Special Splines of Hyperbolic Type for the Solutions of Heat and Mass Transfer 3-D Problems in Porous Multi-Layered Axial Symmetry Domain

Harijs Kalis[1], Andris Buikis[1], Aivars Aboltins[2] and Ilmars Kangro[3]

[1] *Institute of Mathematics and Computer Science of University of Latvia*
Raina bulvāris 29, LV-1459 Rīga, Latvija

[2] *Latvia University of Agriculture, Faculty of Engineering*
5 J. Cakstes blvd., LV-3001 Jelgava, Latvia

[3] *Rezekne Academy of Technologies, Faculty of Engineering*
Atbrīivoŝanas aleja 115, LV-4601 Rēzekne, Latvija

E-mail(*corresp.*): ilmars.kangro@ru.lv

E-mail: kalis@lanet.lv

E-mail: buikis@latnet.lv

E-mail: aivars.aboltins@llu.lv

Abstract. In this paper we study the problem of the diffusion of one substance through the pores of a porous multi layered material which may absorb and immobilize some of the diffusing substances with the evolution or absorption of heat. As an example we consider circular cross section wood-block with two layers in the radial direction. We consider the transfer of heat process. We derive the system of two partial differential equations (PDEs) - one expressing the rate of change of concentration of water vapour in the air spaces and the other - the rate of change of temperature in every layer. The approximation of corresponding initial boundary value problem of the system of PDEs is based on the conservative averaging method (CAM) with special integral splines. This procedure allows reduce the 3-D axis-symmetrical transfer problem in multi-layered domain described by a system of PDEs to initial value problem for a system of ordinary differential equations (ODEs) of the first order.

Keywords: special splines, averaging method, 3D porous axial symmetry domain.

AMS Subject Classification: 65N06; 65N20; 65N40; 65N25; 34C55; 65N35.

1 Introduction

The task of sufficient accuracy numerical simulation of quick solution 3-D problems for mathematical physics in multi-layered media is important in the known areas of the applied sciences. With regard to the numerical analysis several numerical methods are known for solving 3-D problems: FEM, BEM, spectral methods, multigrids and others methods. For simple engineering calculations conservative averaging method (CAM) with special integral hyperbolic type splines is chosed.

CAM by using integral parabolic type splines was developed by A. Buikis in his Doctoral Thesis [3] (in the Russian language and [5]) and in several papers [4], [1] etc. These methods were applied to create the mathematical simulation of the mass transfer 3-D initial boundary-value problems for parabolic type partial differential equations of second order with piece-wise coefficients in multi-layered underground systems.

In this paper the special spline with two different functions of the hyperbolic type, which interpolate middle integral values of piece-wise smooth function, is used (see [5], [9], [2], [10], [11]). With the help of these splines the problems of mathematical physics in 3-D with piece-wise coefficients with respect to one coordinate are reduced to problems for system of PDEs in 2-D and 1-D.
If the parameters of the hyperbolic type spline tend to zero, we get the integral parabolic spline, obtained from A. Buikis.

The study of hydrodynamic flow and heat transfer through a porous media becomes much more interesting due to its vast applications for many decades [15], [17], [20] , [19], [6], [7], [8], [18] . In [19] is considered numerical algorithms for modelling of liquid polymer injection, so that the liquid polymer is flowing through a porous medium. Many mathematical models are developed for the analysis of such processes example mathematical models of moisture movement in wood, when the wood is considered as porous media [13], [14] and [12].

In [13] it is described the moisture movement process in the wood. In the model it is assumed that there is a moisture exchange between two environments and diffusion and convection processes are determining the moisture movement in the wood cells.
In this paper we study the linear heat and moisture transfer processes in the porous multilayered media layer by using CAM with special integral splines. In one layer similar process is analysed and decribed in [15], [16].

2 The mathematical model

The process of diffusion is considered in 3-D domain

$$\Omega = \{(r, z, \phi): \ 0 \le r \le R, \ 0 \le z \le L, \ 0 \le \phi \le 2\pi\}.$$

The domain Ω consist of multilayer medium in the r direction. We will consider the non-stationary axisymmetrical problem of the linear diffusion theory for multilayered piece-wise homogenous materials of N layers in the sub domain

$$\Omega_i = \{(r, z, \phi): \ r \in (r_{i-1}, r_i), \ z \in (0, L), \ \phi \in (0, 2\pi)\}, \quad i = \overline{1, N},$$

where $H_i = r_i - r_{i-1}$ is the height of layer $\Omega_i, r_0 = 0, r_N = R$.

We shall further assume linear dependence on both temperature and moisture content in every layer [15]

$$M_i = const + \sigma_i C_i - \omega_i T_i, \tag{2.1}$$

where $C_i(r, z, t)$ is the concentration of water vapour in the air spaces, M_i is the amount of moisture absorbed by unit mass of fibre, σ_i and ω_i are constants.

We can derive two equations, one expressing the rate of change of concentration and the other - the rate of change of temperature. The water vapour diffusion PDEs in the cylindrical coordinates is in following form

$$m_i \Big(\frac{1}{r} \Big(\frac{\partial}{\partial r} (D_{ir} r \frac{\partial C_i}{\partial r}) \Big) + D_{iz} \frac{\partial^2 C_i}{\partial z^2} \Big)$$
$$= m_i \frac{\partial C_i}{\partial t} + (1 - m_i) \rho_s \frac{\partial M_i}{\partial t}; \quad r \in [r_{i-1}, r_i], \quad i = \overline{1, N}, \quad t > 0,$$

where D_{ir}, D_{iz} are the diffusion coefficients for moisture in air, m_i are the fraction of the total volume of the material occupied by air and $1 - m_i$ is the fraction of the porous material occupied by fibre of density ρ_s, t is the time. If $m_i = 1$, then the equation is diffusion equation for the concentration without fibres.

The heat diffusion PDEs can be rewritten in the following form:

$$c_i \rho_i \frac{\partial T_i}{\partial t} = \frac{1}{r} \Big(\frac{\partial}{\partial r} \big(K_{ir} r \frac{\partial T_i}{\partial r} \big) \Big) + K_{iz} \frac{\partial^2 T_i}{\partial z^2}$$
$$+ q_i \rho_i \frac{\partial M_i}{\partial t}; \quad r \in [r_{i-1}, r_i], \quad i = \overline{1, N}, \quad t > 0,$$

where c_i are the specific heat of the fibres, K_{ir}, K_{iz}, ρ_i are the heat conductivities and the densities of the porous material, q_i is the heat evolved when the water vapour is absorbed by the fibres.

We assume that all coefficients in the PDEs are assumed constant and independent of moisture concentration and temperature. By eliminating M_i from (2.1), we obtain the system of two PDEs

$$D_{ir}^T \frac{1}{r} \Big(\frac{\partial}{\partial r} (r \frac{\partial T_i}{\partial r}) \Big) + D_{iz}^T \frac{\partial^2 T_i}{\partial z^2} = \frac{\partial T_i}{\partial t} - \nu_i \frac{\partial C_i}{\partial t},$$
$$D_{ir}^C \frac{1}{r} \Big(\frac{\partial}{\partial r} (r \frac{\partial C_i}{\partial r}) \Big) + D_{iz}^C \frac{\partial^2 C_i}{\partial z^2} = \frac{\partial C_i}{\partial t} - \lambda_i \frac{\partial T_i}{\partial t}, \quad i = \overline{1, N}, \tag{2.2}$$

where

$$D_{ir}^T = \frac{K_{ir}}{\rho_i(c_i + q_i \omega_i)}, \quad D_{ir}^C = \frac{D_{ir} m_i}{m_i + (1 - m_i)\rho_s \sigma_i}, \quad D_{iz}^T = \frac{K_{iz}}{\rho_i(c_i + q_i \omega_i)},$$

$$D_{iz}^C = \frac{D_{iz} m_i}{m_i + (1-m_i)\rho_s \sigma_i}, \quad \nu_i = \frac{q_i \sigma_i}{c_i + q_i \omega_i}, \quad \lambda_i = \frac{(1-m_i)\omega_i \rho_s}{m_i + (1-m_i)\rho_s \sigma_i}, \quad \lambda_i \nu_i < 1.$$

For the initial condition for $t = 0$ we give $T_i(r, z, 0) = T_0, C_i(r, z, 0) = C_0, i = \overline{1, N}$, where T_0, C_0 are known constants.

We use following boundary and continuous conditions:

$$\frac{\partial Q_1(0,z,t)}{\partial r} = 0, \quad Q_N(R,z,t) = Q_{ar}, \quad Q_N(r,L,t) = Q_{az}, \quad \frac{\partial Q_i(r,0,t)}{\partial z} = 0,$$

$$Q_i(r_i,z,t) = Q_{i+1}(r_i,z,t), \quad D_{ir}^Q \frac{\partial Q_i(r_i,z,t)}{\partial r} = D_{i+1,r}^Q \frac{\partial Q_{i+1}(r_i,z,t)}{\partial r}, \quad (2.3)$$

where $Q_i = Q_i(r,z,t)$, $Q = (T;C)$ are the solutions in every layer, T_{ar}, T_{az}, C_{ar}, C_{az} are the given temperature and concentration on the boundary.

This problem for the system of PDEs is uniquely solvable and stable for $\lambda_i \geq 0$, $\nu_i \geq 0$, $\lambda_i \times \nu_i < 1$, because then the system (2.2) can be represented in the following way:

$$\frac{\partial W_i}{\partial t} = A_{ir} \frac{1}{r} \left(\frac{\partial}{\partial r}(r \frac{\partial W_i}{\partial r}) \right) + A_{iz} \frac{\partial^2 W_i}{\partial z^2},$$

where $W_i = (T_i, C_i)$ is 2-order column-vector and

$$A_{ir} = \frac{1}{1 - \lambda_i \nu_i} \begin{pmatrix} D_{ir}^T & \nu_i D_{ir}^C \\ \lambda_i D_{ir}^T & D_{ir}^C \end{pmatrix}, \quad A_{iz} = \frac{1}{1 - \lambda_i \nu_i} \begin{pmatrix} D_{iz}^T & \nu_i D_{iz}^C \\ \lambda_i D_{iz}^T & D_{iz}^C \end{pmatrix}$$

are matrices with positive eigenvalues.

3   The conservative averaging method in z-direction

We consider CAM of the special integral splines with hyperbolic trigonometrical functions for solving the 3-D initial boundary-value problem in z-direction [10]. This procedure allows reducing the 2-D problem in r,z-directions to a 1D problem in r-direction. Using CAM in z-direction with parameters a_{iz} we have

$$Q_i(r,z,t) = Q_{iv}(r,t) + m_i^Q(r,t)f_{iz1}(z - \overline{z_i}) + e_i^Q(r,t)f_{iz2}(z - \overline{z_i}),$$

where $Q_{iv}(r,t) = \frac{1}{L} \int_0^L Q_i(r,z,t)dz$, $Q_i = (T_i, C_i)$, $f_{iz1}(z) = \frac{0.5L \sinh(a_{iz}(z-L/2))}{\sinh(0.5a_{iz}L)}$, $f_{iz2}(z) = \frac{\cosh(a_{iz}(z-L/2)) - A_{i0}}{8 \sinh^2(a_{iz}L/4)}$, $A_{i0} = \frac{\sinh(a_{iz}L/2)}{a_{iz}L/2}$. The parameters a_{iz} can be choosen to minimize the maximal error. As the parameters $a_{iz} > 0$ tends to zero then as the limit we obtain the integral parabolic spline [2], because

$$A_{i0} \to 1 : \quad Q_i(r,z,t) = Q_{i,v}(r,t) + m_i^Q(z - L/2) + e_i^Q \left(\frac{(z-L/2)^2}{L^2} - \frac{1}{12} \right).$$

The unknown functions $m_i = m_i^Q(r,t), e_i = e_i^Q(r,t)$ can be determined from conditions (2.3): 1) for $z = 0$, $m_i d_i - e_i k_i = 0, m_i = e_i p_i$, $p_i = k_i/d_i$, 2) for $z = L$, $Q_{az} = Q_{iv} + m_i L/2 + e_i b_i, e_i = (Q_{az} - Q_{iv})/g_{0i}$, where

$$d_i = 0.5La_{iz} \coth(0.5a_{iz}L), \quad k_i = 0.25a_{iz} \coth(0.25a_{iz}L),$$
$$b_i = ((\cosh(a_{iz}L/2) - A_{i0})/(8\sinh^2(a_{iz}L/4)), \quad g_{0i} = b_i + 0.5Lp_i.$$

Now the 1D initial-value problem (2.2) is in the following form

$$
\begin{cases}
D_{ir}^T \frac{1}{r} \left(\frac{\partial}{\partial r}\left(r\frac{\partial T_{iv}}{\partial r}\right) \right) + D_{iz}^T a_{0i}^2 (T_{az} - T_{iv}) = \frac{\partial T_{iv}}{\partial t} - \nu_i \frac{\partial C_{iv}}{\partial t}, \\[4pt]
D_{ir}^C \frac{1}{r} \left(\frac{\partial}{\partial r}\left(r\frac{\partial C_{iv}}{\partial r}\right) \right) + D_{iz}^C a_{0i}^2 (C_{az} - C_{iv}) = \frac{\partial C_{iv}}{\partial t} - \lambda_i \frac{\partial T_{iv}}{\partial t}, \\[4pt]
\frac{\partial T_{1v}(0,t)}{\partial r} = 0, \frac{\partial C_{1v}(0,t)}{\partial r} = 0, T_{Nv}(R,t) = T_{ar}, C_{Nv}(R,t) = C_{ar}, \\[4pt]
T_{iv}(r_i,t) = T_{i+1,v}(r_i,t), D_{ir}^T \frac{\partial T_{iv}(r_i,t)}{\partial r} = D_{i+1,r}^T \frac{\partial T_{i+1,v}(r_i,t)}{\partial r}, i = \overline{1, N-1}, \\[4pt]
C_{iv}(r_i,t) = C_{i+1,v}(r_i,t), D_{ir}^C \frac{\partial C_{iv}(r_i,t)}{\partial r} = D_{i+1,r}^C \frac{\partial C_{i+1,v}(r_i,t)}{\partial r}, i = \overline{1, N-1}, \\[4pt]
T_{iv}(r,0) = T_0, C_{iv}(r,0) = C_0, i = \overline{1, N},
\end{cases}
\tag{3.1}
$$

where $a_{0i}^2 = 2k_i/Lg_{0i}$.

4 The aprobation of the hyperbolic approximation in z-direction

In this section we consider averaging methods with hyperbolic type splines for solving special 3-D boundary-value problem in one layer. For multi-layered domains the accuracy of spline method remained, because of the continuous conditions are fulfilled exactly.

With the help of these splines is reducing the 3-D problem in z-direction to 1-D problem for ODEs. This procedure allows the solution obtained analytically. The spline solution is compared with the analytical solution. We consider the process of diffusion in 3-D parallelepiped

$$
\Omega_1 = \{(x,y,z) : 0 \le x \le L_x, 0 \le y \le L_y, 0 \le z \le L_z\}.
$$

We will find the distribution of stationary concentrations $c = c(x,y,z)$ in Ω_1 by solving the following special 3-D boundary value problem for PDE:

$$
\begin{cases}
\frac{\partial}{\partial x}\left(D_x \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_y \frac{\partial c}{\partial y}\right) \\[4pt]
\quad + \frac{\partial}{\partial z}\left(D_z \frac{\partial c}{\partial z}\right) + f_0 \cos\frac{\pi x}{2L_x} \cos\frac{\pi y}{2L_y} = 0, \\[4pt]
\frac{\partial c(0,y,z)}{\partial x} = \frac{\partial c(x,0,z)}{\partial y} = 0, c(L_x,y,z) = 0, c(x,L_y,z) = 0, \\[4pt]
D_z \frac{\partial c(x,y,0)}{\partial z} - \beta_z\left(c(x,y,0) - c_o \cos\frac{\pi x}{2L_x} \cos\frac{\pi y}{2L_y}\right) = 0, \\[4pt]
D_z \frac{\partial c(x,y,L_z)}{\partial z} + \alpha_z\left(c(x,y,L_z) - c_a \cos\frac{\pi x}{2L_x} \cos\frac{\pi y}{2L_y}\right) = 0,
\end{cases}
\tag{4.1}
$$

where f_0, c_o, c_a - are fixed constants, D_x, D_y, D_z are the constant diffusion coefficients, α_z, β_z are the constant mass transfer coefficients with the 3 kind boundary conditions. We can obtained the analytical solution of (4.1) in following form: $c(x,y,z) = g(z) \cos\frac{\pi x}{2L_x} \cos\frac{\pi y}{2L_y}$, where the function $g(z)$ is solution of boundary value problem for following ODEs

$$
\begin{cases}
g''(z) - a_0^2 g(z) + f_1 = 0, \\[4pt]
g'(0) - \beta(g(0) - c_o) = 0, g'(L_z) + \alpha(g(L_z) - c_a) = 0,
\end{cases}
\tag{4.2}
$$

where $f_1 = f_0/D_z$, $\beta = \beta_z/D_z$, $\alpha = \alpha_z/D_z$, $a_0^2 = \pi^2(D_y/L_y^2 + D_x/L_x^2)/4D_z$. We have the following solutions

$$g(z) = C_1 \sinh(a_0 z) + C_2 \cosh(a_0 z) + f_2,$$

where $C_1 = \beta/a_0(C_2 + f_2 - c_o)$, $C_2 = \frac{(c_a - f_2)(\alpha/a_0 + \beta/a_0 c_3)}{\beta/a_0 c_3 + c_4}$, $f_2 = f_1/a_0^2$, $c_3 = \cosh(a_0 L_z) + \alpha/a_0 \sinh(a_0 L_z)$, $c_4 = \sinh(a_0 L_z) + \alpha/a_0 \cosh(a_0 L_z)$.

Next we use the averaged method for (4.2) with fixed parametrical functions f_{z1}, f_{z2}. $g(z) = g_a + m f_{z1}(z - L_z/2) + e f_{z2}(z - L_z/2)$, where $g_a = \frac{1}{L_z}\int_0^{L_z} g(z)dz$ is the averaged value, $\int_0^{L_z} f_{z1}dz = \int_0^{L_z} f_{z2}dz = 0$, $f_{z1} = \frac{0.5 L_z \sinh(a_z(z - L_z/2))}{\sinh(a_z L_z/2)}$, $f_{z2} = \frac{\cosh(a_z(z - L_z/2)) - A_0}{8 \sinh^2(a_z L_z/4)}$, $A_0 = \sinh(a_z L_z/2)/(a_z L_z/2)$, $a_z = a_0$ is the optimal parameter.

If the parameters a_z tend to zero then the limit is the integral parabolic spline [4]. The unknown coefficients m, e we can determine from boundary conditions (4.2):

1) for z=0, $md - ek - \beta(g_a - 0.5 m L_z + eb - c_o) = 0$,

2) for z=L_z, $md + ek + \alpha(g_a + 0.5 m L_z + eb - c_a) = 0$, where $d = (a_z L_z/2)\coth(a_z L_z/2)$, $k = (a_z/4)\coth(a_z L_z/4)$, $b = \frac{\cosh(a_z L_z/2)) - A_0}{8 \sinh^2(a_z L_z/4)}$.
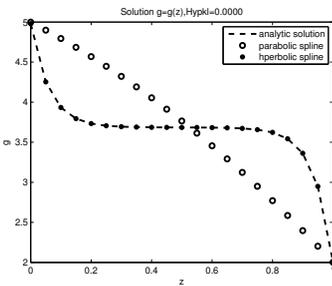
Now the-boundary value problem (4.2) can be written in the form

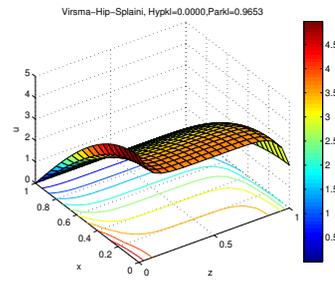$$1/L_z(g'(L_z) - g'(0)) - a_0^2 g_a + f_1 = 0, g'(L_z) - g'(0) = 2ek$$

or

$$g_a = (f_1 L_z + 2da_e)/(a_0^2 L_z + 2k g_e),$$

where the a_e and g_e are given constants.

(a) $g(z)$　　　　　　　　　　　　(b) $c(x, 0, z)$

Figure 1. Solutions

Example 1. We consider the following parameters: $L_x = L_y = L_z = 1$, $f_0 = 0.1$, $\alpha_z = 0.3$, $\beta_z = 0.1$, $c_o = 5$, $c_a = 2$, $D_x = 10^{-2}$, $D_y = 10^{-3}$, $D_z = 10^{-4}$, $a_0 = 16.4747$.

We have following maximal errors: for hyperbolic spline $er_h = 2.10^{-9}$, for parabolic $er_p = 0.9653$ (see $g(z)$ in Figure 1a and $c(x, 0, z)$ in Figure 1b). If $D_y = 0, y = 0$, then $er_h = 8.10^{-8}$, $er_p = 1.0423$.

5 The aprobation of the hyperbolic approximation in r-direction

The process of diffusion is considered in 3-D cylinder

$$\Omega_2 = \{(r, z, \phi) : 0 \le r \le R, 0 \le z \le L_z, 0 \le \phi \le 2\pi\}.$$

We solve the stationary boundary value problem with axial symmetry and find the distribution of concentrations $c = c(r, z)$ in Ω_2 by solving the following special boundary value problem for PDE:

$$D_r \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial c}{\partial r}\right) + \frac{\partial}{\partial z}\left(D_z \frac{\partial c}{\partial z}\right) + f_0 \cos \frac{\pi z}{2L_z} = 0,$$

$$\frac{\partial c(r, 0)}{\partial z} = \frac{\partial c(0, z)}{\partial r} = 0, \quad c(r, L_z) = 0,$$

$$D_r \frac{\partial c(R, z)}{\partial r} + \alpha_r(c(R, z) - c_a \cos \frac{\pi z}{2L_z}) = 0, \tag{5.1}$$

where f_0, c_a - are the fixed constants, D_r, D_z are the diffusion coefficients, α_r is the constant mass transfer coefficient. We can obtaine the analytical solution of (5.1) in following form: $c(r, z) = g(r) \cos(\pi z/2L_z)$, where the function $g(r)$ is solution of boundary value problem for ODE

$$g''(r) + \frac{1}{r} g'(r) - a_0^2 g(r) + f_1 = 0,$$

$$g'(0) = 0, \quad g'(R) + \alpha(g(R) - c_a) = 0, \tag{5.2}$$

where $f_1 = f_0/D_r$, $\alpha = \alpha_r/D_r$, $a_0^2 = \frac{\pi^2 D_z/L_z^2}{4D_r}$. We have following solutions

$$g(r) = C_1 I_0(a_0 r) + f_2,$$

where $f_2 = \frac{f_1}{a_0^2}$, $C_1 = \frac{\alpha(c_a - f_2)}{a_0 I_1(a_0 R) + \alpha I_0(a_0 R)}$, I_0, I_1 are the modified Bessel functions. Next we use the averaged method for (5.2) with fixed parametrical functions f_{r1}, f_{r2}, $g_1(r) = g_a + m f_{r1}(r - R/2) + e f_{r2}(r - R/2)$, where $g_a = \frac{2}{R^2} \int_0^R r g(r) dr$ is the averaged value, $\int_0^R r f_{r1} dr = \int_0^R r f_{r2} dr = 0$, $f_{r1} = \frac{R^2 a_r^2 \sinh(a_r(r - R/2))}{4 \sinh(a_r R/2)(d-1)} - 1$, $f_{r2} = \frac{\cosh(a_r(r - R/2)) - A_0}{8 \sinh^2(a_r R/4)}$, $A_0 = \sinh(a_r R/2)/(a_r R/2)$, $d = Ra_r/2 \coth(a_r R/2)$, $a_r = a_0 + kor$, kor is the correction for optimal parameter. The unknown correction kor can be obtained with Matlab using the operators "fminbnd, fplot" by minimizing the integrals $I_2(kor) = \int_0^R (g(r) - g_1(r))^2 dr$ or $I_1(kor) = \int_0^R |g(r) - g_1(r)| dr$.

We can see that the parameters a_r tend to zero then the limit is the integral parabolic spline. The unknown coefficients m, e can be determined from boundary conditions (3.1):

1) for $r = 0$, $m d_r - e k = 0$,
2) for $r = R$, $m d_r + e k + \alpha(g_a + m b_m + e b_e - c_a)$,

where $k = (a_r/4) \coth(a_r R/4)$, $d_r = 0.5 R d a_r^2/(d-1)$, $b_e = \frac{\cosh(a_r R/2)) - A_0}{8 \sinh^2(a_r R/4)}$, $b_m = \frac{R^2 a_r^2}{4(d-1)} - 1$.

Therefore $e = (c_a - g_a)/g_1, m = ek/d_r, g_1 = 2k/\alpha + b_m k/d_r + b_e$. Now the-boundary value problem (5.2) is given in the form
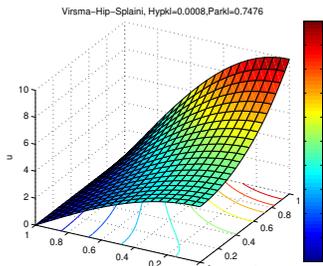
$$2/R^2(Rg'(R)) - a_0^2 g_a + f_1 = 0, g'(R) = 2e\,k$$

or

$$g_a = (f_1 g_1 R + 4kc_a)/(4k + a_0^2 g_1 R).$$

Example 2. We consider the following parameters $L_z = R = 1$, $f_0 = -0.1$, $\alpha_r = 10.01$, $c_a = 10$, $D_r = 10^{-2}$, $D_z = 10^{-2}$, $a_0 = 1.5708$. We get following maximal errors: for hyperbolic spline $er_h = 0.00084, kor = -0.200$ for parabolic $er_p = 0.7476$ (see $g(r)$ in Figure 2a and $c(r, z)$ in Figure 2b).
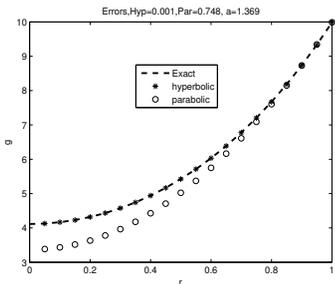
(a) $g(r)$ (b) $c(r, z)$

Figure 2. Solutions

We have piecewise linear graphic for integral I_1 in the interval [-0.4, 0] and parabolic type for I_2. The minimal values for integrals are $I_1 = 6.12e - 4$, $I_2 = 6.9e - 7$. If $\alpha_r = 0.01$ then $er_h = 0.00032$, $er_p = 0.3412$, then the minimal values for integrals are $I_1 = 3.14e - 4$, $I_2 = 1.7e - 7$, but for $f_0 = 1.01$ we have $er_h = 0.00079$, $er_p = 0.8412$ and the minimal values for integrals are $I_1 = 6.9e-4$, $I_2 = 8.25e-7$.

6   CAM in r-direction for N-layers

Using averaged method with parameters a_{ir}^Q we have

$$Q_{iv}(r, t) = Q_{ivv}(t) + m_{ir}^Q(t)f_{ir1}(r) + e_{iz}^Q(t)f_{ir2}(r),$$

where $Q_{ivv}(t) = \frac{1}{H_i \overline{r_i}} \int_{r_{i-1}}^{r_i} rQ_{iv}(r,t)dr$, $f_{ir1}(r) = \frac{0.5H_i \overline{r_i}(a_{ir}^Q)^2 \sinh(a_{ir}^Q(r-\overline{r_i}))}{\sinh(0.5a_{ir}^Q H_i)(d_i-1)} - 1$, $f_{ir2}(r) = \frac{\cosh(a_{ir}^Q(r-\overline{r_i})) - A_{i1}^Q}{8\sinh^2(a_{ir}^Q H_i/4)}$, $\overline{r_i} = (r_{i-1}+r_i)/2$, $r \in [r_{i-1}, r_i]$, $A_{i1}^Q = \frac{sinh(a_{ir}^Q H_i/2)}{a_{ir}^Q H_i/2}$, $d_i^Q = 0.5H_i a_{ir}^Q \coth(0.5a_{ir}^Q H_i), i = \overline{1, N}$. We have used the following values of parameters $a_{ir}^T = a_{0i}\sqrt{D_{iz}^T/D_{ir}^T}$, $a_{ir}^C = a_{0i}\sqrt{D_{iz}^C/D_{ir}^C}$.

We can see that if parameters $a_{ir}^Q > 0$ tend to zero, then the limit is the integral parabolic spline:

$$Q_{iv}(r, t) = Q_{ivv} + m_{ir}^Q\Big(\frac{12\overline{r_i}}{H_i^2}(r - \overline{r_i}) - 1\Big) + e_{ir}^Q\Big(\frac{(r - \overline{r_i})^2}{H_i^2} - \frac{1}{12}\Big).$$

From boundary conditions (2.3) we get the linear system of $2N$-equations for determination unknown functions $m_i = m_{ir}^Q(t)$, $e_i = e_{ir}^Q(t)$:

1) for $r = 0$, $m_1 d_{1r}^Q - e_1 k_{1r}^Q = 0$,

2) for $r = R$, $Q_{ar} = Q_{Nvv} + m_N b_{Nm} + e_N b_{Ne}$,

3) for $r = r_i$, $Q_{ivv} + m_i b_{im} + e_i b_{ie} = Q_{i+1,vv} - m_{i+1}(b_{i+1,m} + 2) + e_{i+1} b_{i+1,e}$,

$D_{ir}^Q(m_i d_{ir}^Q + e_i k_{ir}^Q) = D_{i+1,r}^Q(m_{i+1} d_{i+1,r}^Q - e_{i+1} k_{i+1,r}^Q)$, $i = \overline{1, N-1}$, where

$$d_{ir}^Q = d_i^Q \overline{r_i}(a_{ir}^Q)^2/(d_i^Q - 1), \quad k_{ir}^Q = 0.25 a_{ir}^Q \coth(0.25 a_{ir}^Q H_i),$$

$$b_{ie}^Q = ((\cosh(a_{ir}^Q H_i/2) - A_{i1}^Q)/(8\sinh^2(a_{ir}^Q H_i/4)), \quad b_{im}^Q = \frac{0.5 H_i \overline{r_i}(a_{ir}^Q)^2}{d_i^Q - 1} - 1.$$

Now the 1D initial- value problem (3.1) gives the following ODEs system:

$$
\begin{cases}
D_{ir}^T(m_{ir}^T(t)d_{ir}^T/\overline{r_i} + 2e_{ir}^T(t)k_{ir}^T/H_i) + D_{iz}^T(a0_i^T)^2(T_{az} - T_{ivv}(t)) \\
\quad = \frac{\partial T_{ivv}(t)}{\partial t} - \nu_i \frac{\partial C_{ivv}(t)}{\partial t}, \\
D_{ir}^C(m_{ir}^C(t)d_{ir}^C/\overline{r_i} + 2e_{ir}^C(t)k_{ir}^C/H_i) + D_{iz}^C(a0_i^C)^2(C_{az} - C_{ivv}(t)) \\
\quad = \frac{\partial C_{ivv}(t)}{\partial t} - \lambda_i \frac{\partial T_{ivv}(t)}{\partial t}, \\
T_{ivv}(0) = T_0, C_{ivv}(0) = C_0, \quad i = \overline{1, N}.
\end{cases}
\tag{6.1}
$$

7 CAM for two layers

For $N = 2$ we get from (6.1) the following ODE initial-value problem:

$$
\begin{cases}
b_{i1}^T T_{1vv}(t) + b_{i2}^T T_{2vv}(t) + T_{i0} = \frac{\partial T_{ivv}(t)}{\partial t} - \nu_i \frac{\partial C_{ivv}(t)}{\partial t}, \\
b_{i1}^C C_{1vv}(t) + b_{i2}^C C_{2vv}(t) + C_{i0} = \frac{\partial C_{ivv}(t)}{\partial t} - \lambda_i \frac{\partial T_{ivv}(t)}{\partial t}, \\
T_{ivv}(0) = T_0, \quad C_{ivv}(0) = C_0, \quad i = 1, 2,
\end{cases}
\tag{7.1}
$$

or

$$
\begin{cases}
\dot{T}_{ivv}(t) = \frac{1}{(1-\lambda_i\nu_i)}(b_{i1}^T T_{1vv}(t) + b_{i2}^T T_{2vv}(t) + T_{i0} + \nu_i(b_{i1}^C C_{1vv}(t) \\
\quad + b_{i2}^C C_{2vv}(t) + C_{i0})), \\
\dot{C}_{ivv}(t) = \frac{1}{(1-\lambda_i\nu_i)}(\lambda_i(b_{i1}^T T_{1vv}(t) + b_{i2}^T T_{2vv}(t) + T_{i0}) + b_{i1}^C C_{1vv}(t) \\
\quad + b_{i2}^C C_{2vv}(t) + C_{i0}), \quad T_{ivv}(0) = T_0, \quad C_{ivv}(0) = C_0,
\end{cases}
\tag{7.2}
$$

where b_{ik}^Q, $k = 1, 2$, Q_{i0}, $i = 1, 2$, $Q = (T, C)$ are known coefficients. We rewrite the system of ODEs (7.2) in the following vector form:

$$\dot{W}(t) - AW(t) = F, \quad W(0) = W_0,$$

where $W(t), W_0, F$ are 4th order vector-columns with elements $(T_{1vv}(t), T_{2vv}(t), C_{1vv}(t), C_{2vv}(t))$, (T_0, T_0, C_0, C_0) and $F_1 = G_1(T_{10} + \nu_1 C_{10})$, $F_2 = G_2(T_{20} + \nu_2 C_{20})$, $F_3 = G_1(\lambda_1 T_{10} + C_{10})$, $F_4 = G_2(\lambda_2 T_{20} + C_{20})$, $G_1 = \frac{1}{(1-\lambda_1\nu_1)}$, $G_2 = \frac{1}{(1-\lambda_2\nu_2)}$. A is the 4-order matrix

$$
A = \begin{pmatrix}
G_1 b_{11}^T & G_1 b_{12}^T & G_1 \nu_1 b_{11}^C & G_1 \nu_1 b_{12}^C \\
G_2 b_{21}^T & G_2 b_{22}^T & G_2 \nu_2 b_{21}^C & G_2 \nu_2 b_{22}^C \\
G_1 \lambda_1 b_{11}^T & G_1 \lambda_1 b_{12}^T & G_1 b_{11}^C & G_1 b_{12}^C \\
G_2 \lambda_2 b_{21}^T & G_2 \lambda_2 b_{22}^T & G_2 b_{21}^C & G_2 b_{22}^C
\end{pmatrix}.
$$

The averaged solution is $W(t) = \exp(At)W_0 - (E - \exp(At))A^{-1}F$, $Q_{iv}(r,t) = Q_{ivv}(t) + m_{ir}^Q(t)f_{ir1}(r) + e_{iz}^Q(t)f_{ir2}(r)$, where $i = 1, 2$, $Q = (T, C)$.

8   The conservative averaging method for model equations

In order to apply the averaging method in 2 layers in r-direction and estimate the parameter $a_{ir} = a_{ir}^T$ we consider the stationary 1-D boundary-value problem (3.1) only for temperature $T = T(r)$ with $\nu = 0$, $T_{az} = 0$, $T_{ar} = T_0$, $D_{iz}^T = 1$, $D_{ir}^T = D_i$, $i = 1, 2$:

$$\begin{cases} D_1 \frac{1}{r}(rT_1'(r))' - a_{01}^2 T_1(r) = F_1, r \in [0, H_1], \quad T_1'(0) = 0, \\ D_2 \frac{1}{r}(rT_2'(r))' - a_{02}^2 T_2(r) = F_2, \quad T_2(R) = T_0, \ r \in [H_1, R], \\ T_1(H_1) = T_2(H_1), \quad D_1 T_1'(H_1) = D_2 T_2'(H_1), \quad 0 < H_1 < R, \end{cases}$$

where $T_0, F_i, a_{0i} > 0, D_i > 0$, $i = 1, 2$ are given constants. The analytical solution is defined as $T_1(r) = C_1 I_0(a_1 r) - f_1$, $T_2(r) = C_2 I_0(a_2 r) + C_3 K_0(a_2 r) - f_2$, where $f_i = F_i/a_{0i}^2$, $a_i = a_{0i}/\sqrt{D_i}$, $i = 1, 2$. The unknown constants are obtained from BCs ($I_0(r)' = I_1(r)$, $K_0(r)' = -K_1(r)$):

$$C_2 = \frac{(T_0 + f_2) - C_3 K_0(a_2 R)}{I_0(a_2 R)}, \quad C_3 = \frac{b_1 - f_a}{b_3}, \quad b_1 = \frac{b_0(T_0 + f_2)}{I_0(a_2 R)},$$

$b_0 = D_{2,1} a_2 I_0(a_1 H_1) I_1(a_2 H_1) - a_1 I_1(a_1 H_1) I_0(a_2 H_1)$, $f_a = a_1 I_1(a_1 H_1)(f_1 - f_2)$,

$C_1 = D_{2,1}(C_2 a_2 I_1(a_2 H_1) - C_3 a_2 K_1(a_2 H_1))/(a_1 I_1(a_1 H_1))$, $D_{2,1} = D_2/D_1$,

$b_3 = b_0 K_0(a_2 R)/I_0(a_2 R) + b_2$, $\quad b_2 = D_{2,1} a_2 I_0(a_1 H_1) K_1(a_2 H_1)$
$$+ a_1 I_1(a_1 H_1) K_0(a_2 H_1).$$

Here I_0, K_0, I_1, K_1 are the modified Bessel functions.

With hyperbolic averaging method in two layers we have:

$$S_i(r) = T_{iv} + m_i \left(\frac{0.5 H_i \overline{r_i} a_{ir}^2 \sinh(a_{ir}(r - \overline{r_i}))}{\sinh(0.5 a_{ir} H_i)(d_i - 1)} - 1 \right) + e_i \left(\frac{\cosh(a_{ir}(r - \overline{r_i})) - A_i}{8 \sinh^2(a_{ir} H_i/4)} \right),$$

where $i = 1, 2$, $\overline{r_1} = 0.5 H_1$, $\overline{r_2} = 0.5(H_1 + R)$, $A_i = \frac{\sinh(a_{ir} H_i/2)}{a_{ir} H_i/2}$, $d_i = 0.5 H_i a_{ir} \coth(0.5 a_{ir} H_i)$. The unknown parameters are $a_{ir} = a_i + kor_i$, where kor_i, $i = 1, 2$ are the corrections for the minimal error.

The unknown constants T_{iv} can be determined from stationary equations (7.1) for $i = 1, 2$ by solving the system of linear equations with respect to T_{1v}, T_{2v}. The unknown corrections can be obtained with Matlab using the operators "fminsearch, contour" by minimizing the integrals

$$I_2(kor_1, kor_2) = \int_0^{H_1} (T_1(r) - S_1(r))^2 dr + \int_{H_1}^R (T_2(r) - S_2(r))^2 dr,$$

$$I_1(kor_1, kor_2) = \int_0^{H_1} |T_1(r) - S_1(r)| dr + \int_{H_1}^R |T_2(r) - S_2(r)| dr.$$
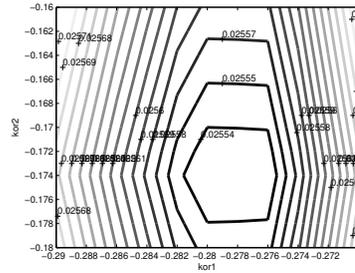
Using Matlab for $D_1 = 10^{-2}$, $D_2 = 10^{-1}$, $F_1 = -2$, $F_2 = 0$, $a_{01} = 2$, $a_{02} = 6$, $a_1 = 20$, $a_2 = 18.9737$, $T_0 = 4$, $R = 3$, $H_1 = 1.8$ we have the following maximal errors:

1) 2.6275 for the parabolic spline ($a_1 = a_2 = 10^{-4}$),

2) 0.0631 for the hyperbolic spline with optimal corrections $kor_1 = -0.2838$, $kor_2 = -0.1700$.

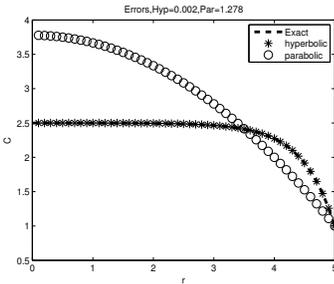The minimal value for integral I_1 is 0.0255, see Figs. 3a and 3b.
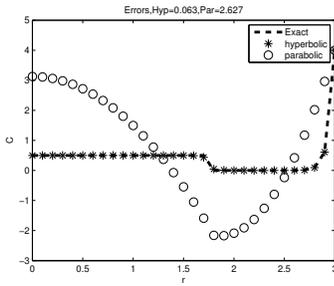
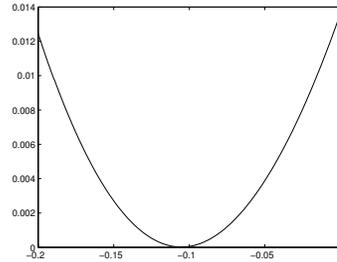(a) Solution of model equations in 2 layers

(b) Levels for minimizing in 2 layers

Figure 3. Solution of model equations and levels for minimizing in 2 layers

(a) Solution of model equations in 1 layer

(b) Minimizing function depending on kor in 1 layer

Figure 4. Solution of model equations and minimizing function depending on kor in 1 layer

For one layer we have:

$$T_1(r) = C_1 I_0(a_1 r) - f_1, \ C_1 = (T_0 + f_1)/I_0(a_1 R), \ H_1 = R.$$

The unknown correction kor can be obtained with Matlab using the operators "fminbnd, fplot" by minimizing the integrals $I_2(kor) = \int_0^R (T_1(r) - S_1(r))^2 dr$ or $I_1(kor) = \int_0^R |T_1(r) - S_1(r)| dr$. For $D_1 = 1$, $F_1 = -10$, $a_{01} = 2$, $a_1 = 1.89$, $T_0 = 1$, $R = 5$ we have the following maximal errors:

1) 1.278 for parabolic spline ($a_1 = 10^{-4}$),

2) 0.0021 for hyperbolic spline with optimal correction $kor = -0.11$, see Figure 4a and minimizing function with minimal value for integral I_2 is 5.10^{-5} see Figure 4b.

In Figs. 5a, 5b we present the spline functions $f_1(r) = f_{ir1}(r)$, $f_2(r) = f_{ir2}(r)$ for $i = 1, \overline{r_i} = R/2$, $R = 1$ with $a = a_{ir} \in [0, 25]$.

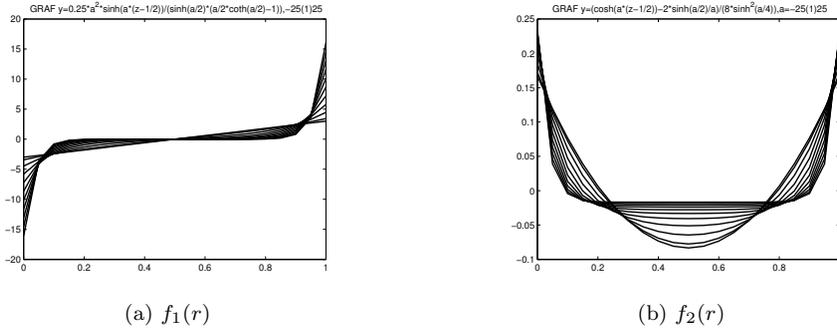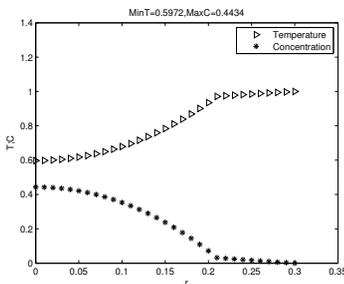(a) $f_1(r)$

(b) $f_2(r)$

Figure 5. Functions for cylindr. coord. depending on $a \in [0, 25]$, $R = 1$

9   Some numerical results

The results of calculations are obtained by MATLAB. We use the discrete values $r_j = jh_r$, $j = \overline{0, N_r}$, $N_r h_r = R$, $z_k = kh_z$, $k = \overline{0, N_z}$, $N_z h_z = L$, $t_n = n\tau$, $n = \overline{0, N_t}$, $N_t \tau = t_f$, $N_t = 1000$, $N_r = 30$, $N_z = 10$, $t_f = 1000 - 20000s$.

We consider the drying process in the two laered wood-block (see Figure 6b).

(a) the T_v, C_v for drying process depending on r by $t = 1000$

(b) the wood-block

Figure 6. The T_v, C_v for drying process depending on r by $t = 1000$ and the wood-block

The following parameters and diffusion coefficients are used in computations: $L = 0.5m$, $R = 0.3m$, $H_1 = 0.21m$, $H_2 = 0.09m$, $D_{1z}^C = 10^{-5}$, $D_{2z}^C = 10^{-4}$, $D_{1r}^C = 10^{-7}$, $D_{2r}^C = 10^{-6}$, $D_{2z}^T = D_{1z}^T = D_{2r}^T = D_{1r}^T = 10^{-3}$ $\left(\frac{m^2}{s}\right)$.

The dimensionless value for temperature is defined as $T = \frac{T' - T_0'}{T_{max}' - T_0'}$ and for the concentration of water vapour $C = \frac{C'}{C_{max}'}$, where $T_0' = 20^0 C$, $T_{max}' = 80^0 C$,

$C'_{max} = 20\%$, $T'[^0C]$, $C'[\%]$ are the corresponding dimensional values.

For the drying process we have used the following initial and boundary conditions:

$$T_0 = 0, \ \ C_0 = 1, \ \ C_{0r} = C_{0z} = 0, \ \ T_{0r} = T_{0z} = 1.$$

The results of calculation for $\lambda = 0$, $\nu = 1$, $a_{1z} = 10$, $a_{2z} = 1$ are represented in the Table 1 and Figs. (7a)-(8a), where MC_v, MC are the maximal values of concentration C_v, C and mC_{1vv}, mC_{2vv} are the minimal value of averaged concentration C_{1vv}, C_{2vv} in every layers, obtained in the time moment t_f. For $t_f = 30000$ these values are equal to 10^{-5}. For increasing λ the concentration of water vapour in the air spaces can be increased (see Table 1).

Table 1. The values of mC_{1vv}, mC_{2vv}, MC_v, MC

t_f	mC_{1vv}	mC_{2vv}	MC_v	MC	λ	ν
1000	0.776	0.160	0.868	0.909	0	1
2000	0.571	0.041	0.668	0.824	-	-
5000	0.233	0.010	0.275	0.339	-	-
8000	0.095	0.004	0.112	0.138	-	-
10000	0.052	0.002	0.061	0.076	-	-
20000	0.003	.0001	0.003	0.004	-	
1000	0.843	0.177	0.955	1.170	0.1	1
5000	0.257	0.011	0.302	0.373	0.1	1
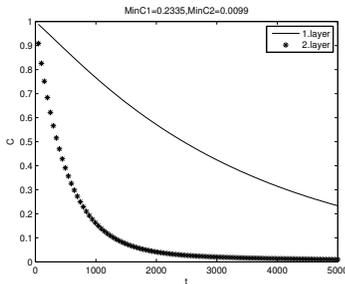
(a) values C_{1vv}, C_{2vv} depending on t
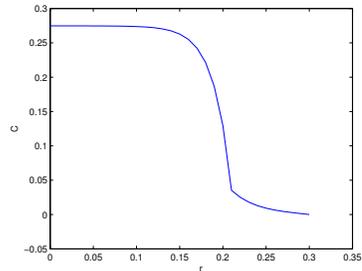
(b) values C_{vv} depending on r by $t = 5000$

Figure 7. The averaged values for drying process

In following Table 2 we can see the depending mC_{1vv}, mC_{2vv}, $MC_v = MC$ on $D_{1z}^C = D_{1r}^C$, $D_{2z}^C = D_{2r}^C$, $D_{1z}^T = D_{1r}^T$, $D_{2z}^T = D_{2r}^T$, by $\lambda = 0.1$, $\nu = 9$, $t_f = 10000$ and $\frac{\partial T_i(r,L,t)}{\partial z} = \frac{\partial C_i(r,L,t)}{\partial z} = 0$ ($MT_v = \max T \approx 1$).

In Figs. 6a and 8b we represent also the averaged concentration T_{1vv}, T_{2vv} depending on time in every layer and $T_v = T$ depending on r, obtained in the time moment $t_f = 1000$, by $D_{1z}^C = D_{1r}^C = 10^{-5}, D_{2z}^C = D_{2r}^C = 10^{-4}$, $D_{1z}^T = D_{1r}^T = 10^{-4}, D_{2z}^T = D_{2r}^T = 10^{-3}$.
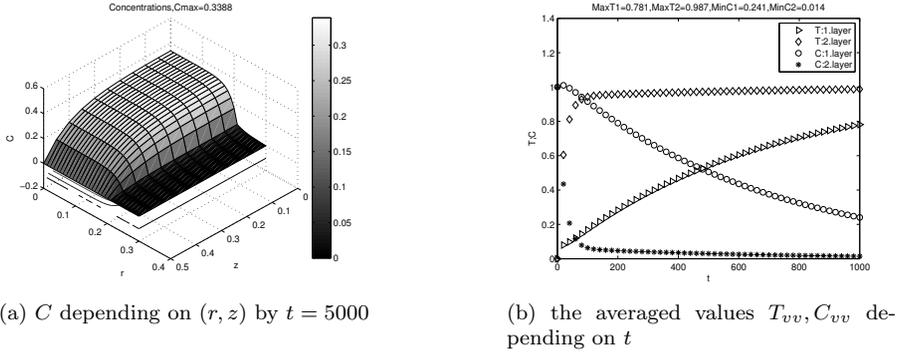
(a) C depending on (r, z) by $t = 5000$

(b) the averaged values T_{vv}, C_{vv} depending on t

Figure 8. Values for drying process

Table 2. The minimal and maximal values of temperature for $\lambda = 0.1, \nu = 9$

D_{1r}^C	D_{2r}^C	D_{1r}^T	D_{2r}^T	mC_{1vv}	mC_{2vv}	MC_v
10^{-7}	10^{-6}	10^{-3}	$.510^{-3}$	.2294	.0135	.4227
-	—	10^{-4}	10^{-5}	.2321	.0137	.4277
-	—	10^{-5}	10^{-4}	.2306	.0136	.4250
10^{-6}	10^{-6}	10^{-5}	10^{-4}	.0015	.0004	.0020
10^{-6}	10^{-7}	—	—	.4872	.2180	.5125
10^{-5}	10^{-7}	—	—	.4668	.2201	.4693
10^{-7}	10^{-6}	—	—	.1774	.0012	.3472
10^{-7}	10^{-7}	—	—	.6470	.1875	.8820

In the Table 3 are given maximal and minimal values of temperature and concentration MT, mT, MC, mC. We see that if parameter λ is increased, then inside the layers the concentration of water vapour in the air spaces is also increased.

Table 3. The maximal and minimal values for $t_f = 1000$

λ	ν	mC_{1vv}	mC_{2vv}	MC_v	MT_{1vv}	MT_{2vv}	mT_v
0.1	9.0	.2407	.0142	.4434	.7814	.9871	.5972
0.2	4.5	.2627	.0155	.4841	.8807	.9930	.7802

10　Conclusions

For the transfer of moisture content and heat in porous multi-layered 3-D domain is considered the system of two PDEs for determination the concentration C of water vapour in the air spaces and the temperature T.

The approximation of corresponding initial boundary-value problem of the system of PDEs is based on the conservative averaging method (CAM), where

is used with new hyperbolic type splines. For these splines the best parameter for the minimal error is calculated.

For the stationary 1D boundary value problem in one and two layers the spline parameters for error corrections are caculated with minimizing method using Matlab.

The problem of the system of PDEs with constant coefficients are reduced to the initial value problem of a system of ODEs of the first order. Such a procedure allows us to obtain analytical solution with a simple engineering algorithm for mass transfer equations for different substances in layered domain. It is possible to model round and angular wood-blocks and give some new physical conclusions about the drying an also about moistening processes in this blocks.

Acknowledgement

References

[1] M. Buike and A. Buikis. Modelling 3-D transport processes in anisotropic layered stratum by conservative averaging method. *WSEAS Transactions on Heat and Mass Transfer*, **1**(4):430–437, 2006.

[2] M. Buike, A. Buikis and H. Kalis. Time direct and time inverse problems for wave energy and steel quenching models, solved exactly and approximately. *WSEAS Transactions on Heat and Mass Transfer*, **10**:30–43, 2015.

[3] A. Buikis. *Modelling of Filtration Processes in Layered Porous Media by the Conservative Averaging Method.* Kazan, 1987. 374 p., Dr. Thesis. In Russian, 1987.

[4] A. Buikis. Definition and calculation of a generalized integral parabolic spline. *Proceedings of the Latvian Academy of Sciences. Section B*, **7/8**(576/577):97–100, 1995.

[5] A. Buikis and M. Buike. The conservative averaging method: applications, theory and new hyperbolic approximation. In North Atlantic University Union Imre J.Rudas(Ed.), *Mathematical and Computation Methods in Applied Sciences,Proc. of 3-rd Int. Conf. on Applied, Numerical and Computational Mathematics(ICANCM'15), Sliema,Malta Aug. 17-19, 2015*, volume 51 of *Mathematics and Computers in Science and Engineering Series 51*, pp. 58–67, ISSN: 2227-4588, ISBN:978-1-61804-328-3, 2015. WSEAS Press.

[6] A. Buikis, J. Cepitis, H. Kalis and A. Reinfelds. Non-isothermal mathematical model of wood and paper drying. In *Progress in Industrial Mathematics at ECMI 2000*, pp. 488–492. Springer, 2002.

[7] A. Buikis, J. Cepitis, H. Kalis and A. Reinfelds. Mathematical model of save timber drying. *Proceeding of the Latvian Academy of Sciences, Sec B*, **57**(3/4(626)):128–132, 2003.

[8] A. Buikis, J. Cepitis and S. Kostjukova. The mathematical model of the plywood production. In *Applied and Computational Mathematics. Proc. of the*

13-th WSEAS International Conference and Applied Mathematics, pp. 365–369. WSEAS Press, 2008.

[9] A. Buikis and H. Kalis. Hyperbolic type approximation for the solutions of the hyperbolic heat conduction equation in 3-D domain. In North Atlantic University Union Imre J.Rudas(Ed.), *Mathematical and Computation Methods in Applied Sciences,Proc. of 3-rd Int. Conf. on Applied, Numerical and Computational Mathematics(ICANCM'15), Sliema,Malta Aug. 17-19, 2015*, volume 51 of *Mathematics and Computers in Science and Engineering Series 51*, pp. 42–51, ISSN: 2227-4588, ISBN:978-1-61804-328-3, 2015. WSEAS Press.

[10] A. Buikis, H. Kalis and I. Kangro. Special hyperbolic type spline for mass transfer problems in multi-layer 3-D domains. In North Atlantic University Union Imre J.Rudas(Ed.), *Mathematical and Computation Methods in Applied Sciences,Proc. of 3-rd Int. Conf. on Applied, Numerical and Computational Mathematics(ICANCM'15), Sliema,Malta Aug. 17-19, 2015*, volume 51 of *Mathematics and Computers in Science and Engineering Series 51*, pp. 25–34, ISSN: 2227-4588, ISBN:978-1-61804-328-3, 2015. WSEAS Press.

[11] A. Buikis, H. Kalis and I. Kangro. Special splines of exponential type for the solutions of mass transfer problems in multilayer domains. *Mathematical Modelling and Analysis*, **21**(4):450–465, 2016. https://doi.org/10.3846/13926292.2016.1182594.

[12] R. Čiegis and V. Starikovičius. The finite difference scheme for 3d mathematical modelling of wood drying process. *Computational methods in applied mathematics*, **1**(2):123–137, 2001. https://doi.org/10.2478/cmam-2001-0009.

[13] R. Čiegis and V. Starikovičius. The finite difference scheme for wood drying process. *Mathematical Modelling and Analysis*, **6**(1):48–57, 2001.

[14] R. Čiegis and V. Starikovičius. Mathematical modelling of wood drying process. *Mathematical Modelling and Analysis*, **7**(2):177–190, 2002.

[15] J. Crank. *The mathematics of diffusion.* Oxford, at the Clarendon Press, 1956.

[16] H. Kalis and I. Kangro. Calculation of heat and moisture sistribution in the porous media layer. *Mathematical Modelling and Analysis*, **12**(1):91–100, 2007. https://doi.org/10.3846/1392-6292.2007.12.91-100.

[17] M. Kaviany. *Principles of heat transfer in porous media.* Springer-Verlag, Berlin, 1991. https://doi.org/10.1007/978-1-4684-0412-8.

[18] S. Kostjukova and A. Buikis. Mathematical model for diffusion-sorption processes in layered strata with interlayers. *Environmental Science, Ecosystems and Development, WSEAS Press*, pp. 7–12, 2007.

[19] R.Čiegis and O. Iliev. On numerical simulation of liquid polymer moulding. *Mathematical Modelling and Analysis*, **8**(3):101–202, 2003.

[20] J. Salin. Modelling of wood drying. *Drying Technology*, **9**(3):775–793, 1991. https://doi.org/10.1080/07373939108916709.