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Discrepancy Sets for Combined Least Squares Projection and Tikhonov Regularization

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Abstract. To solve a linear ill-posed problem, a combination of the finite dimensional least squares projection method and the Tikhonov regularization is considered. The dimension of the projection is treated as the second parameter of regularization. A two-parameter discrepancy principle defines a discrepancy set for any data error bound. The aim of the paper is to describe this set and to indicate its subset such that for regularization parameters from this subset the related regularized solution has the same order of accuracy as the Tikhonov regularization with the standard discrepancy principle but without any discretization.

Keywords: linear ill-posed problem, discrepancy principle, LSQ projection, Tikhonov regularization.

AMS Subject Classification: 65J20; 47A52; 65F22.

1 Introduction

Consider an ill-posed linear equation

$$Au = f \tag{1.1}$$

with a linear, injective and bounded operator $A \in L(X, Y)$ between real infinite dimensional Hilbert spaces X and Y with a nonclosed range $R(A)$. We assume that $f \in R(A)$, so that (1.1) has a unique solution $u \in X$. Moreover, we assume that f is unknown and a noisy right-hand side $f^\delta \in Y$ with

$$\|f - f^\delta\| \le \delta \tag{1.2}$$

is available only. A stable solution of (1.1) can be obtained via regularization methods. For numerical calculations, we have to look for methods which can be realized in finite dimensional spaces.

In this study we are concerned with a combination of finite dimensional least squares (LSQ) projections of (1.1) and the Tikhonov regularization method. There is an extensive literature concerning regularization by projection onto finite dimensional subspaces, beginning with the work of Natterer [13]. The mathematical analysis of projection schemes and their self-regularization can be found e.g. in [6, 11, 19] in the Hilbert space setting and with a posteriori choice of the discretization level. These investigations have been extended to regularization by projection in Banach spaces (see [7] and references therein) as well as to nonlinear ill-posed equations (see e.g. [8, 9]). Generally, without additional assumptions it cannot be guaranteed that a solution u_n obtained by a projection method approximates the exact solution even in the error free case (see [2] for an example given by T.I. Seidman (1980)). So, an additional regularization method should usually be applied.

In our analysis, for regularization of a discrete problem we apply the ordinary Tikhonov regularization, which is one of the most widely applied regularization methods. See for instance the monographs [2, 17, 18, 20]. For convergence analysis of regularization with LSQ projection see [3]. Finite dimensional approximation of (1.1) by the LSQ projection regularized by the standard regularization method generated by Borel measurable functions has been studied in [4, 5, 15]. This class of regularization methods contains also the Tikhonov method. From the very beginning, there has been much interest in a posteriori rules for choosing regularization parameters. The basic one is Morozov's discrepancy principle [12]. In [14] this principle is applied to some finite dimensional version of Tikhonov's regularization for severely ill-posed problems.

In the present paper, following [16], a discretization level n as well as a Tikhonov regularization parameter α are treated as a pair of regularization prameters. We consider an a posteriori choice of both the parameters based on Morozov's discrepancy principle. In such a situation, for any δ we get a set of parameter pairs (n, α), further called the *discrepancy set*. The aim of the paper is to describe this set and to indicate its subset such that for regularization parameters from this subset the combination of LSQ projection and Tikhonov's method has the same rate of convergence under standard source conditions. The discrepancy set is an analog of the discrepancy curve defined and investigated in [10] for multiple penalty regularization of Tikhonov - type.

In the recently published paper [1] the authors consider a similar approach to a nonlinear ill-posed problem in Banach space where a discretization level and a Tikhonov regularization parameter are chosen by a relaxed version of Morozov's discrepancy principle. However, in such a general setting a discretization set is not analyzed.

Let us take into account the LSQ projection method where $\{X_n\}_{n=1}^{\infty}$, $X_n \subset X$, is a finite dimensional approximation of X and $Y_n := AX_n$. Then $\{Y_n\}_{n=1}^{\infty}$ is an approximation of $\overline{R(A)}$. Let P_n, Q_n denote orthogonal the projectors of X and Y onto X_n and Y_n, respectively. The finite dimensional approximation of (1.1) by the LSQ projection has then the form

$$A_n u_n^{\delta} = f_n^{\delta}, \tag{1.3}$$

where

$$A_n := Q_n A|_{X_n} = A|_{X_n} \text{ and } \mathrm{f}_{\mathrm{n}}^{\delta} = \mathrm{Q}_{\mathrm{n}}\mathrm{f}^{\delta}.$$

It is known that u_n^{δ} does not have to be an approximation of u.

The Tikhonov regularization method applied to (1.3) consists in minimization of the Tikhonov functional

$$F^{\delta}_{n,\alpha}(v) = \|A_n v - f^{\delta}_n\|^2 + \alpha\|v\|^2$$

over X_n, i.e.

$$u^{\delta}_{n,\alpha} = \arg\min_{v\in X_n} F^{\delta}_{n,\alpha}(v). \tag{1.4}$$

We omit the index δ if the noisy data f^{δ} are replaced by the exact data f. The regularized solution $u^{\delta}_{n,\alpha}$ satisfies the equation

$$(A_n^* A_n + \alpha I_n)\, u^{\delta}_{n,\alpha} = A_n^* Q_n f^{\delta}$$

while $u_{n,\alpha}$ satisfies

$$(A_n^* A_n + \alpha I_n)\, u_{n,\alpha} = A_n^* Q_n f.$$

Thus the regularized solution $u^{\delta}_{n,\alpha}$ depends on two parameters, n and α.

In this paper, the following set of parameters will be investigated:

Definition 1. Let C be a fixed constant greater than 1. For any δ and f^{δ} satisfying (1.2) define

$$DC(\delta, f^{\delta}) := \left\{(n,\alpha) : n \in \mathbb{N}, \alpha \in \mathbb{R}^+, n = \min\{k : \|Au^{\delta}_{k,\alpha} - f^{\delta}\| \le C\delta\}\right\}. \tag{1.5}$$

For simplicity this set will be further denoted by $DC(\delta)$ and called the *discrepancy set.*

In [16] a subset of $DC(\delta)$ denoted by $DS(\delta)$

$$DS(\delta) := \{(n,\alpha) : n \in \mathbb{N}, \alpha(n) \in \mathbb{R}^+, \|Au^{\delta}_{n,\alpha(n)} - f^{\delta}\| = C\delta\} \tag{1.6}$$

was introduced and analysed. The set $DS(\delta)$ was there described in detail for the discretization given by the truncated SVD. For the LSQ method, the desired error bound was proved for parameter pairs from $DS(\delta)$, provided n is sufficiently large.

In the present paper we answer the question what is the error rate for parameters n and $\alpha \in (\alpha(n-1), \alpha(n))$, i.e. for pairs of parameters belonging to $DC(\delta) \setminus DS(\delta)$. In order to investigate the convergence rate we will assume (as in [16]) that the exact solution u satisfies the standard source condition of the form

$$u \in X_{\mu,\rho} := \{v \in X : v = (A^*A)^{\mu} w \text{ and } \|w\| \le \rho\}. \tag{1.7}$$

This work is organized as follows: In Section 2 properties of $DC(\delta)$ are investigated. A certain additional assumption on α is introduced in order to prove an auxiliary lemma which plays an important role in proving the desired order of convergence. In Section 3 an error bound is derived. It is shown that for pairs of regularization parameters from $DC(\delta)$ satisfying this additional assumption, the combination of the LSQ projection and Tikhonov's method has the same order of accuracy as Tikhonov's regularization with the standard discrepancy principle and without discretization.

2 Discrepancy set

Let $\delta > 0$ be fixed and

$$\phi(n,\alpha) := \|Au^\delta_{n,\alpha} - f^\delta\|^2.$$

Let us observe that for every $n > 0$, $\phi(n,\alpha)$ is increasing with respect to α. To see this, let us take the singular system $\{\mu_{n,j}, \varphi_{n,j}, \psi_{n,j}\}$ of A_n. Then

$$\phi(n,\alpha) = \sum_{j=1}^{n} \left(\frac{\alpha}{\mu^2_{n,j} + \alpha}\right)^2 (f^\delta, \psi_{n,j})^2 + \|(I - Q_n)f^\delta\|^2.$$

So $\phi(n,\alpha)$ is increasing, since $\frac{\alpha}{\mu^2_{n,j}+\alpha}$ is increasing with respect to $\alpha \in \mathbb{R}^+$. This fact will be exploited in the next two lemmas:

Lemma 1. *Let*

$$n_0(\delta) := \min\{n : \|(I - Q_n)f^\delta\| \le \delta\}.$$

If $C > 1$ in (1.6), then for every $n \ge n_0(\delta)$ there exists a unique $\alpha = \alpha(n)$ such that

$$\|Au^\delta_{n,\alpha(n)} - f^\delta\| = C\delta, \tag{2.1}$$

i.e. $(n, \alpha(n)) \in DS(\delta)$.

Proof. For every $n \in \mathbb{N}$ there exists $u^\delta_n \in X_n$: $Au^\delta_n = Q_n f^\delta$. Thus

$$\phi(n, 0) = \|Au^\delta_n - f^\delta\|^2 = \|(I - Q_n)f^\delta\|^2 \le \delta^2$$

for $n \ge n_0(\delta)$. On the other hand, existence of $\widetilde{\alpha}(n)$ such that $\|Au^\delta_{n,\widetilde{\alpha}(n)} - f^\delta\| = \sqrt{2}C$ follows from the existence of a unique solution of the standard discrepancy principle for A_n: $\|A_n u^\delta_{n,\alpha} - Q_n f^\delta\| = \gamma\delta$ with $\gamma > 1$ (see [3], Thm. 3.3.1). This means that

$$\phi(n, \widetilde{\alpha}(n)) = 2C^2\delta^2.$$

Since $\phi(n,\alpha)$ is increasing with respect to α, there exists $\alpha(n) \in (0, \widetilde{\alpha}(n))$ such that (2.1) holds for $n \ge n_0(\delta)$. □

First, we collect some properties of $DS(\delta)$ in the case of A_n related to the LSQ method, which were not proved in [16].

Lemma 2. *If $C > 1$ in (1.6) and $n > n_0(\delta)$ then $\alpha(n-1) < \alpha(n)$.*

Proof. Let $(n, \alpha(n)) \in DS(\delta)$ and let $\widetilde{\alpha} < \alpha(n)$ be such that

$$n = \min\{k : \ \phi(k, \widetilde{\alpha}) \le (C\delta)^2\}.$$

This means that

$$\phi\,(n - 1, \widetilde{\alpha}) > (C\delta)^2.$$

On the other hand, from Lemma 1 it follows that there exists $\alpha(n-1)$ such that

$$\phi(n - 1, \alpha(n - 1)) = (C\delta)^2.$$

Since $\phi(n-1,\alpha)$ is increasing, $\alpha(n-1) < \widetilde{\alpha} < \alpha(n)$, which ends the proof. □

Similarly to the case when the truncated SVD is chosen for discretization (see [16] Lemma 3.3), (2.1) implies the following estimation for A_n related to the LSQ method and the exact data:

Lemma 3. *If* $\|f - f^\delta\| \le \delta$ *and* $(n, \alpha(n)) \in DS(\delta)$, *then*

$$(C-1)\delta \le \|Au_{n,\alpha(n)} - f\| \le (C+1)\delta.$$

Proof. We have

$$Au_{n,\alpha} - f = \left[Au^\delta_{n,\alpha} - f^\delta\right] + \left[A(u_{n,\alpha} - u^\delta_{n,\alpha}) - (f - f^\delta)\right]. \tag{2.2}$$

For $\alpha = \alpha(n)$ the norm of the first term on the right hand side is equal to $C\delta$. Using the singular system $\{\mu_{n,j}, \varphi_{n,j}, \psi_{n,j}\}$ of the operator A_n we have

$$A(u_{n,\alpha} - u^\delta_{n,\alpha}) - (f - f^\delta) = \sum_{j=1}^{n} \frac{-\alpha}{\mu^2_{n,j} + \alpha}(f_{n,j} - f^\delta_{n,j})\psi_{n,j} - (I - Q_n)(f - f^\delta),$$

where $f_{n,j} = (f, \psi_{n,j})$ and $f^\delta_{n,j} = (f^\delta, \psi_{n,j})$. Since $\frac{\alpha}{\mu^2_{n,j}+\alpha} < 1$, the norm of the second term on the right hand side of (2.2) is bounded by δ. The triangle inequality applied to (2.2) ends the proof. □

Now, let us consider the new set $DC(\delta)$ introduced by (1.5). We have

$$DS(\delta) \subset DC(\delta).$$

Basing on Definition 1, let us denote

$$n(\alpha) := \min\{n : \ \|Au^\delta_{n,\alpha} - f^\delta\| \le C\delta\},$$

where $u^\delta_{n,\alpha}$ is given by (1.4). Of course $n(\alpha(n)) = n$. A similar estimation to that in Lemma 3 holds for parameters $(n(\alpha), \alpha)$ from $DC(\delta)$ but under some additional assumption on α.

Lemma 4. *Let* $\|f - f^\delta\| \le \delta$ *and* $(n(\alpha), \alpha) \in DC(\delta)$. *If* α *is such that*

$$\tilde{C}\delta \le \|Au^\delta_{n(\alpha),\alpha} - f^\delta\| \tag{2.3}$$

with $1 < \tilde{C} < C$, *then*

$$(\tilde{C} - 1)\delta \le \|Au_{n(\alpha),\alpha} - f\| \le (C+1)\delta.$$

Proof. Let us consider the equality (2.2) for $(n(\alpha), \alpha)$. Then the first term on the right hand side of (2.2) is bounded from above, and from below according to (2.3). The second term is bounded by δ (see the proof of Lemma 3). The triangle inequality applied to (2.2) ends the proof. □

Note that in general, (2.3) does not necessarily occur for all values $\alpha \in (\alpha(n-1), \alpha(n))$. However, for any $\tilde{C} < C$ there exists a neighborhood $B_{\tilde{C}}$ of $\alpha(n)$ such that (2.3) holds for $\alpha \in (\alpha(n-1), \alpha(n)] \cap B_{\tilde{C}}$.

Let $\tilde{\alpha}(n)$ be such that

$$\|Au^\delta_{n,\tilde{\alpha}(n)} - f^\delta\| = \tilde{C}\delta.$$

If $\tilde{C} > 1$ then $\tilde{\alpha}(n)$ exists according to Lemma 1. Since $\|Au_{n,\alpha} - f^\delta\|$ is increasing with respect to α (see Lemma 2), $\tilde{\alpha}(n) < \alpha(n)$. If $\tilde{\alpha}(n) \le \alpha(n-1)$, then

$$\forall \alpha \in (\alpha(n-1), \alpha(n)] \quad \tilde{C}\delta \le \|Au^\delta_{n(\alpha),\alpha} - f^\delta\|.$$

However, if $\tilde{\alpha}(n) > \alpha(n-1)$, then this inequality holds only for $\alpha \in (\tilde{\alpha}(n), \alpha(n)]$.

2.1 Example

Let us consider the truncated singular value decomposition (TSVD) as the exceptional projection method which is simultaneously the LSQ and the dual LSQ method. Let A be compact and let $\{\mu_j, \varphi_j, \psi_j\}_{j=1}^\infty$ be the singular system for A where $\mu_1 \ge \mu_2 \ge \cdots$, $A^*A\varphi_j = \mu_j^2\varphi_j$, $A\varphi_j = \mu_j\psi_j$ and φ_j, ψ_j are normalized. Let

$$X_n := span\{\varphi_1, \ldots, \varphi_n\}; \quad \text{then } Y_n := span\{\psi_1, \ldots, \psi_n\} = AX_n.$$

In this case $u^\delta_{n,\alpha}$ and u^δ_n have the forms

$$u^\delta_{n,\alpha} = \sum_{j=1}^n \frac{\mu_j}{\mu_j^2 + \alpha} f^\delta_j \varphi_j \quad \text{and} \quad u^\delta_n = \sum_{j=1}^n \frac{1}{\mu_j} f^\delta_j \varphi_j,$$

where $f^\delta_j = (f^\delta, \psi_j)$, while the solution of the Tikhonov regularization (without discretization) has the form

$$u^\delta_\alpha = \sum_{j=1}^\infty \frac{\mu_j}{\mu_j^2 + \alpha} f^\delta_j \varphi_j.$$

Let $\alpha(\delta)$ be such that $\|Au^\delta_{\alpha(\delta)} - f^\delta\| = C\delta$.

It was proved in [16] that in the case $C > 1$ for every $n \ge \min\{n \in \mathbb{N} : \|Au^\delta_n - f^\delta\| \le C\delta\}$ there exists a unique $\alpha = \alpha(n)$ such that $(n, \alpha(n)) \in DS(\delta)$. Moreover, $\alpha(n-1) < \alpha(n)$ and $\alpha(n) \to \alpha(\delta)$ as $n \to \infty$. In order to verify the assumption (2.3) for $(n(\alpha), \alpha) \in DC(\delta)$ let us see that $\|Au^\delta_{n,\alpha} - f^\delta\|$ is increasing with respect to α, and

$$\|Au^\delta_{n,\alpha(n-1)} - f^\delta\|^2 = \sum_{j=1}^{n-1} \left(\frac{\alpha(n-1)}{\mu_j^2 + \alpha(n-1)}\right)^2 (f^\delta_j)^2 + \left(\frac{\alpha(n-1)}{\mu_n^2 + \alpha(n-1)}\right)^2 (f^\delta_n)^2$$

$$+ \sum_{j=n+1}^\infty (f^\delta_j)^2 = \|Au^\delta_{n-1,\alpha(n-1)}\|^2 + (f^\delta_n)^2 \left[\left(\frac{\alpha(n-1)}{\mu_n^2 + \alpha(n-1)}\right)^2 - 1\right].$$

The first term on the right hand side is equal to $(C\delta)^2$ by (1.6). Since

$$(f_n^\delta)^2 \le \|(I - Q_{n-1})f^\delta\|^2$$

and for $n > n_0(\delta)$, $\|(I - Q_{n-1})f^\delta\| \le \delta$, we have

$$\|Au^\delta_{n,\alpha(n-1)} - f^\delta\|^2 \ge (C^2 - 1)\delta^2.$$

Therefore, for every $\alpha \in (\alpha(n-1), \alpha(n))$ and $n > n_0(\delta)$,

$$\sqrt{C^2 - 1}\delta \le \|Au^\delta_{n,\alpha} - f^\delta\| \le C\delta. \tag{2.4}$$

This holds for any fixed $C > 1$. If $C > \sqrt{2}$ then $\tilde{C} = \sqrt{C^2 - 1} > 1$ and Lemma 4 holds for every $\alpha \in (\alpha(n_0(\delta)), \alpha(\delta))$.

This result can also be obtained in the case when $\{x_i\}_{i=1}^\infty$ is any orthogonal basis of X and $X_n = span\{x_1, \cdots, x_n\}$.

3 Order of convergence

In [16] (Theorem 4.4) it was proved that if the exact solution u satisfies the source condition (1.7) then for $(n, \alpha(n)) \in DS(\delta)$ (see (1.6)) and for n sufficiently large we have

$$\|u^\delta_{n,\alpha(n)} - u\| \le C_1\delta^{\frac{2\mu}{2\mu+1}}, \quad \text{for } \mu \le 1/2, \tag{3.1}$$

$$\|u^\delta_{n,\alpha(n)} - u\| \le C_2\sqrt{\delta}, \quad \text{for } \mu > 1/2, \tag{3.2}$$

where the constants C_1, C_2 depend on μ, ρ, C and $\|A\|$.

Now, we are going to estimate $\|u^\delta_{n(\alpha),\alpha} - u\|$ in the case of parameters $(n(\alpha), \alpha)$ belonging to the set $DC(\delta)$ (1.5). The estimations (3.1) and (3.2) are proved in [16] by using the inequality $\|u^\delta_{n,\alpha(n)}\| \le \|u\|$, which holds for $C \ge 2$ and $(n, \alpha(n)) \in DS(\delta)$ when n is sufficiently large.

However, the inequality $\|u^\delta_{n(\alpha),\alpha}\| \le \|u\|$ may not occur for some

$$\alpha(n-1) < \alpha < \alpha(n).$$

For a fixed n, in terms of the singular system $\{\mu_{n,j}, \varphi_{n,j}, \psi_{n,j}\}$ of A_n, $\|u^\delta_{n,\alpha}\|$ has the form

$$\|u^\delta_{n,\alpha}\|^2 = \sum_{j=1}^{n} \left(\frac{\mu_{n,j}}{\mu^2_{n,j} + \alpha} f^\delta_{n,j} \right)^2,$$

from which it follows that $\|u^\delta_{n,\alpha}\|$ is a decreasing function of α. Hence, for $\alpha < \alpha(n)$,

$$\|u^\delta_{n,\alpha}\| > \|u^\delta_{n,\alpha(n)}\|.$$

We are going to show that if α belongs to a sufficiently small left hand neighborhood of $\alpha(n)$ then the desired estimation holds:

Lemma 5. *Let u be the solution of (1.1). Define*

$$m(\delta) := \min\{n : \|A(I - P_n)u\| \le \delta\}.$$

If $(n, \alpha) \in DC(\delta)$ with $C > 2$, $n \ge m(\delta)$ and $\alpha \in (\alpha(n-1), \alpha(n))$ is such that

$$2\delta \le \|Au^\delta_{n,\alpha} - f^\delta\|, \tag{3.3}$$

then

$$\|u^\delta_{n,\alpha}\| \le \|u\|.$$

Proof. Since $u^\delta_{n,\alpha}$ is the minimizer of the Tikhonov functional $F_\alpha(z)$ over X_n,

$$F_\alpha(u^\delta_{n,\alpha}) \le F_\alpha(P_n u) = \|AP_n u - f^\delta\|^2 + \alpha\|P_n u\|^2.$$

If $(n(\alpha), \alpha) \in DC(\delta)$ and (3.3) is satisfied, then

$$(2\delta)^2 + \alpha\|u^\delta_{n(\alpha),\alpha}\|^2 \le \|AP_n u - f^\delta\|^2 + \alpha\|P_n u\|^2.$$

Thus

$$\alpha\|u^\delta_{n(\alpha),\alpha}\|^2 \le \alpha\|u\|^2 + \left[\|AP_{n(\alpha)}u - f^\delta\|^2 - (2\delta)^2\right].$$

If $n(\alpha) \ge m(\delta)$, then the term in the square brackets is nonpositive because

$$\|AP_n u - f^\delta\| \le \|A(I - P_n)u\| + \|f - f^\delta\| \le 2\delta,$$

which ends the proof. □

Remark 1. Let A be compact and let the projection method be generated by subspaces X_n spanned by eigenfunctions of A^*A (see SubSection 2.1).

a) If $C \ge \sqrt{5}$, then according to (2.4), (3.3) holds for any $\alpha \in (\alpha(n-1), \alpha(n))$.

b) If $u \in X_{\mu,\rho}$, then

$$\sum_{j=1}^{\infty} \frac{1}{\mu_j^{2(1+2\mu)}} f_j^2 \le \rho^2,$$

where $f_j = (f, \psi_j)$ and thus

$$\|A(I - P_n)u\|^2 = \sum_{n+1}^{\infty} f_j^2 \le \rho^2 \left(\mu_{n+1}^{1+2\mu}\right)^2.$$

So, $m(\delta) \le \min\{n : \rho\mu_{n+1}^{1+2\mu} \le \delta\}$.

Theorem 1. *Let $u \in X_{\mu,\rho}$ and $(n, \alpha) \in DC(\delta)$ with $C > 2$. If $n \ge m(\delta)$ and α satisfies (3.3), then*

$$\|u^\delta_{n,\alpha} - u\| \le c_1 \delta^{\frac{2\mu}{2\mu+1}}, \quad \text{for } \mu \le 1/2, \tag{3.4}$$

where $c_1 = \left(2\rho(C+1)^{2\mu}\right)^{\frac{1}{2\mu+1}}$ and

$$\|u^\delta_{n,\alpha} - u\| \le c_2\sqrt{\delta}, \quad \text{for } \mu > 1/2, \tag{3.5}$$

where $c_2 = \left(2\rho\|A\|^{2\mu-1}(C+1)\right)^{\frac{1}{2}}$.

Proof. The idea is the same as in the proof of Theorem 4.4 in [16]. By Lemma 5 we have

$$\|u^\delta_{n(\alpha),\alpha} - u\|^2 = \|u^\delta_{n(\alpha),\alpha}\|^2 - 2(u^\delta_{n(\alpha),\alpha}, u) + \|u\|^2$$

$$\leq 2(\|u\|^2 - (u^\delta_{n(\alpha),\alpha}, u)) = 2(u - u^\delta_{n(\alpha),\alpha}, u).$$

By the assumption $u = (A^*A)^\mu w$ with $\|w\| \leq \rho$. Thus

$$\|u^\delta_{n(\alpha),\alpha} - u\|^2 \leq 2\rho\|((A^*A)^{\frac{1}{2}})^{2\mu}(u - u^\delta_{n(\alpha),\alpha})\|. \tag{3.6}$$

Let B be a bounded selfadjoint operator in X and let P_λ be its spectral family. Then

$$\|B^\tau x\|^2 = \int_0^\infty \lambda^{2\tau} d(P_\lambda x, x).$$

Since $(P_\lambda x, x)$ is a nonnegative measure, we can apply the Hölder inequality to this integral with $p = \frac{1}{\tau}$, $q = \frac{1}{1-\tau}$ and $0 < \tau \leq 1$ to obtain

$$\|B^\tau x\|^2 \leq \left(\int_0^\infty \lambda^{2\tau p} d(P_\lambda x, x)\right)^{\frac{1}{p}} \left(\int_0^\infty 1^q d(P_\lambda x, x)\right)^{\frac{1}{q}} = \|Bx\|^{2\tau}\|x\|^{2(1-\tau)}.$$

Using this inequality for $B = (A^*A)^{\frac{1}{2}}$ and $\tau = 2\mu$ we get from (3.6)

$$\|u^\delta_{n,\alpha} - u\|^2 \leq 2\rho\|(A^*A)^{\frac{1}{2}}(u - u^\delta_{n,\alpha})\|^{2\mu}\|u - u^\delta_{n,\alpha}\|^{1-2\mu}. \tag{3.7}$$

For every $w \in X$ we have

$$\|(A^*A)^{1/2}w\|^2 = ((A^*A)^{1/2}w, (A^*A)^{1/2}w) = (A^*Aw, w) = \|Aw\|^2,$$

therefore, for $w = u - u_{n(\alpha),\alpha}$ it holds

$$\|(A^*A)^{\frac{1}{2}}(u - u^\delta_{n(\alpha),\alpha})\| = \|(A^*A)^{-\frac{1}{2}}A^*A(u - u^\delta_{n(\alpha),\alpha})\| \leq \|Au^\delta_{n(\alpha),\alpha} - f\|.$$

Moreover, if $(n(\alpha), \alpha) \in DC(\delta)$, then

$$\|Au^\delta_{n(\alpha),\alpha} - f\| \leq C\delta + \|f - f^\delta\| \leq (C+1)\delta.$$

Combining these inequalities with (3.7) we see that

$$\|u^\delta_{n(\alpha),\alpha} - u\|^{1+2\mu} \leq 2\rho(C+1)^{2\mu}\delta^{2\mu},$$

which establishes (3.4). If $u = (A^*A)^\mu w$ for $\mu > \frac{1}{2}$ and $\|w\| \leq \rho$, then $u = (A^*A)^{\frac{1}{2}}v$, where

$$\|v\| = \|(A^*A)^{\mu-\frac{1}{2}}w\| \leq \rho\|A\|^{2\mu-1} =: \widetilde{\rho}.$$

Thus $u \in X_{\frac{1}{2},\widetilde{\rho}}$ and (3.5) follows from (3.4). □

Conclusions

In [16] the set $DS(\delta)$ of pairs $(n, \alpha(n))$ of regularization parameters determined by Morozov's type discrepancy principle was introduced for Tikhonov's regularization applied to the LSQ projection of a linear ill-posed problem. It was proved there that under the standard source condition $u = (A^*A)^{\mu}v$ with $\|v\| \leq \rho$ and for arbitrary choice of $(n, \alpha(n)) \in DS(\delta)$, if $\mu \leq \frac{1}{2}$, then the convergence is of optimal order. If $\mu > \frac{1}{2}$, the sub-optimal rate of convergence $O(\sqrt{\delta})$ is only obtained.

In the present paper a greater parameter set $DC(\delta) \supset DS(\delta)$ containing $(n(\alpha), \alpha)$ is considered. The convergence result obtained in Theorem 1 means that the optimal rate of convergence for $\mu \leq \frac{1}{2}$ as well as $O(\sqrt{\delta})$ for $\mu > \frac{1}{2}$ is stable with respect to the choice of Tikhonov's regularization parameter α in a certain left hand neighborhood of $\alpha(n)$ for n sufficiently large. Sometimes, as in the case of discretization given by truncated SVD, the desired order of convergence occurs for all pairs (n, α) where $\alpha \in (\alpha(n-1), \alpha(n)]$ and n is sufficiently large.

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