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Asymptotic Dynamics of a New Mechanochemical Model in Biological Patterns

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Abstract. In this paper, we prove the existence of attractor for a new mechanochemical model with Neumann boundary conditions on a bounded domain of space dimension $n \leq 3$. Based on the regularity estimates for the semigroups and the classical existence theorem of global attractors, we prove that the mechanochemical model possesses a global attractor and H^k attractor.

Keywords: mechanochemical model, biological patterns, attractor.

AMS Subject Classification: 35B41; 35K41; 92C15.

1 Introduction

In this paper, we study the following mechanochemical model

$$\frac{\partial \phi}{\partial t} = \Delta\phi + \alpha\phi - \phi^3 - \varepsilon\phi\psi, \tag{1.1}$$

$$\frac{\partial \psi}{\partial t} = -\lambda(\Delta + 1)^2\psi - \gamma\psi + g\psi^2 - \psi^3 - \frac{\varepsilon}{2}\phi^2 \tag{1.2}$$

for $(x,t) \in \Omega \times (0,\infty)$, where $\Omega \subset R^n (n \leq 3)$ is a bounded domain with smooth boundary, and all the parameters are arbitrarily given positive constants.

It was Morales, Rojas, Torres and Rubio [10] who first derived the system (1.1)-(1.2), which is to model ternary mixtures by using the theory of pattern formation and of polyelectrolytes, with mean-field approximations. Recently, Morales, Rojas, Oliveros and Hernández derived a new mechanochemical model based on Coupled Ginzburg-Landau and Swift-Hohenberg equations in biological patterns of marine animals [9]. The new model was proposed to describe some cellular interactions in three out layers of animal (such as the fish) marine skin. $\phi(x,t)$ represents the concentration difference of at least two pigment, and $\psi(x,t)$ is the difference of dermal cellular densities of at least two types of cells. The $\overrightarrow{J_\phi} = -\nabla\phi$ is the flux of pigment concentration difference ϕ in the epidermis, and $\overrightarrow{J_\psi} = 2\lambda\nabla\psi + \lambda\nabla\Delta\psi$ is the density of cell flux ψ of the dermis. The system (1.1)-(1.2) is supplemented by the zero flux boundary condition,

$$\overrightarrow{J_\phi} \cdot \nu = 0, \quad \overrightarrow{J_\psi} \cdot \nu = 0, \quad x \in \partial\Omega,$$

where ν is the outward unit normal to $\partial\Omega$, that is

$$\frac{\partial\phi}{\partial\nu} = 0, \quad 2\lambda\frac{\partial\psi}{\partial\nu} + \lambda\frac{\partial\Delta\psi}{\partial\nu} = 0, \quad x \in \partial\Omega \tag{b1}$$

and the natural boundary condition

$$\frac{\partial\psi}{\partial\nu} = 0, \quad x \in \partial\Omega. \tag{b2}$$

It follows from (b2) that (b1) can be replaced by

$$\frac{\partial\Delta\psi}{\partial\nu}(x,t) = 0, \quad x \in \partial\Omega.$$

Hence, we consider the Neumann boundary conditions

$$\frac{\partial\phi}{\partial\nu}(x,t) = \frac{\partial\Delta\psi}{\partial\nu}(x,t) = \frac{\partial\psi}{\partial\nu}(x,t) = 0, \quad t > 0, \quad x \in \partial\Omega \tag{1.3}$$

and the initial condition

$$\phi(x,0) = \phi_0, \quad \psi(x,0) = \psi_0, \quad x \in \Omega. \tag{1.4}$$

The dynamic properties of the reaction-diffusion system (1.1)-(1.2), such as the global asymptotical behaviors of solutions and existence of global attractors are important. During the past years, many authors had paid much attention to the higher order equation ([1, 4, 6, 7, 16]) or the reaction-diffusion systems ([2, 3, 11]). You ([17, 18, 19]) had proved the existence of global attractor for some Gray-Scott type systems. The main difficulties for treating the problem (1.1)-(1.2) are caused by the nonlinearity of low order terms, and linear higher order terms are not homogeneous. The source type nonlinear low terms and Neumann boundary conditions can not make us use Poincaré type inequality directly, thanks to strong absorptive terms $-\phi^3$ and $-\varphi^3$, which guarantees the existence of a global solution and will not blow up.

The paper is arranged as follows. In Section 2, some notations and the main results are stated. We present some estimates in Section 3, and then we prove that problem (1.1)-(1.4) possesses global attractors on $L^2(\Omega) \times H^2(\Omega)$ in Section 4. Based on this result, we prove the existence of global attractors for problem (1.1)-(1.4) in H^k $(k \geq 0)$ space in Section 5.

2 Statement of main results

We first introduce the following abbreviations.

The notation $(\cdot,\cdot)$ for L^2-inner product will also be used for the notation of duality pairing between dual spaces, $\|\cdot\| = \|\cdot\|_{L^2}$. We use the same letter C to denote different positive constants, and $C(\cdot,\cdot,\cdot)$ to denote positive constants depending on the quantities appearing in the parenthesis.

Theorem 1. *For any positive parameters $\alpha, \lambda, \gamma, g, \varepsilon$, any $(\phi_0, \psi_0)^T \in L^2(\Omega) \times H^2(\Omega)$, and $n \leq 3$, there exists a global attractor $\mathcal{A}$ in the phase space $L^2(\Omega) \times H^2(\Omega)$ for the solution semiflow $\{S(t)\}_{t\geq 0}$ on $L^2(\Omega) \times H^2(\Omega)$ generated by system (1.1)-(1.2) with the Neumann boundary conditions (1.3).*

The basic theory of infinite dimensional dynamical systems and global attractors can be seen in [11, 15] and references therein. A few definitions are listed for clarity.

Definition 1. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a real Banach space X. A bounded subset B_0 of X is called an absorbing set in X for this semiflow, if for any bounded subset $B \subset X$ there is some finite time $t_0 \geq 0$ depending on B such that $S(t)B \subset B_0$ for all $t \geq t_0$.

Definition 2. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a real Banach space X whose norm-induced metric is denoted by $d(\cdot,\cdot)$. A subset $\mathcal{A}$ of X is called a global attractor for this semiflow, if the following properties are satisfied:

(H1) $\mathcal{A}$ is a nonempty, compact, invariant set in the sense that $S(t)\mathcal{A}=\mathcal{A}$ for any $t \geq 0$.

(H2) $\mathcal{A}$ attracts any bounded set B of X with respect to the Hausdorff distance,

$$\operatorname{dist}(S(t)B, \mathcal{A}) = \sup_{x\in B} \inf_{y\in\mathcal{A}} d(S(t)x, y) \to 0, \quad as \quad t \to \infty.$$

Definition 3. A semiflow on a real Banach space X is asymptotically compact if for any bounded sequence u_n in X and any sequence $t_n \subset (0, \infty)$ with $t_n \to \infty$, there exist subsequences u_{n_k} of u_n and t_{n_k} of t_n, such that $\lim_{k\to\infty} S(t_{n_k})u_{n_k}$ exists in X.

Lemma 1. ([15]) *Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a real Banach space X. If the following properties are satisfied:*

(1) there exists a bounded absorbing set $B_0 \subset X$ for $\{S(t)\}_{t\geq 0}$,

(2) $\{S(t)\}_{t\geq 0}$ is asymptotically compact on X,

then there exists a global attractor $\mathcal{A}$ for $\{S(t)\}_{t\geq 0}$ in X, which is given by

$$\mathcal{A} = \omega(B_0) := \bigcap_{\tau\geq 0} Cl_X \bigcup_{t\geq\tau} (S(t)B_0). \tag{2.1}$$

Let us write (1.1)-(1.2) as an evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = F(u), \quad t > 0, \\ u(0) = u_0, \end{cases}$$

where $u = (\phi, \psi)^T$, $H := L^2(\Omega) \times L^2(\Omega)$,

$$\begin{aligned} F(u) &= \begin{pmatrix} F_1(\phi,\psi) \\ F_2(\phi,\psi) \end{pmatrix} \\ &= \begin{pmatrix} \alpha\phi - \phi^3 - \varepsilon\phi\psi \\ -2\lambda\Delta\psi - (\gamma+\lambda)\psi + g\psi^2 - \psi^3 - \frac{\varepsilon}{2}\phi^2 \end{pmatrix} : D(A^{\frac{1}{2}}) \to H, \end{aligned}$$

and

$$A = \begin{pmatrix} -\Delta & 0 \\ 0 & \lambda \Delta^2 \end{pmatrix} : D(A) \to H.$$

$F(u)$ and the operator A is considered on the Hilbert space L^2 with dense domain

$$D(A) = \{u \in H^2(\Omega) \times H^4(\Omega) : \frac{\partial \phi}{\partial \nu} = \frac{\partial \Delta \psi}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = 0, \text{ on } \partial\Omega\},$$
$$D(A^{\frac{1}{2}}) = \{u \in H^1(\Omega) \times H^2(\Omega) : \frac{\partial \phi}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = 0, \text{ on } \partial\Omega\}.$$

Let

$$E := \{u \in H^1(\Omega) \times H^1(\Omega) : \frac{\partial \phi}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = 0, \text{ on } \partial\Omega\},$$
$$E_1 := \{u \in L^2(\Omega) \times H^2(\Omega) : \frac{\partial \psi}{\partial \nu} = 0, \text{ on } \partial\Omega\},$$
$$E_2 := \{u \in H^1(\Omega) \times H^3(\Omega) : \frac{\partial \phi}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = \frac{\partial \Delta \psi}{\partial \nu} = 0, \text{ on } \partial\Omega\}.$$

In order to prove the existence of solutions, we shall show A is sectorial, $F(u)$ is locally Lipschitz continuous as the operation between the space $D(A^{1/2})$ and $L^2 \times L^2$, denoting by $\langle \cdot, \cdot \rangle$ the scalar product in L^2 and

$$T := A + \delta_0 \Gamma, \quad \Gamma := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

where $\delta_0 > 0$ is taken sufficient large. For any $u = (\phi, \psi)^T, v = \left(\overline{\phi}, \overline{\psi}\right)^T$, $\{u, v\} \in D(A)$, noticing Γ is symmetric we find that

$$\begin{aligned} \langle Tu, v\rangle &= \int_\Omega \left(-\Delta\phi\overline{\phi} + \lambda\Delta^2\psi\overline{\psi}\right) dx + \int_\Omega \delta_0 u^T \Gamma v dx \\ &= \int_\Omega \left(\nabla\phi\nabla\overline{\phi} + \lambda\Delta\psi\Delta\overline{\psi}\right) dx \int_\Omega \delta_0 u^T \Gamma v dx \\ &= \int_\Omega \left(-\phi\Delta\overline{\phi} + \psi\lambda\Delta^2\overline{\psi}\right) dx + \int_\Omega \delta_0 v^T \Gamma u dx = \langle u, Tv\rangle, \end{aligned}$$

which proves the symmetry of A. Next, since Γ is positive definite, the operator A is bounded below, that is, for each $u \in D(A)$,

$$\begin{aligned} \langle Tu, u\rangle &= \int_\Omega \left(-\Delta\phi\phi + \lambda\Delta^2\psi\psi\right) dx + \int_\Omega \delta_0 u^T \Gamma u dx \\ &= \int_\Omega ((\nabla\phi)^2 + \lambda(\Delta\psi)^2) dx + \int_\Omega \delta_0 u^T \Gamma u dx \geq \delta_0 \langle u, u\rangle. \end{aligned}$$

A is self-adjoint and bounded below, which means that A is itself sectorial. We prove the local Lipschitz continuity of nonlinear function $F(u)$

$$\forall\, u, v \in U \subset D\left(A^{1/2}\right), \quad \|F(u) - F(v)\|_{L^2} \leq K_U \|u - v\|_{D\left(A^{1/2}\right)}.$$

Notice $H^2 \hookrightarrow L^\infty$ and $H^1 \hookrightarrow L^6$ for $n \le 3$.

We find by differentiation that, for $\forall u, v \in U \subset D\left(A^{1/2}\right)$

$$\begin{aligned}
&\|F_1(u) - F_1(v)\|_{L^2} \\
&\quad = \|\left(\alpha - \phi^2 - \phi\overline{\phi} - \overline{\phi}^2\right)\left(\phi - \overline{\phi}\right) + \varepsilon\psi\left(\phi - \overline{\phi}\right) + \varepsilon\overline{\phi}\left(\psi - \overline{\psi}\right)\|_{L^2} \\
&\quad \le \|\left(\alpha - \phi^2 - \phi\overline{\phi} - \overline{\phi}^2\right)\left(\phi - \overline{\phi}\right)\|_{L^2} + \|\varepsilon\psi\left(\phi - \overline{\phi}\right)\|_{L^2} + \|\varepsilon\overline{\phi}\left(\psi - \overline{\psi}\right)\|_{L^2} \\
&\quad \le C\left(|\alpha| + \|\overline{\phi}\|_{L^6}^2 + \|\phi\|_{L^6}^2 + \|\psi\|_{L^\infty}\right)\|\phi - \overline{\phi}\|_{L^6} + \varepsilon\|\overline{\phi}\|_{L^2}\|\psi - \overline{\psi}\|_{L^\infty} \\
&\quad \le C\|\phi - \overline{\phi}\|_{H^1} + C\|\psi - \overline{\psi}\|_{H^2} \le K_U\|u - v\|_{D\left(A^{1/2}\right)}
\end{aligned}$$

and

$$\begin{aligned}
\|F_2(u) - F_2(v)\|_{L^2} &= \| - 2\lambda\Delta\left(\psi - \overline{\psi}\right) - (\gamma + \lambda)\left(\psi - \overline{\psi}\right) + g\left(\psi + \overline{\psi}\right)\left(\psi - \overline{\psi}\right) \\
&\qquad + \left(-\psi^2 - \psi\overline{\psi} - \overline{\psi}^2\right)\left(\psi - \overline{\psi}\right) - \frac{\varepsilon}{2}\left(\phi + \overline{\phi}\right)\left(\phi - \overline{\phi}\right)\|_{L^2} \\
&\le C\left(\|\overline{\psi}\|_{L^6}^2 + \|\psi\|_{L^6}^2\right)\|\phi - \overline{\psi}\|_{L^6} + 2\lambda\|\psi - \overline{\psi}\|_{H^2} \\
&\qquad + C\left(\|\phi\|_{L^4}^2 + \|\overline{\phi}\|_{L^4}^2\right)\|\phi - \overline{\phi}\|_{L^4} \le K_U\|u - v\|_{D\left(A^{1/2}\right)}.
\end{aligned}$$

The general theory of ([5]) guarantees the existence of a local solution for system (1.1)-(1.2). The priori estimate Lemma 2-Lemma 5 implying the local solution can be a global one for t. Moreover, the family of operators $S(t)_{t\ge 0}$ forms a strongly continuous semigroup on the space $D(A^{1/2})$, which has the property

$$u \in C([0, T_{max}); H^1 \times H^2) \cap C^1((0, T_{max}); H^1 \times H^2) \cap L^2([0, T_{max}); L^2 \times L^2).$$

3 Absorbing sets

Lemma 2. *For any given $R > 0$ there exists a constant $M_1(R) > 0$ such that if the initial data $u_0 = (\phi_0, \psi_0)^T \in H$ and $\|u_0\|_H^2 \le R$, then $S(t)u_0 = (\phi, \psi)^T \in H$ for all $t \ge 0$, and*

$$\|S(t)u_0\|_H^2 \le M_1(R), \quad for \quad t \ge 0.$$

Proof. Taking the inner products $\langle(1.1), 2\phi\rangle$, and $\langle(1.2), 2\psi\rangle$, then summing up the resulting equalities and by the Neumann boundary conditions, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|\phi\|^2 + \|\psi\|^2\right) + 2\|\nabla\phi\|^2 + 2\lambda\int_\Omega[(\Delta + 1)\psi]^2dx \\
&\qquad = 2\alpha\|\phi\|^2 + 2g\int_\Omega\psi^3dx - 3\varepsilon\int_\Omega\phi^2\psi dx - 2\|\phi\|_{L^4}^4 - 2\|\psi\|_{L^4}^4 - 2\gamma\|\psi\|^2,
\end{aligned}$$

that is

$$\begin{aligned}
&\frac{d}{dt}\left(\|\phi\|^2 + \|\psi\|^2\right) + \kappa\left(\|\phi\|^2 + \|\psi\|^2\right) + 2\|\nabla\phi\|^2 + 2\lambda\int[(\Delta + 1)\psi]^2dx \\
&\qquad \le 2\Lambda(\phi, \psi), \qquad (3.1)
\end{aligned}$$

where

$$\Lambda(\phi,\psi) = -\int\left(\frac{\sqrt{2}}{2}\phi^2 - \frac{3\sqrt{2}}{4}\psi\right)^2 dx + \left(\frac{9\varepsilon^2}{8}+\frac{\kappa}{2}\right)\|\psi\|^2 + \frac{\kappa}{2}\|\phi\|^2 - \frac{1}{2}\|\phi\|_{L^4}^4 - \gamma\|\psi\|^2 + g\|\psi\|_{L^3}^3 - \|\psi\|_{L^4}^4.$$

The Young inequalities yield

$$\frac{\kappa}{2}\|\phi\|^2 \le \frac{\kappa\varepsilon_1}{4}\|\phi\|_{L^4}^4 + \frac{\kappa}{4\varepsilon_1}|\Omega|, \quad g\|\psi\|_{L^3}^3 \le g\varepsilon_2\|\psi\|_{L^4}^4 + g\varepsilon_2^{-3}|\Omega|,$$
$$\left(\frac{9\varepsilon^2}{8}+\frac{\kappa}{2}\right)\|\psi\|^2 \le \left(\frac{9\varepsilon^2}{8}+\frac{\kappa}{2}\right)\frac{\varepsilon_3}{2}\|\psi\|_{L^4}^4 + \left(\frac{9\varepsilon^2}{8}+\frac{\kappa}{2}\right)\frac{1}{2\varepsilon_3}|\Omega|,$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are arbitrarily positive constants. Hence, we have

$$\Lambda(\phi,\psi) \le \left(g\varepsilon_2 + \left(\frac{9\varepsilon^2}{8}+\frac{\kappa}{2}\right)\frac{\varepsilon_3}{2} - 1\right)\|\psi\|_{L^4}^4 - \int\left(\frac{\sqrt{2}}{2}\phi^2 - \frac{3\sqrt{2}}{4}\psi\right)^2 dx - \gamma\|\psi\|^2 + \left(\frac{\kappa\varepsilon_1}{4} - \frac{1}{2}\right)\|\phi\|_{L^4}^4 + \left(\frac{\kappa}{4\varepsilon_1} + g\varepsilon_2^{-3} + \left(\frac{9\varepsilon^2}{8}+\frac{\kappa}{2}\right)\frac{1}{2\varepsilon_3}\right)|\Omega|.$$

We take $\varepsilon_1 = \frac{1}{2}$, $\varepsilon_2 = \frac{1}{2g}$, $\varepsilon_3 = \frac{1}{(9\varepsilon^2/8+\kappa/2)}$ and $0 < \kappa \le 2$, then

$$\Lambda(\phi,\psi) \le \left(\frac{\kappa}{4\varepsilon_1} + g\varepsilon_2^{-3} + \left(\frac{9\varepsilon^2}{8}+\frac{\kappa}{2}\right)\frac{1}{2\varepsilon_3}\right)|\Omega| \equiv \frac{1}{2}C^*\left(\kappa, g, \varepsilon, |\Omega|\right). \quad (3.2)$$

From (3.1) and (3.2), we obtain

$$\frac{d}{dt}\left(\|\phi\|^2 + \|\psi\|^2\right) + \kappa\left(\|\phi\|^2 + \|\psi\|^2\right) \le C^*\left(\kappa, g, \varepsilon, |\Omega|\right), \quad for\ t \in [0, T_{max}),$$

then, applying the Gronwall inequality, we deduce that

$$\left(\|\phi\|^2 + \|\psi\|^2\right) \le e^{-\kappa t}\left(\|\phi_0\|^2 + \|\psi_0\|^2\right) + \frac{C^*}{\kappa}, \quad for \quad t \in [0, T_{max}).$$

Let M_1 be the constant $M_1 = \frac{C^*}{\kappa} + 1$. The proof is completed. □

Lemma 3. *For any given $R > 0$ there exists a constant $M_2(R) > 0$ such that if the initial data $u_0 = (\phi_0, \psi_0)^T \in E$ and $\|u_0\|_E^2 \le R$, then $S(t)u_0 = (\phi, \psi)^T \in E$ for all $t \ge 0$, and*

$$\|S(t)u_0\|_E^2 \le M_2(R), \quad for \quad t \ge 0.$$

Proof. Take the inner products $\langle(1.1), -2\Delta\phi\rangle$, and $\langle(1.2), -2\Delta\psi\rangle$. Then sum up the resulting equalities

$$\frac{d}{dt}\left(\|\nabla\phi\|^2 + \|\nabla\psi\|^2\right) = -2\|\Delta\phi\|^2 - 6\int_\Omega \phi^2(\nabla\phi)^2 dx + 2\alpha\|\nabla\phi\|^2 - 2\|\nabla\Delta\psi\|^2 - 2(\gamma+\lambda)\|\nabla\psi\|^2 - 6\int_\Omega \psi^2(\nabla\psi)^2 dx + 4g\int_\Omega \psi(\nabla\psi)^2 dx + 4\lambda\|\Delta\psi\|^2 - 2\varepsilon\int_\Omega \phi\nabla\phi\nabla\psi dx + 2\varepsilon\int_\Omega \psi\phi\Delta\phi dx,$$

that is

$$\frac{d}{dt}\left(\|\nabla\phi\|^2+\|\nabla\psi\|^2\right)+\kappa_2\left(\|\nabla\phi\|^2+\|\nabla\psi\|^2\right)\equiv \Lambda_2(\phi,\psi), \tag{3.3}$$

where

$$\begin{aligned}\Lambda_2(\phi,\psi)=&-\int_\Omega\left(\sqrt{6}\phi\nabla\phi+\frac{\varepsilon\nabla\psi}{\sqrt{6}}\right)^2dx-\int_\Omega\left(\sqrt{6}\psi\nabla\psi+\frac{\sqrt{6}}{3}g\nabla\psi\right)^2dx\\&+(2\alpha+\kappa_2)\|\nabla\phi\|^2+\left(\kappa_2+\frac{\varepsilon^2}{6}+\frac{2}{3}g^2-2\gamma-2\lambda\right)\|\nabla\psi\|^2\\&-2\|\Delta\phi\|^2-2\lambda\|\nabla\Delta\psi\|^2+2\varepsilon\int_\Omega\psi\phi\Delta\phi dx+4\lambda\|\Delta\psi\|^2.\end{aligned}$$

Then, applying the Nirenberg inequality ($n\le 3$), we deduce that

$$\begin{aligned}&\|\Delta\psi\|\le C_1\|\nabla\Delta\psi\|^{1/2}\|\psi\|^{1/2}+C_2\|\psi\|,\\&\|\phi\|_{L^4}\le C_1\|\Delta\phi\|^{3/8}\|\phi\|^{5/8}+C_2\|\phi\|,\\&\|\psi\|_{L^4}\le C_1\|\nabla\Delta\psi\|^{1/4}\|\psi\|^{3/4}+C_2\|\psi\|,\\&\|\nabla\phi\|\le C_1\|\Delta\phi\|^{1/2}\|\phi\|^{1/2}+C_2\|\phi\|,\\&\|\nabla\psi\|\le C_1\|\nabla\Delta\psi\|^{1/3}\|\psi\|^{2/3}+C_2\|\psi\|.\end{aligned}$$

Using the Young inequality and Lemma 2, we have

$$\begin{aligned}2\varepsilon\int_\Omega\psi\phi\Delta\phi dx&\le\frac{1}{4}\|\Delta\phi\|^2+C(\varepsilon)\|\phi\|_{L^4}^4+C(\varepsilon)\|\psi\|_{L^4}^4\\&\le\frac{1}{2}\|\Delta\phi\|^2+\frac{\lambda}{2}\|\nabla\Delta\psi\|^2+C(\varepsilon),\\4\lambda\|\Delta\psi\|^2&\le\frac{\lambda}{4}\|\nabla\Delta\psi\|^2+C(\lambda),\quad(2\alpha+\kappa_2)\|\nabla\phi\|^2\le\frac{1}{2}\|\Delta\phi\|^2+C(\alpha,\kappa_2)\end{aligned}$$

and

$$\left(\kappa_2+\frac{\varepsilon^2}{6}+\frac{2}{3}g^2-2\gamma-2\lambda\right)\|\nabla\psi\|^2\le\frac{\lambda}{4}\|\nabla\Delta\psi\|^2+C\left(\kappa_2,\varepsilon,g,\gamma,\lambda\right).$$

Summing up the resulting equalities, we have

$$\begin{aligned}\Lambda_2(\phi,\psi)\le&-\int_\Omega\left(\sqrt{6}\phi\nabla\phi+\frac{\varepsilon\nabla\psi}{\sqrt{6}}\right)^2dx-\int_\Omega\left(\sqrt{6}\psi\nabla\psi+\frac{\sqrt{6}}{3}g\nabla\psi\right)^2dx\\&-\|\Delta\phi\|^2-\lambda\|\nabla\Delta\psi\|^2+C_2^*\left(\alpha,\gamma,g,\lambda,\kappa_2,\varepsilon,|\Omega|\right).\end{aligned}\tag{3.4}$$

From (3.3) and (3.4), we find that

$$\frac{d}{dt}\left(\|\nabla\phi\|^2+\|\nabla\psi\|^2\right)+\kappa_2\left(\|\nabla\phi\|^2+\|\nabla\psi\|^2\right)\le C_2^*\left(\alpha,\gamma,g,\lambda,\kappa_2,\varepsilon,|\Omega|\right),$$

then, applying the Gronwall inequality, we deduce that

$$\left(\|\nabla\phi\|^2+\|\nabla\psi\|^2\right)\le e^{-\kappa_2 t}\left(\|\nabla\phi_0\|^2+\|\nabla\psi_0\|^2\right)+\frac{C_2^*}{\kappa_2},\ for\ t\in[0,T_{max}).$$

Let M_2 be the constant $M_2=\frac{C_2^*}{\kappa_2}+1$. The proof is completed. □

Lemma 4. *For any given $R>0$ there exists a constant $M_3(R)>0$ such that if the initial data $u_0=(\phi_0,\psi_0)^T\in E_1$ and $\|u_0\|_{E_1}\le R$, then $S(t)u_0=(\phi,\psi)^T\in E_1$ for all $t\ge 0$, and*

$$\|S(t)u_0\|_E^2\le M_3(R),\quad for\quad t\ge 0.$$

Proof. Taking the inner products $\langle(1.1),2\phi\rangle$, and $\langle(1.2),2\Delta^2\psi\rangle$, by the Neumann boundary conditions, we get

$$\begin{aligned}\frac{d}{dt}\|\phi\|^2&=-2\|\nabla\phi\|^2+2\alpha\|\phi\|^2-2\|\phi\|_{L^4}^4-2\varepsilon\int_\Omega\phi^2\psi dx,\\ \frac{d}{dt}\|\Delta\psi\|^2&=-2\lambda\|\Delta^2\psi\|^2+4\lambda\|\nabla\Delta\psi\|^2-2(\lambda+\gamma)\|\Delta\psi\|^2\\ &\quad+2\alpha\int_\Omega\psi^2\Delta^2\psi dx-2\int_\Omega\psi^3\Delta^2\psi dx-\varepsilon\int_\Omega\phi^2\Delta^2\psi dx.\end{aligned}$$

Then summing up the resulting equalities, we see that

$$\begin{aligned}&\frac{d}{dt}\left(\|\phi\|^2+\|\Delta\psi\|^2\right)+\kappa_3\left(\|\phi\|^2+\|\Delta\psi\|^2\right)\\ =&-2\|\nabla\phi\|^2+(2\alpha+\kappa_3)\|\phi\|^2-2\|\phi\|_{L^4}^4-2\varepsilon\int_\Omega\phi^2\psi dx-2\lambda\|\Delta^2\psi\|^2\\ &+4\lambda\|\nabla\Delta\psi\|^2-2\left(\lambda+\gamma-\frac{\kappa_3}{2}\right)\|\Delta\psi\|^2+2\alpha\int_\Omega\psi^2\Delta^2\psi dx\\ &-2\int_\Omega\psi^3\Delta^2\psi dx-\varepsilon\int_\Omega\phi^2\Delta^2\psi dx\equiv\Lambda_3(\phi,\psi).\end{aligned}\tag{3.5}$$

By the Nirenberg inequalities, we get

$$\begin{aligned}&\|\phi\|_{L^4}^4\le C_1\|\nabla\phi\|^{3/4}\|\phi\|^{1/4}+C_2\|\phi\|,\\ &\|\nabla\Delta\psi\|\le C_1\|\Delta^2\psi\|^{3/4}\|\psi\|^{1/4}+C_2\|\psi\|,\\ &\|\psi\|_{L^4}\le C_1\|\Delta^2\psi\|^{3/16}\|\psi\|^{13/16}+C_2\|\psi\|,\\ &\|\psi\|_{L^6}\le C_1\|\Delta^2\psi\|^{1/4}\|\psi\|^{3/4}+C_2\|\psi\|.\end{aligned}$$

Using the Young inequalities, Lemma 2 and above inequalities, we obtain

$$2\varepsilon\int_\Omega\phi^2\psi dx\le\|\phi\|_{L^4}^4+\varepsilon\|\psi\|^2\le\|\nabla\phi\|^2+C(\varepsilon),\tag{3.6}$$

$$2\alpha\int_\Omega\psi^2\Delta^2\psi dx\le\frac{\lambda}{4}\|\Delta^2\psi\|^2+C(\lambda,\alpha)\|\psi\|_{L^4}^4\le\frac{\lambda}{2}\|\Delta^2\psi\|^2+C(\lambda,\alpha,|\Omega|)$$

and

$$2\int_\Omega \psi^3\Delta^2\psi dx \le \frac{\lambda}{4}\|\Delta^2\psi\|^2 + C(\lambda)\|\psi\|_{L^6}^6 \le \frac{\lambda}{2}\|\Delta^2\psi\|^2 + C(\lambda,|\Omega|).$$

Similarly, we have

$$\varepsilon\int_\Omega \phi^2\Delta^2\psi dx \le \frac{\lambda}{4}\|\Delta^2\psi\|^2 + C(\varepsilon)\|\phi\|_{L^4}^4, \le \frac{\lambda}{8}\|\Delta^2\psi\|^2 + \frac{1}{2}\|\nabla\phi\|^2 + C(\varepsilon,|\Omega|),$$
$$2\left(\lambda+\gamma-\frac{\kappa_3}{2}\right)\|\Delta\psi\|^2 \le \frac{\lambda}{8}\|\Delta^2\psi\|^2 + C\left(\lambda,\kappa_3,\gamma,|\Omega|\right). \tag{3.7}$$

From (3.5) and (3.6)-(3.7), we obtain

$$\begin{aligned}\Lambda_3(\phi,\psi) \le & -\frac{\lambda}{4}\|\Delta^2\psi\|^2 - \frac{1}{2}\|\nabla\phi\|^2 + C_3^*\left(\gamma,\varepsilon,\lambda,\alpha,\kappa_3,|\Omega|\right)\\ \le & C_3^*\left(\gamma,\varepsilon,\lambda,\alpha,\kappa_3,|\Omega|\right),\end{aligned}$$

that is

$$\frac{d}{dt}\left(\|\phi\|^2 + \|\Delta\psi\|^2\right) + \kappa_3\left(\|\phi\|^2 + \|\Delta\psi\|^2\right) \le C_3^*.$$

Applying the Gronwall inequality, we deduce that

$$\left(\|\phi\|^2 + \|\Delta\psi\|^2\right) \le e^{-\kappa_3 t}\left(\|\phi_0\|^2 + \|\Delta\psi_0\|^2\right) + C_3^*/\kappa_3, \quad for \quad t\in[0,T_{max}).$$

Taking $M_3 = C_3^*/\kappa_3 + 1$, the proof is completed. □

4 Asymptotic compactness

Lemma 5. *For any given $R>0$ there exists a constant $M_4(R)>0$ such that if the initial data $u_0=(\phi_0,\psi_0)^T\in E_2$ and $\|u_0\|_{E_2}^2\le R$, then $S(t)u_0=(\phi,\psi)^T\in E_2$ for all $t\ge 0$, and*

$$\|S(t)u_0\|_{E_2}^2 \le M_4(R), \quad for \quad t\ge 0.$$

Proof. Take the inner products $\langle(1.1),-2\Delta\phi\rangle$, and $\langle\Delta(1.2),-2\Delta^2\psi\rangle$. By the Neumann boundary conditions, we get

$$\begin{aligned}&\frac{d}{dt}\|\nabla\phi\|^2 = -2\|\Delta\phi\|^2 + 2\alpha\|\nabla\phi\|^2 - 6\int_\Omega \phi^2(\nabla\phi)^2 dx + 2\varepsilon\int_\Omega \phi\psi\Delta\phi dx,\\ &\frac{d}{dt}\|\nabla\Delta\psi\|^2 = -2\lambda\|\nabla\Delta^2\psi\|^2 + 4\lambda\|\Delta^2\psi\|^2 - 2(\lambda+\gamma)\|\nabla\Delta\psi\|^2\\ &+4g\int_\Omega \psi\nabla\psi\nabla\Delta^2\psi dx - 6\int_\Omega \psi^2\nabla\psi\nabla\Delta^2\psi dx - 2\varepsilon\int_\Omega \phi\nabla\phi\nabla\Delta^2\psi dx.\end{aligned}$$

Then summing up the resulting equalities, we see that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla\phi\|^2+\|\nabla\Delta\psi\|^2\right)+\kappa_4\left(\|\nabla\phi\|^2+\|\nabla\Delta\psi\|^2\right)=-2\|\Delta\phi\|^2\\
&\quad+(2\alpha+\kappa_4)\|\nabla\phi\|^2-6\int_\Omega\phi^2(\nabla\phi)^2dx+2\varepsilon\int_\Omega\phi\psi\Delta\phi dx-2\lambda\|\nabla\Delta^2\psi\|^2\\
&\quad+4\lambda\|\Delta^2\psi\|^2-2\left(\lambda+\gamma-\frac{\kappa_4}{2}\right)\|\nabla\Delta\psi\|^2+4g\int_\Omega\psi\nabla\psi\nabla\Delta^2\psi dx\\
&\quad-6\int_\Omega\psi^2\nabla\psi\nabla\Delta^2\psi dx-2\varepsilon\int_\Omega\phi\nabla\phi\nabla\Delta^2\psi dx\equiv\Lambda_4(\phi,\varphi).
\end{aligned}\tag{4.1}$$

Using the Nirenberg inequalities ($n\le 3$), we obtain

$$\begin{aligned}
&\|\nabla\phi\|\le C_1\|\Delta\phi\|^{1/2}\|\phi\|^{1/2}+C_2\|\phi\|,\\
&\|\phi\|_{L^\infty}\le C_1\|\Delta\phi\|^{3/4}\|\phi\|^{1/4}+C_2\|\phi\|,\\
&\|\nabla\Delta\psi\|\le C_1\|\nabla\Delta^2\psi\|^{3/5}\|\psi\|^{1/4}+C_2\|\psi\|,\\
&\|\Delta^2\psi\|\le C_1\|\nabla\Delta^2\psi\|^{4/5}\|\psi\|^{1/4}+C_2\|\psi\|.
\end{aligned}$$

From Lemma 3, and noticing $H^1(\Omega)\hookrightarrow L^4(\Omega)$ for $n\le 3$, we deduce that

$$2\varepsilon\int_\Omega\psi\phi\Delta\phi dx\le\frac14\|\Delta\phi\|^2+C(\varepsilon)\|\phi\|_{L^4}^4+C(\varepsilon)\|\psi\|_{L^4}^4\le\frac12\|\Delta\phi\|^2+C(\varepsilon),\tag{4.2}$$

$$\begin{aligned}
&4\lambda\|\Delta^2\psi\|^2\le\frac{\lambda}{4}\|\nabla\Delta^2\psi\|^2+C(\lambda,|\Omega|),\\
&2\left(\lambda+\gamma-\frac{\kappa_4}{2}\right)\|\nabla\Delta\psi\|^2\le\frac{\lambda}{4}\|\nabla\Delta^2\psi\|^2+C\left(\lambda,\gamma,\kappa_4,|\Omega|\right),\\
&4g\int_\Omega\psi\nabla\psi\nabla\Delta^2\psi dx\le\frac{\lambda}{4}\|\nabla\Delta^2\psi\|^2+C(\lambda)\|\psi\|_{L^4}^4+C(\lambda)\|\nabla\psi\|_{L^4}^4\\
&\qquad\le\frac{\lambda}{4}\|\nabla\Delta^2\psi\|^2+C(\lambda,g,|\Omega|).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&6\int_\Omega\psi^2\nabla\psi\nabla\Delta^2\psi dx\le\frac{\lambda}{8}\|\nabla\Delta^2\psi\|^2+C\|\psi\|_{L8}^8+C\|\nabla\psi\|\le\frac{\lambda}{4}\|\nabla\Delta^2\psi\|^2+C,\\
&2\varepsilon\int_\Omega\phi\nabla\phi\nabla\Delta^2\psi dx\le\frac{\lambda}{4}\|\nabla\Delta^2\psi\|^2+C(\lambda,\varepsilon)\|\phi\|_{L^\infty}^2\|\nabla\phi\|^2\\
&\qquad\le\frac{\lambda}{4}\|\nabla\Delta^2\psi\|^2+\|\Delta\phi\|+C(\lambda,\varepsilon,|\Omega|).
\end{aligned}\tag{4.3}$$

From (4.1) and (4.2)-(4.3), we obtain

$$\Lambda_4(\phi,\varphi)\le-\frac34\|\Delta\phi\|^2-\frac34\|\nabla\Delta^2\psi\|^2+C_4^*\left(\lambda,\varepsilon,\alpha,\kappa_4,|\Omega|\right),$$

that is

$$\frac{d}{dt}\left(\|\nabla\phi\|^2+\|\nabla\Delta\psi\|^2\right)+\kappa_4\left(\|\nabla\phi\|^2+\|\nabla\Delta\psi\|^2\right)\le C_4^*.$$

Applying the Gronwall inequality, we deduce that

$$\left(\|\nabla\phi\|^2+\|\nabla\Delta\psi\|^2\right)\le e^{-\kappa_4 t}\left(\|\nabla\phi_0\|^2+\|\nabla\Delta\psi_0\|^2\right)+\frac{C_4^*}{\kappa_4},\ \textit{for } t\in[0,T_{max}).$$

Taking $M_4=C_4^*/\kappa_4+1$, the proof is completed. □

We now finish the proof of Theorem 1.

Proof. [Proof of Theorem 1] First, by Lemma 4, the solution semiflow $S(t)_{t\ge0}$ of reaction-diffusion system (1.1)-(1.2) has a bounded absorbing set B_0 in E_1. Second, according to Lemma 5 and due to that Sobolev imbedding $E_2\hookrightarrow E_1$ is compact, this solution semiflow $S(t)_{t\ge0}$ is asymptotically compact in E_1, then by Lemma 1, there exists a global attractor $\mathcal{A}$ for $S(t)_{t\ge0}$ in E_1, which is given by (2.1). □

5 The H^k global attractor

In order to consider the global attractor for the system (1.1)-(1.2) in H^k space, we introduce the definition as follows:

$$H=L^2(\Omega),\quad H_{\frac12}=\left\{u\in H^2(\Omega),\ \frac{\partial u}{\partial n}|_{\partial\Omega}=0\right\},$$
$$H_1=\left\{u\in H^4(\Omega),\ \frac{\partial u}{\partial n}|_{\partial\Omega}=\frac{\partial\Delta u}{\partial n}|_{\partial\Omega}=0\right\}.$$

In this paper, we used to assume that the linear operator

$$L=-\lambda\Delta^2:\quad H_1\to H$$

is a sectorial operator, which generates an analytic semigroup e^{tL}, and L induces the fractional power operators and fractional order spaces as follows

$$\mathcal{L}^\alpha=(-L)^\alpha:H_\alpha\to H,\ \ (i=1,2),\ \alpha\in\mathbb{R},$$

where $H_\alpha=D\left(\mathcal{L}^\alpha\right)$ is the domain of $\mathcal{L}^\alpha$. By the semigroup theory of linear operators, $H_\beta\subset H_\alpha$ is a compact inclusion for any $\beta>\alpha$. For details of the space H_α see [8].

Then, we have the following lemma on the existence of global attractor which is equivalent to Lemma 1 and the proof is similar to [12, 13, 14].

Lemma 6. *Assume that $(\phi(t),\psi(t))=S(t)(\phi_0,\psi_0)\ \ ((\phi_0,\psi_0))\in H\times H),t\ge0)$ is a solution of (1.1) and $S(t)$ the semigroup generated by (1.1). Assume further that H_α is the fractional order space generated by L and*

(1) For some $\alpha\ge0$, there is a bounded set $B\subset H_{\alpha+\frac14}\times H_{\alpha+\frac34}$, which means that for any $(\phi_0,\psi_0)\in H_{\alpha+\frac14}\times H_{\alpha+\frac34}$, there exists $t_0\ge0$ such that

$$(\phi(t),\psi(t))\in B,\ \forall\ t>t_0;$$

(2) There is a $\beta > \alpha$, such that for any bounded set $U \subset H_{\beta+\frac{1}{4}} \times H_{\beta+\frac{3}{4}}$, there are $T > 0$ and $C > 0$,

$$\|u(t,u_0)\|_{H_{\beta+\frac{1}{4}} \times H_{\beta+\frac{3}{4}}} \leq C, \ \ \forall t > T \ \ and \ \ (\phi_0, \psi_0) \in U.$$

Then (1.1) has a global attractor $\mathcal{A} \subset H_{\alpha+\frac{1}{4}} \times H_{\alpha+\frac{3}{4}}$ which attracts any bounded set of $H_{\alpha+\frac{1}{4}} \times H_{\alpha+\frac{3}{4}}$ in the $H_{\alpha+\frac{1}{4}} \times H_{\alpha+\frac{3}{4}}$ norm.

For sectorial operators, we also have the following lemma which is important for this paper and can be founded in [12, 13, 14].

Lemma 7. *Assume that L is a sectorial operator which generates an analytic semigroup $T(t) = e^{tL}$. If all eigenvalues λ of L satisfy $Re\lambda < -\lambda_0$ for some real number $\lambda_0 > 0$, then for $\mathcal{L}^\alpha$ ($\mathcal{L} = -L$) we have*

(1) $T(t) : H \to H_\alpha$ is bounded for all $\alpha \in \mathbb{R}$ and $t > 0$;

(2) $T(t)\mathcal{L}^\alpha x = \mathcal{L}T(t)x$, $\forall x \in H_\alpha$;

(3) For each $t > 0$, $\mathcal{L}^\alpha T(t) : H \to H$ is bounded, and

$$\|\mathcal{L}^\alpha T(t)\| \leq C_\alpha t^{-\alpha} e^{-\delta t},$$

where some $\delta > 0$ and $C_\alpha > 0$ is a constant depending only on α;

(4) The H_α−norm can be defined by $\|x\|_{H_\alpha} = \|\mathcal{L}^\alpha x\|_H$.

The main result of this paper is given by the following theorem, which provides the existence of global attractors of Eq.(1.1) in any kth space H^k.

Theorem 2. *Assume that Ω denotes an open bounded domain in R^3, then for any $k \geq 0$, the initial-boundary value problem (1.1)-(1.2) has a global attractor $\mathcal{A}$ in $H^k \times H^{k+2}$, and $\mathcal{A}$ attracts any bounded subset of $H^k \times H^{k+2}$ in the $H^k \times H^{k+2}$-norm.*

By Lemma 6, in order to prove Theorem 2, we first prove the following lemma.

Lemma 8. *For any $\sigma \geq 0$, the solution (ϕ, ψ) of (1.1)-(1.2) is uniformly bounded in $H_{\sigma+\frac{1}{4}} \times H_{\sigma+\frac{3}{4}}$, i.e. for any bounded set $U \subset H_{\sigma+\frac{1}{4}} \times H_{\sigma+\frac{3}{4}}$, there exists $C > 0$ such that*

$$\|(\phi,\psi)\|_{H_{\sigma+\frac{1}{4}} \times H_{\sigma+\frac{3}{4}}} \leq C, \ \forall t > 0, (\phi_0, \psi_0) \in U \subset H_{\sigma+\frac{1}{4}} \times H_{\sigma+\frac{3}{4}}, \sigma \geq 0.$$

Proof. For any $(\phi_0, \psi_0) \in L^2(\Omega) \times H^2(\Omega)$, the solutions (ϕ, ψ) of (1.1)-(1.2) can be expressed as

$$\phi(t,\phi_0) = e^{tL_1}\phi_0 + \int_0^t e^{(t-\tau)L_1} F_1(\phi,\psi) d\tau,$$
$$\psi(t,\psi_0) = e^{tL_2}\psi_0 + \int_0^t e^{(t-\tau)L_2} F_2(\phi,\psi) d\tau,$$

where $L_1 = \Delta, L_2 = -\lambda\Delta^2$. By Theorem 1, there exists attractor in the phase space $L^2 \times H^2$. Using the same way, it is not difficult to prove that there exists attractor in $H^1 \times H^3$, which means $(\phi, \psi) \in H_{\frac{1}{4}} \times H_{\frac{3}{4}}$.

Step1. We shall prove that for any bounded set $U \subset H_{\sigma+\frac{1}{4}} \times H_{\sigma+\frac{3}{4}} \left(0 \le \sigma < \frac{1}{4}\right)$, there exists $C > 0$ such that

$$\|(\phi, \psi)\|_{H_{\sigma+\frac{1}{4}} \times H_{\sigma+\frac{3}{4}}} \le C,\ (\phi_0, \psi_0) \in U \subset H_{\sigma+\frac{1}{4}} \times H_{\sigma+\frac{3}{4}}, 0 \le \sigma < \frac{1}{4}. \quad (5.1)$$

We claim that $F_2 : H_{\frac{1}{4}} \times H_{\frac{1}{2}} \to H$ is bound. Based on Lemma 5 and embedding theorem $H^1 \hookrightarrow L^6$ for $n \le 3$, we have

$$\begin{aligned}
&\|F_2(\phi, \psi)\|_H = \|-2\lambda\Delta\psi - (\gamma + \lambda)\psi + g\psi^2 - \psi^3 - \frac{\varepsilon}{2}\phi^2\|_H \\
&\le C\left(\|\Delta\psi\|_{L^2} + \|\psi\|_{L^4}^2 + \|\psi\|_{L^6}^3 + \|\phi\|_{L^4}^2\right) \le C\left(\|\psi\|_{H_{\frac{1}{2}}}^2 + \|\phi\|_{H_{\frac{1}{4}}}^2\right) \le C,
\end{aligned}$$

where C depends on $\lambda, \gamma, g, \varepsilon, |\Omega|$ but independent of ϕ_0 and ψ_0. Hence, we obtain

$$\begin{aligned}
\|\psi(t, \psi_0)\|_{H_{\sigma+\frac{3}{4}}} &= \left\|e^{tL_2}\psi_0 + \int_0^t e^{(t-\tau)L_2} F_2(\phi, \psi) d\tau\right\|_{H_{\sigma+\frac{3}{4}}} \\
&\le \|\psi_0\|_{H_{\sigma+\frac{3}{4}}} + \left\|\int_0^t e^{(t-\tau)L_2} F_2(\phi, \psi) d\tau\right\|_{H_{\sigma+\frac{3}{4}}} \\
&\le \|\psi_0\|_{H_{\sigma+\frac{3}{4}}} + \int_0^t \|(-L_2)^{\sigma+\frac{3}{4}} e^{(t-\tau)L_2}\| \cdot \|F_2(\phi, \psi)\|_H d\tau \\
&\le \|\psi_0\|_{H_{\sigma+\frac{3}{4}}} + C\int_0^t (t-\tau)^{-\left(\sigma+\frac{3}{4}\right)} e^{-\delta(t-\tau)} d\tau \\
&\le \|\psi_0\|_{H_{\sigma+\frac{3}{4}}} + C\int_0^t \tau^{-\beta} e^{-\delta\tau} d\tau \\
&\le C\|(\phi_0, \psi_0)\|_{H_{\sigma+\frac{1}{4}} \times H_{\sigma+\frac{3}{4}}},\quad \left(0 \le \sigma < \frac{1}{4}\right), \quad (5.2)
\end{aligned}$$

where $\beta = \sigma + \frac{3}{4}$ and $0 \le \beta < 1$.

Similarly, we claim that $F_1 : H_{\frac{1}{4}} \times H_{\frac{1}{4}} \to H$ is bound. Based on Lemma 5 and embedding theorem $H^1 \hookrightarrow L^6$, $H^1 \hookrightarrow L^4$ for $n \le 3$, we have

$$\begin{aligned}
&\|F_1(\phi, \psi)\|_H = \|\alpha\phi - \phi^3 - \varepsilon\phi\psi\|_H \\
&\quad \le C\left(\|\phi\|_{L^2} + \|\psi\|_{L^4}^2 + \|\phi\|_{L^6}^3 + \|\phi\|_{L^4}^2\right) \le C\left(\|\psi\|_{H_{\frac{1}{4}}}^2 + \|\phi\|_{H_{\frac{1}{4}}}^2\right) \le C,
\end{aligned}$$

where C depends on $\alpha, \varepsilon, |\Omega|$ but independent of ϕ_0 and ψ_0. Hence,

$$\begin{aligned}
\|\phi(t, \phi_0)\|_{H_{\sigma+\frac{1}{4}}} &= \left\|e^{tL_1}\phi_0 + \int_0^t e^{(t-\tau)L_1} F_1(\phi, \psi) d\tau\right\|_{H_{\sigma+\frac{1}{4}}} \\
&\le \|\phi_0\|_{H_{\sigma+\frac{1}{4}}} + \left\|\int_0^t e^{(t-\tau)L_1} F_1(\phi, \psi) d\tau\right\|_{H_{\sigma+\frac{1}{4}}} \\
&\le \|\phi_0\|_{H_{\sigma+\frac{1}{4}}} + \int_0^t \|(-L_1)^{2\sigma+\frac{1}{2}} e^{(t-\tau)L_1}\| \cdot \|F_1(\phi, \psi)\|_H d\tau
\end{aligned}$$

$$\begin{aligned}&\leq \|\phi_0\|_{H_{\sigma+\frac{1}{4}}} + \int_0^t \tau^{-2\sigma-\frac{1}{2}} e^{-\delta\tau} d\tau \\ &\leq C\|(\phi_0,\psi_0)\|_{H_{\sigma+\frac{1}{4}}\times H_{\sigma+\frac{3}{4}}},\ \ (0\leq\sigma<1/4)\,,\end{aligned} \tag{5.3}$$

where $0 \leq 2\sigma + \frac{1}{2} < 1$. From (5.2)-(5.3), then (5.1) is proved.

For $\sigma = 1/8$, we have $\phi \subset H_{3/8}$, this meaning $\phi \subset H^{3/2}(\Omega)$(for fractional order Sobolev space). By the embedding theorems of fractional order spaces, we deduce that

$$\phi(t.x) \subset H^{3/2} \hookrightarrow C^0(\Omega) \cap H^1(\Omega). \tag{5.4}$$

Step2. We shall prove that for any bounded set $U \subset H_{\sigma+\frac{1}{4}} \times H_{\sigma+\frac{3}{4}}$ $\left(\frac{1}{4} \leq \sigma < \frac{1}{2}\right)$, there exists $C > 0$ such that

$$\|(\phi,\psi)\|_{H_{\sigma+\frac{1}{4}}\times H_{\sigma+\frac{3}{4}}} \leq C,\ (\phi_0,\psi_0) \in U \subset H_{\sigma+\frac{1}{4}} \times H_{\sigma+\frac{3}{4}}, \frac{1}{4} \leq \sigma < \frac{1}{2}. \tag{5.5}$$

We claim that $F_2 : H_{\frac{1}{4}} \times H_{\frac{3}{4}} \to H_{\frac{1}{4}}$ is bound. Based on Lemma 5 and embedding theorem $H^3 \hookrightarrow L^\infty$ for $n \leq 3$, we have

$$\begin{aligned}\|F_2(\phi,\psi)\|_{H_{\frac{1}{4}}} &= \|\nabla\left(-2\lambda\Delta\psi - (\gamma+\lambda)\psi + g\psi^2 - \psi^3 - \frac{\varepsilon}{2}\phi^2\right)\|_H \\ &= \|-2\lambda\nabla\Delta\psi - (\gamma+\lambda)\nabla\psi + 2g\nabla\psi\psi - 3\psi^2\nabla\psi - \varepsilon\phi\nabla\phi)\|_H \\ &\leq C\{\|\nabla\Delta\psi\|_{L^2}^2 + \left(\|\psi\|_{L^\infty}^4 + 1\right)\|\nabla\psi\|_{L^2}^2 + \sup_{\Omega}|\phi|^2\|\nabla\phi\|_{L^2}^2\} \\ &\leq C\left(\|\psi\|_{H_{\frac{3}{4}}}^2 + \|\phi\|_{H_{\frac{1}{4}}}^2\right) \leq C,\end{aligned}$$

where we used (5.4).

$$\begin{aligned}\|\psi(t,\psi_0)\|_{H_{\sigma+\frac{3}{4}}} &= \left\|e^{tL_2}\psi_0 + \int_0^t e^{(t-\tau)L_2}F_2(\phi,\psi)d\tau\right\|_{H_{\sigma+\frac{3}{4}}} \\ &\leq \|\psi_0\|_{H_{\sigma+\frac{3}{4}}} + \left\|\int_0^t e^{(t-\tau)L_2}F_2(\phi,\psi)d\tau\right\|_{H_{\sigma+\frac{3}{4}}} \\ &\leq \|\psi_0\|_{H_{\sigma+\frac{3}{4}}} + \int_0^t \|(-L_2)^{\sigma+\frac{1}{2}}e^{(t-\tau)L_2}\|\cdot\|F_2(\phi,\psi)\|_{H_{\frac{1}{4}}}d\tau \\ &\leq \|\psi_0\|_{H_{\sigma+\frac{3}{4}}} + C\int_0^t (t-\tau)^{-\left(\sigma+\frac{1}{2}\right)}e^{-\delta(t-\tau)}d\tau \\ &\leq \|\psi_0\|_{H_{\sigma+\frac{3}{4}}} + C\int_0^t \tau^{-\left(\sigma+\frac{1}{2}\right)}e^{-\delta\tau}d\tau \\ &\leq C\|(\phi_0,\psi_0)\|_{H_{\sigma+\frac{1}{4}}\times H_{\sigma+\frac{3}{4}}}, (1/4 \leq \sigma < 1/2)\,,\end{aligned} \tag{5.6}$$

where $0 \leq \sigma + \frac{1}{2} < 1$.

Similarly, we claim that $F_1 : H_{\frac{1}{4}} \times H_{\frac{1}{4}} \to H_{\frac{1}{4}}$ is bound.

$$\|F_1(\phi,\psi)\|_{H_{\frac{1}{4}}} = \|\nabla\left(\alpha\phi - \phi^3 - \varepsilon\phi\psi\right)\|_H$$

$$\leq C\left(\|\nabla\phi\|_{L^2}^2+\|\phi\|_{L^\infty}^4\|\nabla\phi\|_{L^2}^2+\|\phi\|_{L^\infty}^2\|\nabla\psi\|_{L^2}^2+\|\psi\|_{L^\infty}^2\|\nabla\phi\|_{L^2}^2\right)$$
$$\leq C\left(\|\psi\|_{H_{\frac{1}{4}}}^2+\|\phi\|_{H_{\frac{1}{4}}}^2\right)\leq C,$$

where C depends on $\alpha,\varepsilon,|\Omega|$ but independent of ϕ_0 and ψ_0. Therefore

$$\begin{aligned}
\|\phi(t,\phi_0)\|_{H_{\sigma+\frac{1}{4}}} &= \left\|e^{tL_1}\phi_0+\int_0^t e^{(t-\tau)L_1}F_1(\phi,\psi)d\tau\right\|_{H_{\sigma+\frac{1}{4}}}\\
&\leq \|\phi_0\|_{H_{\sigma+\frac{1}{4}}}+\left\|\int_0^t e^{(t-\tau)L_1}F_1(\phi,\psi)d\tau\right\|_{H_{\sigma+\frac{1}{4}}}\\
&\leq \|\phi_0\|_{H_{\sigma+\frac{1}{4}}}+\int_0^t \|(-L_1)^{2\sigma}e^{(t-\tau)L_1}\|\cdot\|F_1(\phi,\psi)\|_{H_{\frac{1}{4}}}d\tau\\
&\leq \|\phi_0\|_{H_{\sigma+\frac{1}{4}}}+\int_0^t \tau^{-2\sigma}e^{-\delta\tau}d\tau\\
&\leq C\|(\phi_0,\psi_0)\|_{H_{\sigma+\frac{1}{4}}\times H_{\sigma+\frac{3}{4}}},\ (1/4\leq\sigma<1/2)\,,
\end{aligned}\tag{5.7}$$

where $0\leq 2\sigma<1$. From (5.6)-(5.7), then (5.5) is proved. □

Lemma 9. *For any $\sigma>0$, (1.1)-(1.2) has a bounded absorbing set in $H_{\sigma+\frac{1}{4}}\times H_{\sigma+\frac{3}{4}}$. That is, for any bounded set $U\subset H_{\sigma+\frac{1}{4}}\times H_{\sigma+\frac{3}{4}}$ there are $T>0$ and a constant $C>0$ independent of ϕ_0 and ψ_0, such that*

$$\|(\phi,\psi)\|_{H_{\sigma+\frac{1}{4}}\times H_{\sigma+\frac{3}{4}}}\leq C,\ \forall t>T,\ (\phi_0,\psi_0)\in U\subset H_{\sigma+\frac{1}{4}}\times H_{\sigma+\frac{3}{4}},\sigma\geq 0. \tag{5.8}$$

Proof. Step1. We shall show that for any $0\leq\sigma<\frac{1}{4}$, (1.1)-(1.2) has a bounded absorbing set in $H_{\sigma+\frac{1}{4}}\times H_{\sigma+\frac{3}{4}}$. The solution (ϕ,ψ) can be expressed as

$$\phi(t,\phi_0)=e^{(t-T)L_1}\phi(T,\phi_0)+\int_T^t e^{(t-\tau)L_1}F_1(\phi,\psi)d\tau,$$
$$\psi(t,\psi_0)=e^{(t-T)L_2}\psi(T,\psi_0)+\int_T^t e^{(t-\tau)L_2}F_2(\phi,\psi)d\tau.$$

On the other hand, note that

$$\|e^{tL_1}\|\leq Ce^{-d\lambda_1 t},\quad \|e^{tL_2}\|\leq Ce^{-d\lambda_1^2 t},$$

where $\lambda_1>0$ is the first eigenvalue of the equation

$$-\Delta u=\lambda u,\quad u|_{\partial\Omega}=0.$$

By assertion (1) of lemma 7, for any given $T>0$ and $0\leq\sigma<\frac{1}{4}$, we have

$$\|e^{(t-T)L_i}\psi(T,\psi_0)\|_{H_\sigma}\to 0,\quad i=1,2,\quad \text{as } t\to\infty.$$

Using assertion (3) of lemma 7, we have

$$\|\psi(t,\psi_0)\|_{H_{\sigma+\frac{3}{4}}}=\left\|e^{(t-T)L_2}\psi(T,\psi_0)+\int_T^t e^{(t-\tau)L_2}F_2(\phi,\psi)d\tau\right\|_{H_{\sigma+\frac{3}{4}}}$$

$$\le \|e^{(t-T)L_2}\psi(T,\psi_0)\|_{H_{\sigma+\frac{3}{4}}} + \Big\|\int_T^t e^{(t-\tau)L_2}F_2(\phi,\psi)d\tau\Big\|_{H_{\sigma+\frac{3}{4}}}$$
$$\le \|e^{(t-T)L_2}\psi(T,\psi_0)\|_{H_{\sigma+\frac{3}{4}}} + \int_T^t \|(-L_2)^{\sigma+\frac{3}{4}}e^{(t-\tau)L_2}\|\cdot\|F_2(\phi,\psi)\|_H d\tau$$
$$\le \|e^{(t-T)L_2}\psi(T,\psi_0)\|_{H_{\sigma+\frac{3}{4}}} + C\int_T^t (t-\tau)^{-\left(\sigma+\frac{3}{4}\right)}e^{-\delta(t-\tau)}d\tau$$
$$\le \|e^{(t-T)L_2}\psi(T,\psi_0)\|_{H_\beta} + C\int_T^t \tau^{-\beta}e^{-\delta\tau}d\tau \le C,$$

where $\beta=\sigma+\frac{3}{4}$, $0\le\beta<1$, $C>0$ is a constant independent of ψ_0. Similarly, we have

$$\|\phi(t,\phi_0)\|_{H_{\sigma+\frac{1}{4}}} = \Big\|e^{(t-T)L_1}\phi(T,\phi_0) + \int_T^t e^{(t-\tau)L_1}F_1(\phi,\psi)d\tau\Big\|_{H_{\sigma+\frac{1}{4}}}$$
$$\le \|e^{(t-T)L_1}\phi(T,\phi_0)\|_{H_{\sigma+\frac{1}{4}}} + \Big\|\int_T^t e^{(t-\tau)L_1}F_1(\phi,\psi)d\tau\Big\|_{H_{\sigma+\frac{1}{4}}}$$
$$\le \|e^{(t-T)L_1}\phi(T,\phi_0)\|_{H_{\sigma+\frac{1}{4}}} + \int_T^t \|(-L_1)^{2\sigma+\frac{1}{2}}e^{(t-\tau)L_1}\|\cdot\|F_1(\phi,\psi)\|_H d\tau$$
$$\le \|e^{(t-T)L_1}\phi(T,\phi_0)\|_{H_{\sigma+\frac{1}{4}}} + \int_T^t \tau^{-2\sigma-\frac{1}{2}}e^{-\delta\tau}d\tau \le C,$$

where $0\le 2\sigma+\frac{1}{2}<1$, $C>0$ is a constant independent of ϕ_0.

Step2. We shall show that for any $\frac{1}{4}\le\sigma<\frac{1}{2}$, (1.1)-(1.2) has a bounded absorbing set in $H_\sigma\times H_{\sigma+\frac{3}{4}}$.

$$\|\psi(t,\psi_0)\|_{H_{\sigma+\frac{3}{4}}} = \Big\|e^{(t-T)L_2}\psi(T,\psi_0) + \int_T^t e^{(t-\tau)L_2}F_2(\phi,\psi)d\tau\Big\|_{H_{\sigma+\frac{3}{4}}}$$
$$\le \|e^{(t-T)L_2}\psi(T,\psi_0)\|_{H_{\sigma+\frac{3}{4}}} + \Big\|\int_T^t e^{(t-\tau)L_2}F_2(\phi,\psi)d\tau\Big\|_{H_{\sigma+\frac{3}{4}}}$$
$$\le \|e^{(t-T)L_2}\psi(T,\psi_0)\|_{H_{\sigma+\frac{3}{4}}} + \int_T^t \|(-L_2)^{\sigma+\frac{1}{2}}e^{(t-\tau)L_2}\|\cdot\|F_2(\phi,\psi)\|_{H_{\frac{1}{4}}} d\tau$$
$$\le \|e^{(t-T)L_2}\psi(T,\psi_0)\|_{H_{\sigma+\frac{3}{4}}} + C\int_T^t (t-\tau)^{-\left(\sigma+\frac{1}{2}\right)}e^{-\delta(t-\tau)}d\tau$$
$$\le \|e^{(t-T)L_2}\psi(T,\psi_0)\|_{H_{\sigma+\frac{3}{4}}} + C\int_T^t \tau^{-\left(\sigma+\frac{1}{2}\right)}e^{-\delta\tau}d\tau$$
$$\le \|e^{(t-T)L_2}\psi(T,\psi_0)\|_{H_{\sigma+\frac{3}{4}}} + C,$$

where $C>0$ is a constant independent of ψ_0.

$$\|\phi(t,\phi_0)\|_{H_{\sigma+\frac{1}{4}}} = \Big\|e^{(t-T)L_1}\phi(T,\phi_0) + \int_T^t e^{(t-\tau)L_1}F_1(\phi,\psi)d\tau\Big\|_{H_{\sigma+\frac{1}{4}}}$$
$$\le \|e^{(t-T)L_1}\phi(T,\phi_0)\|_{H_{\sigma+\frac{1}{4}}} + \Big\|\int_T^t e^{(t-\tau)L_1}F_1(\phi,\psi)d\tau\Big\|_{H_{\sigma+\frac{1}{4}}}$$

$$\leq \|e^{(t-T)L_1}\phi(T,\phi_0)\|_{H_{\sigma+\frac{1}{4}}} + \int_T^t \|(-L_1)^{2\sigma} e^{(t-\tau)L_1}\| \cdot \|F_1(\phi,\psi)\|_{H_{\frac{1}{4}}} d\tau$$
$$\leq \|e^{(t-T)L_1}\phi(T,\phi_0)\|_{H_{\sigma+\frac{1}{4}}} + \int_T^t \tau^{-2\sigma} e^{-\delta\tau} d\tau$$
$$\leq \|e^{(t-T)L_1}\phi(T,\phi_0)\|_{H_{\sigma+\frac{1}{4}}} + C,$$

by iteration, we can obtain (5.8). □

Proof. [Proof of Theorem 2.] By Lemma 8 and Lemma 9, we immediately conclude the proof of Theorem 2 is completed. □

Conclusions

Based on the regularity estimates for the semigroups and the classical existence theorem of global attractors, we prove that the system possesses a global attractor in the space $H_{k+\frac{1}{4}} \times H_{k+\frac{3}{4}}$. Comparing this paper with [12,13,14]. The system (1.1)-(1.2) is a two-component model. We define the product Hilbert spaces, using the Lumer-Phillips theorem and the generation theorem for analytic semigroups. The main difficulties for treating the problem (1.1)-(1.2) are caused by the nonlinearity of low order terms, and linear higher order terms are not homogeneous. The existence of the attractor in $H^k \times H^{k+2}$, guarantee a solution of the model equations for any value of the control parameters. This explains the following: 1) the solutions are robust and not sensitive to changes in the value of its control parameters and 2) the diversity of patterns that explain different biological systems (pigmentation vertebrate) and inhere animals (membranes porous medium and ternary mixtures with surfactants).

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