_1(1+t)^{-\frac{k}{2}}.$$

To estimate the nonlinear $J$, as in the previous section, we divide as $J = J_1 + J_2$ where $J_1$ and $J_2$ correspond to the time intervals $[0, t/2]$ and $[t/2, t]$, respectively. For the term $J_1$, we use (2.23) with $p = 1$ and $j = l = 0$ and deduce that

$$\begin{aligned}J_1 &\leq C\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}}\|Z(u,u_t)(\tau)\|_{L^1} d\tau \\ &\quad + C\int_0^{t/2} e^{-c(t-\tau)}\|\partial_x^k(Z(u,u_t)(\tau)\|_{L^2} d\tau \ =: J_{11} + J_{12}.\end{aligned}$$

By (3.1), we have $\|Z(u,u_t)(\tau)\|_{L^1} \leq CR^2$. Thus we can estimate the $J_{11}$ as

$$\begin{aligned}J_{11} &\leq CR^2\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}}(1+\tau)^{-\frac{n}{2}} d\tau \\ &\leq CR^2(1+t)^{-\frac{k}{2}}\int_0^{t/2}(1+\tau)^{-\frac{n}{4}-\frac{1}{2}} d\tau \ \leq \ CR^2(1+t)^{-\frac{k}{2}}.\end{aligned}$$

By applying (3.1) and the Gagliardo-Nirenberg inequality, we get

$$\begin{aligned} \|\partial_x^k f(u)\|_{L^2} &\leq C\|u\|_{L^\infty}\|\partial_x^k u\|_{L^2} \leq C(1+t)^{\frac{n}{4}-\frac{k}{2}}R^2, \\ |\partial_x^k g(u_t)\|_{L^2} &\leq C\|u_t\|_{L^\infty}\|\partial_x^k u_t\|_{L^2} \leq C(1+t)^{\frac{n}{4}-\frac{k}{2}}R^2. \end{aligned} \tag{4.2}$$

Thus we have

$$J_{12} \leq CR^2\int_0^{t/2} e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{4}-\frac{k}{2}}d\tau \leq CR^2 e^{-ct}.$$

It follows from (2.23) with $p=1, j=k$ and $l=2$ that

$$\begin{aligned} J_2 &\leq C\int_{t/2}^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\|\partial_x^k(Z(u,u_t)(\tau)\|_{L^1}d\tau \\ &\quad +C\int_{t/2}^t e^{-c(t-\tau)}\|\partial_x^{k+2}(Z(u,u_t)(\tau)\|_{L^2} =: J_{21}+J_{22}. \end{aligned}$$

We have from (3.2) that

$$\begin{aligned} J_{21} &\leq C\int_{t/2}^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}(\|u\|_{L^2}\|\|\partial_x^k u\|_{L^2}+\|u_t\|_{L^2}\|\partial_x^k u_t\|_{L^2})d\tau \\ &\leq CR^2\int_{t/2}^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}(1+\tau)^{-\frac{k}{2}}d\tau \\ &\leq CR^2(1+t)^{-\frac{k}{2}}\int_{t/2}^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}d\tau \leq CR^2(1+t)^{-\frac{k}{2}}. \end{aligned}$$

To estimate the term $J_{22}$, we have from (4.2) that

$$\begin{aligned} J_{22} &\leq C\int_{t/2}^t e^{-c(t-\tau)}\|\partial_x^{k+2}(Z(u,u_t)(\tau)\|_{L^2}d\tau \\ &\leq CR^2\int_{t/2}^t e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{4}-\frac{k+2}{2}}d\tau \leq CR^2(1+t)^{-\frac{k}{2}}. \end{aligned}$$

The above inequality shows that

$$(1+t)^{\frac{k}{2}}\|\partial_x^k\Phi(u)\| \leq CE_1+CR^2. \tag{4.3}$$

We deduce from (3.3) that

$$(\Phi(u))_t = G_t(t)*u_1+H_t(t)*u_0+\int_0^t G_t(t-\tau)*\Delta(Z(u,u_t)(\tau)\|_{L^2}d\tau. \tag{4.4}$$

Applying $\partial_x^k$ to $\Phi(u)_t$ and taking $L^2$-norm we have

$$\begin{aligned} \|\partial_x^k\Phi(u)_t\|_{L^2} &\leq \|\partial_x^k G_t(t)*u_1\|_{L^2}+\|\partial_x^k H_t(t)*u_0\|_{L^2} \\ &\quad +C\int_0^t \|\partial_x^k G_t(t-\tau)*\Delta(Z(u,u_t)(\tau)\|_{L^2}d\tau =: \acute{I}_1+\acute{I}_2+\acute{J}. \end{aligned}$$

To estimate the term $\acute{I}_1$, apply (2.21) with $p=2, l=j=0$. It yields

$$\acute{I}_1 \le C(1+t)^{-\frac{k+1}{2}}\|u_1\|_{\dot{W}^{-1,2}} + Ce^{-ct}\|\partial_x^k u_1\|_{L^2} \le CE_1(1+t)^{-\frac{k+1}{2}}.$$

Similarly, using (2.22) with $p=2, j=l=0$, we have

$$\acute{I}_2 \le C(1+t)^{-\frac{k+1}{2}}\|u_0\|_{L^2} + Ce^{-ct}\|\partial_x^{k+2} u_0\|_{L^2} \le CE_1(1+t)^{-\frac{k+1}{2}}.$$

To estimate the nonlinear term $\acute{J}$, let

$$\begin{aligned}\acute{J} = C\int_0^{t/2} \|\partial_x^k G_t(t-\tau) * \Delta(Z(u,u_t)(\tau)\|_{L^2} d\tau \\ + C\int_{t/2}^{t} \|\partial_x^k G_t(t-\tau) * \Delta(Z(u,u_t)(\tau)\|_{L^2} d\tau \;=:\; \acute{I}_1 + \acute{I}_2 + \acute{J}.\end{aligned}$$

It yields from (2.24) with $p=1$ and $j=l=0$ that

$$\begin{aligned}\acute{J}_1 \;\;\le\;\; & C\int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}}\|Z(u,u_t)(\tau)\|_{L^1} d\tau \\ & + C\int_0^{t/2} e^{-c(t-\tau)}\|\partial_x^{k+2}(Z(u,u_t)(\tau)\|_{L^2} d\tau \;:=\; \acute{J}_{11} + \acute{J}_{12}.\end{aligned}$$

We obtain from (3.1) that

$$\|Z(u,u_t)(\tau)\|_{L^1} \le CR^2.$$

Thus we can estimate $\acute{J}_{11}$ as

$$\begin{aligned}\acute{J}_{11} &\le CR^2\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} d\tau \\ &\le CR^2(1+t)^{-\frac{k+1}{2}}\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} d\tau \;\le\; CR^2(1+t)^{-\frac{k+1}{2}}.\end{aligned}$$

For the term $\acute{J}_{12}$, we have from (4.2) that

$$\acute{J}_{12} \;\;\le\;\; CR^2\int_0^{t/2} e^{-c(t-\tau)}(1+t)^{-\frac{n}{4}-\frac{k+2}{2}} d\tau \le CR^2 e^{-ct}.$$

Applying (2.24) with $p=2, j=k+2$ and $l=0$, we get

$$\begin{aligned}\acute{J}_2 \;\;\le\;\; & C\int_{t/2}^{t} \|\partial_x^{k+2}(Z(u,u_t)(\tau))\|_{L^2} d\tau \\ & + C\int_{t/2}^{t} e^{-c(t-\tau)}\|\partial_x^{k+2}(Z(u,u_t)(\tau))\|_{L^2} d\tau \\ \le\;\; & CR^2\int_{t/2}^{t}(1+\tau)^{-\frac{n}{4}-\frac{k+2}{2}} d\tau \\ \le\;\; & CR^2(1+t)^{-\frac{k+1}{2}}\int_{t/2}^{t}(1+\tau)^{-\frac{n}{4}-\frac{1}{2}} d\tau \le CR^2(1+t)^{-\frac{k+1}{2}}.\end{aligned}$$

Thus we have

$$(1+t)^{-\frac{k+1}{2}}\|\partial_x^k \Phi(u)_t\|_{L^2} \le CE_1 + CR^2. \tag{4.5}$$

Combining (4.3) and (4.5) and taking $E_0$ and $R$ suitably small, we obtain $\|\Phi(u)\|_X \le R$.

For $u, \tilde{u} \in X_R$, by using (3.3) we obtain

$$\begin{aligned}
&\|\partial_x^k(\Phi(u) - \Phi(\tilde{u}))\|_{L^2} \\
&\quad = \int_0^t \|\partial_x^k G(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t)) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\
&\quad = \int_0^{t/2} \|\partial_x^k G(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t)) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\
&\qquad + \int_{t/2}^t \|\partial_x^k G(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t)) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\
&\quad =: J_1 + J_2.
\end{aligned}$$

By applying (2.23) with $p = 1, j = 0$ and $l = 0$, we have

$$\begin{aligned}
J_1 &\le C \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} \|(f(u) - f(\tilde{u})(\tau) - \beta(g(u_t) - g(\tilde{u}_t))(\tau)\|_{L^1} d\tau \\
&\quad + C \int_0^{t/2} e^{-c(t-\tau)} \|\partial_x^k(f(u) - f(\tilde{u}) - \beta(g(u_t - g(\tilde{u}_t)))(\tau)\|_{L^2} d\tau \\
&=: J_{11} + J_{12}.
\end{aligned}$$

Using (3.2), we get

$$\begin{aligned}
J_{11} &\le CR\|u - \tilde{u}\|_X \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+1}{2}} d\tau \\
&\le CR\|u - \tilde{u}\|_X (1+t)^{-\frac{k}{2}} \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} d\tau. \\
&\le CR\|u - \tilde{u}\|_X (1+t)^{-\frac{k}{2}}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
J_{12} &\le \int_0^{t/2} e^{-c(t-\tau)} \Big[ (\|\partial_x^k u\|_{L^2} + \|\partial_x^k \tilde{u}\|_{L^2}) \|u - \tilde{u}\|_{L^\infty} \\
&\quad + (\|u\|_{L^\infty} + \|\tilde{u}\|_{L^\infty}) \|\partial_x^k(u - \tilde{u})\|_{L^2} + (\|\partial_x^k u_t\|_{L^2} + \|\partial_x^k \tilde{u}_t\|_{L^2}) \|u_t - \tilde{u}_t\|_{L^\infty} \\
&\quad + (\|u_t\|_{L^\infty} + \|\tilde{u}_t\|_{L^\infty}) \|\partial_x^k(u_t - \tilde{u}_t)\|_{L^2} \Big] d\tau \\
&\le CR\|u - \tilde{u}\|_X \int_0^{t/2} e^{-c(t-\tau)} (1+\tau)^{-\frac{n}{4}-\frac{k}{2}} d\tau \le CR\|u - \tilde{u}\|_X e^{-ct}.
\end{aligned}$$

To estimate the term $J_2$, apply (2.23) with $p = 1, j = k, l = 2$. We obtain

$$\begin{aligned}
J_2 &\le C \int_{t/2}^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} \|\partial_x^k(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^1} d\tau \\
&\quad + C \int_{t/2}^t e^{-c(t-\tau)} \|\partial_x^{k+2}(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t))(\tau))\|_{L^2} d\tau =: J_{21} + J_{22}.
\end{aligned}$$

By using (3.2), we get

$$
\begin{aligned}
J_{21} \leq & \int_{t/2}^{t} (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} \Big[ (\|\partial_x^k u\|_{L^2} + \|\partial_x^k \tilde{u}\|_{L^2}) \|u-\tilde{u}\|_{L^2} \\
& + (\|u\|_{L^2} + \|\tilde{u}\|_{L^2})) \|\partial_x^k (u-\tilde{u})\|_{L^2} + (\|\partial_x^k u_t\|_{L^2} + \|\partial_x^k \tilde{u}_t\|_{L^2}) \|u_t - \tilde{u}_t\|_{L^2} \\
& + (\|u_t\|_{L^2} + \|\tilde{u}_t\|_{L^2}) \|\partial_x^k (u_t - \tilde{u}_t)\|_{L^2} \Big] d\tau \\
& \leq CR\|u-\tilde{u}\|_X \int_{t/2}^{t} (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}} (1+\tau)^{-\frac{k}{2}} d\tau \\
& \leq CR\|u-\tilde{u}\|_X (1+t)^{-\frac{k}{2}}.
\end{aligned}
$$

Finally, we estimate the term $J_{22}$ as

$$
\begin{aligned}
J_{22} \leq & \int_{t/2}^{t} e^{-c(t-\tau)} \Big[ (\|\partial_x^{k+2} u\|_{L^2} + \|\partial_x^{k+2} \tilde{u}\|_{L^2}) \|u-\tilde{u}\|_{L^2} \\
& + (\|u\|_{L^2} + \|\tilde{u}\|_{L^2}) \|\partial_x^k (u-\tilde{u})\|_{L^2} + (\|\partial_x^k u_t\|_{L^2} + \|\partial_x^k \tilde{u}_t\|_{L^2}) \|u_t - \tilde{u}_t\|_{L^2} \\
& + (\|u_t\|_{L^2} + \|\tilde{u}_t\|_{L^2}) \|\partial_x^k (u_t - \tilde{u}_t)\|_{L^2} \Big] d\tau \\
& \leq CR\|u-\tilde{u}\|_X \int_{t/2}^{t} e^{-c(t-\tau)} (1+\tau)^{-\frac{n}{4}-\frac{k+2}{2}} d\tau \\
& \leq CR\|u-\tilde{u}\|_X (1+\tau)^{\frac{k}{2}}.
\end{aligned}
$$

Thus we have shown that

$$
(1+t)^{\frac{k}{2}} \|\partial_x^k (\Phi(u) - \Phi(\tilde{u}))\|_{L^2} \leq CR\|u-\tilde{u}\|_X. \tag{4.6}
$$

Suppose that $u, \tilde{u} \in X_R$. It follows from (3.3) that

$$
\begin{aligned}
& \|\partial_x^k (\Phi(u) - \Phi(\tilde{u})_t\|_{L^2} \\
& \quad = \int_0^t \|\partial_x^k G_t(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t)) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\
& \quad = \int_0^{t/2} \|\partial_x^k G_t(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t)) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\
& \qquad + \int_{t/2}^{t} \|\partial_x^k G_t(t-\tau) * \Delta(f(u) - f(\tilde{u}) - \beta(g(u_t)) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\
& \quad =: \acute{J}_1 + \acute{J}_2.
\end{aligned}
$$

By using (2.24) with $p=1$, $j=0$, we have

$$
\begin{aligned}
\acute{J}_1 \leq & C \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} \|f(u) - f(\tilde{u})(\tau) - \beta(g(u_t) - g(\tilde{u}_t))(\tau)\|_{L^1} d\tau \\
& + C \int_0^{t/2} e^{-c(t-\tau)} \|\partial_x^{k+2} (f(u) - f(\tilde{u}_t) - \beta(g(u_t) - g(\tilde{u}_t))(\tau)\|_{L^2} d\tau \\
& =: \acute{J}_{11} + \acute{J}_{12}.
\end{aligned}
$$

By (3.2), we obtain

$$\begin{aligned} J_{11}' &\leq \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{k+2}{2}} (\|u\|_{L^2} + \|\tilde{u}_{L^2})\|u-\tilde{u}\|_{L^2} \\ &\quad \times (\|u_t\|_{L^2} + \|\tilde{u}_t\|_{L^2})(\|u_t - \tilde{u}_t\|_{L^2}) d\tau \\ &\leq CR\|u-\tilde{u}\|_X (1+t)^{-\frac{k+1}{2}} \int_0^{t/2} (1+t\tau)^{-\frac{n}{4}-\frac{1}{2}} d\tau \\ &\leq CR|u-\tilde{u}\|_X (1+t)^{-\frac{k+1}{2}}. \end{aligned}$$

For the term $J_{12}'$, by (3.2) we get

$$\begin{aligned} J_{12}' \leq \int_0^{t/2} & e^{-c(t-\tau)} \big[ (\|\partial_x^{k+2} u\|_{L^2} + \|\partial_x^{k+2} \tilde{u}\|_{L^2}) \|u-\tilde{u}\|_{L^\infty} \\ & + (\|u\|_{L^\infty} + \|\tilde{u}\|_{L^\infty})) \|\partial_x^{k+2}(u-\tilde{u})\|_{L^2} \\ & + (\|\partial_x^{k+2} u_t\|_{L^2} + \|\partial_x^{k+2} \tilde{u}_t\|_{L^2}) \|u_t - \tilde{u}_t\|_{L^\infty} \\ & + (\|u_t\|_{L^\infty} + \|\tilde{u}_t\|_{L^\infty}) \|\partial_x^{k+2}(u_t - \tilde{u}_t)\|_{L^2} \big] d\tau \\ \leq CR\|u-\tilde{u}\|_X & \int_0^{t/2} e^{-c(t-\tau)} (1+\tau)^{-\frac{k+2}{2}} (1+\tau)^{-\frac{n}{4}} d\tau \\ \leq CR\|u-\tilde{u}\|_X & e^{-ct}. \end{aligned}$$

By applying (2.24) with $p=2$, $j=k+2$ and $l=0$, we conclude that

$$\begin{aligned} J_2' \leq\ & C \int_{t/2}^t \|\partial_x^{k+2}(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t)(\tau)))\|_{L^2} d\tau \\ & + C \int_{t/2}^t e^{-c(t-\tau)} \|\partial_x^{k+2}(f(u) - f(\tilde{u}) - \beta(g(u_t) - g(\tilde{u}_t)(\tau)))\|_{L^2} d\tau \\ \leq\ & \int_{t/2}^t \big[ (\|\partial_x^{k+2} u\|_{L^2} + \|\partial_x^{k+2} \tilde{u}\|_{L^2}) \|u-\tilde{u}\|_{L^\infty} \\ & + (\|u\|_{L^\infty} + \|\tilde{u}\|_{L^\infty}) \|\partial_x^{k+2}(u-\tilde{u})\|_{L^2} \\ & + (\|\partial_x^{k+2} u_t\|_{L^2} + \|\partial_x^{k+2} \tilde{u}_t\|_{L^2}) \|u - \tilde{u}_t\|_{L^\infty} \\ & + (\|u_t\|_{L^\infty} + \|\tilde{u}_t\|_{L^\infty}) \|\partial_x^{k+2}(u - \tilde{u}_t)\|_{L^2} \big] d\tau \\ \leq\ & CR\|u-\tilde{u}\|_X \int_{t/2}^t (1+\tau)^{-\frac{n}{4}-\frac{k+2}{2}} d\tau \\ \leq\ & CR\|u-\tilde{u}\|_X (1+t)^{-\frac{k+1}{2}} \int_{t/2}^t (1+\tau)^{-\frac{n}{4}-\frac{1}{2}} d\tau \\ \leq\ & CR\|u-\tilde{u}\|_X (1+t)^{-\frac{k+1}{2}}. \end{aligned}$$

Consequently, we have shown that

$$(1+t)^{\frac{k+1}{2}} \| \ \partial_x^k (\Phi(u) - \Phi(\tilde{u}))_t \|_X \leq CR\|u-\tilde{u}\|_X. \tag{4.7}$$

Using (4.5) and (4.7) and taking $R$ suitably small, it yields

$$\|\Phi(u) - \Phi(\tilde{u})\|_X \leq \frac{1}{2}\|u-\tilde{u}\|_X. \tag{4.8}$$

From (4.8), we conclude that $\Phi$ is a contracting mapping. Then there exists a fixed point $u \in X_R$ of mapping $\Phi$, which is a solution (1.1) and (1.2) and the proof is completed. □

Finally we study the asymptotic linear profile of the solution.

Suppose that $u_L$ given by the formula $u_L(t) = G(t) * u_1 + H(t) * u_0$. In the previous two section, we have shown that the solution $u$ to problem (1.1) and (1.2) can be approximated by the linear solution $u_L$. Now the aim is to derive a simpler asymptotic profile of the linear solution $u_L$.

In the Fourier space, we obtain $\hat{u}_L(\xi,t) = \hat{G}(\xi,t)\hat{u}_1(\xi,t) + \hat{H}(\xi,t)\hat{u}_0(\xi)$, where $\hat{G}(\xi,t)$ and $\hat{H}(\xi,t)$ are given explicitly in (2.6) and (2.7). First we give the asymptotic expansions of $\hat{G}(\xi,t)$ and $\hat{H}(\xi,t)$ for $\xi \to 0$. By using the Taylor expansion to (2.4), we obtain

$$\begin{aligned}\lambda_\pm(\xi) &= \frac{1}{2}(\alpha|\xi|^2 - |\xi|^4) \pm \frac{|\xi|i}{2}(2 + |\xi|^2 - \frac{\alpha^2}{4}|\xi|^4 + O(|\xi|^4)\\ &= \pm i|\xi| + \frac{\alpha}{2}|\xi|^2 + O(|\xi|^3)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\lambda_+ - \lambda_-} &= \frac{1}{i|\xi|\sqrt{4 + 4|\xi|^2 - |\xi|^6 - \alpha^2|\xi|^4 + 2\alpha|\xi|^5}}\\ &= \frac{1}{2i|\xi|}(1 - \frac{1}{2}|\xi|^2 + O(|\xi|^4)).\end{aligned}$$

Substituting these expansions to (2.6) and (2.7), we obtain

$$\begin{aligned}\hat{G}(\xi,t) &= \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\\ &= \frac{1}{2i|\xi|}\Big(e^{\frac{\alpha}{2}|\xi|^2 t}(e^{|\xi|ti} - e^{-|\xi|ti}) + e^{\frac{\alpha}{2}|\xi|^2 t}(O(|\xi|^2) + O(|\xi|^3 t))\Big)\end{aligned}$$

and

$$\begin{aligned}\hat{H}(\xi,t) &= \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\\ &= \frac{1}{2}e^{\frac{\alpha}{2}|\xi|^2 t}(e^{|\xi|ti} + e^{-|\xi|ti}) + e^{\frac{\alpha}{2}|\xi|^2 t}(O(|\xi|^2) + O(|\xi|^3 t))\end{aligned}$$

for $\xi \to 0$. Let

$$\hat{G}_0(\xi,t) = \frac{1}{2i|\xi|}e^{\frac{\alpha}{2}|\xi|^2 t}(e^{i|\xi|t} - e^{-i|\xi|t}), \quad \hat{H}_0(\xi,t) = \frac{1}{2}e^{\frac{\alpha}{2}|\xi|^2 t}(e^{i|\xi|t} + e^{-i|\xi|t}).$$

Thus for $|\xi| \le r_0$ we obtain

$$|(\hat{G} - \hat{G}_0)(\xi,t)| \le Ce^{-c|\xi|^2 t}, \quad |(\hat{H} - \hat{H}_0)(\xi,t)| \le C|\xi|e^{-c|\xi|^2 t},$$

where $r_0$ is a small positive constant. Now we define $\overline{u}_L$ by

$$\overline{u}_L(t) = G_0(t) * u_1 + H_0(t) * u_0. \tag{4.9}$$

$\overline{u}_L$ gives an asymptotic profile of the linear solution $u_L$.

**Theorem 3.** *Suppose that* $n \geq 1$, $s \geq 0$ *and* $u_0 \in H^{s+2} \bigcap L^1$ *and* $u_1 \in H^s \bigcap \dot{W}^{-1,1}$. *Put* $E_0 = \|u_0\|_{L^1} + \|u_1\|_{\dot{W}^{-1,1}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}$. *Let* $u_L$ *be the linear solution and* $\overline{u}_L$ *be defined by* (4.9). *Thus we have*

$$\|\partial_x^k (u_L - \overline{u}_L)(t)\|_{L^2} \leq C E_0 (1+t)^{-\frac{n}{4} - \frac{k+1}{2}}$$

*for* $0 \leq k \leq s+2$.

*Proof.* It follows from definition that

$$(u_L - \overline{u}_L)(t) = (G - G_0)(t) * u_1 + (H - H_0)(t) * u_0.$$

So it suffices to show the following estimates:

$$\begin{aligned}\|\partial_x^k (G - G_0)(t) * u_1\|_{L^2} \leq\ & C(1+t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+1-j}{2}} \|\partial_x^j u_1\|_{\dot{W}^{-1,p}} \\ & + Ce^{-ct} \|\partial_x^{k+l-2} u_1\|_{L^2},\end{aligned}$$

$$\begin{aligned}\|\partial_x^k (H - H_0)(t) * u_0\|_{L^2} \leq\ & C(1+t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+1-j}{2}} \|\partial_x^j u_0\|_{L^p} \\ & + Ce^{-ct} \|\partial_x^{k+l} \phi\|_{L^2},\end{aligned}$$

where $1 \leq p \leq 2$, and $k, j$ and $l$ are nonnegative integers such that $0 \leq j \leq k+1$. We assumed $k + l - 2 \geq 0$ in the first estimate. These estimates can be proved similarly as in the proof of Lemma 5 by using (4.1) for $|\xi| \leq r_0$ and (2.17) and (2.17) and (4.4) for $|\xi| \geq r_0$. We omit the details. □

## 5 Conclusions

In the body of the paper, we have considered a dissipative Boussinesq-type equation. This equation (see Equation (1.1)) arises in the study of the stability of one-dimensional periodic patterns in systems with Galilean invariance and also the oscillations of elastic beams. Here we have shown that the Cauchy problem associated to this equation has a unique global solution in some suitable Sobolev space. The main difficulty was how to control the nonlinear term of the equation which is a combination of $u$ and $u_t$. Under small condition on the initial value, we proves the existence by the contraction mapping principle due to small nonlinear terms. We also obtained some asymptotic behavior of global solutions in some time weighted spaces. In the presence of the dissipation and small nonlinearities, we established that the solution the nonlinear initial value problem behaves like the linear solution, and thus it can be approximated by the linear solution.

In the future, we plan to study this equation with more general nonlinearity. The sharpness of the decay estimates and low regularity of the solutions will be also very important and interesting issues.

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