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A Joint Elliott Type Theorem for Twists of L-Functions of Elliptic Curves

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Abstract. We consider a collection of L-functions of elliptic curves twisted by a Dirichlet character modulo q (q is a prime number), and prove for this collection a joint limit theorem for weakly convergent probability measures in the space of analytic functions as $q \to \infty$. The limit measure is given explicitly.
Keywords: Dirichlet character, elliptic curve, L-function of elliptic curve, probability measure, weak convergence.

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1 Introduction

Let E be an elliptic curve over the field of rational numbers given by the Weierstrass equation

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

with discriminant $\varDelta = -16(4a^3 + 27b^2) \neq 0$. For each prime number p, denote by E_p the reduction modulo p of the curve E which is a curve over the finite field $\mathbb{F}_p$, and define the integer $\lambda(p)$ by the equality

$$|E(\mathbb{F}_p)| = p + 1 - \lambda(p),$$

where $|E(\mathbb{F}_p)|$ is the number of points of curve E_p. The L-function $L_E(s)$, $s = \sigma + it$, of the curve E is defined by the Euler product

$$L_E(s) = \prod_{p|\varDelta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p\nmid\varDelta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1},$$

which, in virtue of the estimate

$$|\lambda(p)| \leqslant 2\sqrt{p}$$

is absolutely convergent for $\sigma > 3/2$. Moreover, the function $L_E(s)$ is analytically continued to an entire function, see, for example, [12].

Now let χ be a Dirichlet character modulo q. Then the twist $L_E(s,\chi)$ of the function $L_E(s)$ is defined, for $\sigma > \frac{3}{2}$, by the Euler product

$$L_E(s,\chi) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta}\left(1 - \frac{\lambda(p)\chi(p)}{p^s} + \frac{\chi^2(p)}{p^{2s-1}}\right)^{-1}$$

and can be expanded in the Dirichlet series

$$\sum_{m=1}^{\infty} \frac{\lambda(m)\chi(m)}{m^s}.$$

In the sequel, we assume that q is a prime number. Then it was observed in [12] that the function $L_E(s,\chi)$, as $L_E(s)$, is also entire one.

Limit theorems for $L_E(s,\chi)$ with increasing q were began to study in [6], [7], [8], [9], however, only in the half-plane of absolute convergence $\sigma > \frac{3}{2}$. In [12], a limit theorem in the space of analytic functions $H(D)$, $D = \{s \in \mathbb{C} : \sigma > 1\}$, for the function $L_E(s,\chi)$ has been obtained. For its statement, we need some notation and definitions. For $Q \geqslant 2$, let

$$M_Q = \sum_{q \leqslant Q} \sum_{\substack{\chi = \chi(\mathrm{mod} q) \\ \chi \neq \chi_0}} 1,$$

where, as usual, χ_0 denotes the principal character modulo q. It is well known that

$$M_Q = \frac{Q^2}{2\log Q} + \mathcal{O}\left(\frac{Q^2}{\log^2 Q}\right).$$

Denote by γ the unite circle $\{s \in \mathbb{C} : |s| = 1\}$, and define $\Omega = \prod_p \gamma_p$, where $\gamma_p = \gamma$ for all primes p. The infinite-dimensional torus Ω with the product topology and operation of pointwise multiplication is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(X)$ is the Borel σ-field of the space X, the probability Haar measure m_H exists, and this gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of an element $\omega \in \Omega$ to the coordinate space γ_p, and, on $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $L_E(s,\omega)$ by the formula

$$L_E(s,\omega) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)\omega(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta}\left(1 - \frac{\lambda(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}}\right)^{-1}.$$

Let P_{L_E} be the distribution of $L_E(s,\omega)$, i. e.,

$$P_{L_E}(A) = m_H\{\omega \in \Omega : L_E(s,\omega) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Then the main result of [12] is the following limit theorem.

Theorem 1. *Suppose that $Q \to \infty$. Then*

$$\frac{1}{M_Q}\#\{\chi(\text{mod} q), q \leqslant Q, \chi \neq \chi_0 : L_E(s,\chi) \in A\}, \quad A \in \mathcal{B}(H(D))$$

converges weakly to P_{L_E}.

Here $\#A$ stands for the cardinality of the set A.

In [8], a joint limit theorem of type of Theorem 1 has been obtained for a collection of moduli of twists of L-functions of elliptic curves, however, in the region $\sigma > \frac{3}{2}$, only. We note that P. D. T. A. Elliott was the first who began to study limit theorems with increasing modulus for Dirichlet L-functions [4], [5].

The aim of this paper is a multidimensional analogue of Theorem 1. For $j = 1, ..., r$, let E_j be an elliptic curve over the field of rational numbers given by the equation

$$y^2 = x^3 + a_j x + b_j, \quad a_j, b_j \in \mathbb{Z}$$

with discriminant $\varDelta_j = -16(4a_j^3 + 27b_j^2) \neq 0$. Consider the corresponding L-function

$$L_{E_j}(s) = \prod_{p|\varDelta_j}\left(1 - \frac{\lambda_j(p)}{p^s}\right)^{-1} \prod_{p\nmid\varDelta_j}\left(1 - \frac{\lambda_j(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}.$$

Suppose that N_j is the conductor of the curve E_j. Then, by the Weil-Shimura-Taniyama conjecture proved in [3], see also Theorem 14.6 of [10], the function $L_{E_j}(s)$ coincides with L-function of a new cusp form of weight 2 and level N_j. This shows that $L_{E_j}(s)$ is an entire function.

By Theorem 14.20 of [10], the twist $L_{E_j}(s,\chi)$ of $L_{E_j}(s)$ with a character χ modulo q is again a new cusp form of weight 2 and level $N_j q^2$. Therefore, the function $L_{E_j}(s,\chi)$ is also an entire function.

Let, for brevity, $\underline{E} = (E_1, ..., E_r)$, $\underline{L}_{\underline{E}}(s,\chi) = (L_{E_1}(s,\chi), ..., L_{E_r}(s,\chi))$ and $A_Q = \{\chi(\text{mod} q) : q \leqslant Q,\ \chi \neq \chi_0\}$. Moreover, define

$$\underline{L}_{\underline{E}}(s,\omega) = (L_{E_1}(s,\omega), ..., L_{E_r}(s,\omega)),$$

where, for $j = 1, ..., r$,

$$L_{E_j}(s,\omega) = \prod_{p|\varDelta_j}\left(1 - \frac{\lambda_j(p)\omega(p)}{p^s}\right)^{-1} \prod_{p\nmid\varDelta_j}\left(1 - \frac{\lambda_j(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}}\right)^{-1}.$$

Denote by $P_{\underline{L}_{\underline{E}}}$ the distribution of the $H^r(D)$-valued random element $\underline{L}_{\underline{E}}(s,\omega)$, i. e.,

$$P_{\underline{L}_{\underline{E}}}(A) = m_H\left\{\omega \in \varOmega : \underline{L}_{\underline{E}}(s,\omega) \in A\right\}, \quad A \in \mathcal{B}(H^r(D)).$$

Then we have the following statement.

Theorem 2. *Suppose that $Q \to \infty$. Then*

$$P_{Q,\underline{E}}(A) \overset{\text{def}}{=} \frac{1}{M_Q}\#\{\chi \in A_Q : \underline{L}_{\underline{E}}(s,\chi) \in A\}, \quad A \in \mathcal{B}(H^r(D))$$

converges weakly to $P_{\underline{L}_{\underline{E}}}$.

2 Auxiliary results

We start with a joint limit theorem which is a generalization of Lemma 1 from [12]. We extend the function $\omega(p)$ to the set $\mathbb{N}$ by the formula

$$\omega(m) = \sum_{p^l | m,\, p^{l+1} \nmid m} \omega^l(p), \quad m \in \mathbb{N}.$$

Lemma 1. *For $j = 1, ..., r$, let $\{a_{mj} : m \in \mathbb{N}\}$ be a sequence of complex numbers such that*

$$\sum_{m \leqslant n} |a_{mj}|^2 = \mathcal{O}\left(n^{2\alpha}\right), \quad \alpha > 0$$

as $n \to \infty$, and, for $\omega \in \Omega$ and $\sigma > \alpha + \frac{1}{2}$, let $X_j(s,\omega) = \sum_{m=1}^{\infty} \frac{a_{mj}\omega(m)}{m^s}$. Suppose that $\{A_m : m \in \mathbb{N}\}$ is a sequence of finite subsets of the torus Ω such that, for each $\omega \in \bigcup_{m=1}^{\infty} A_m$, $X_j(s,\omega)$ has an analytic continuation to the half plane $D_\alpha = \{s \in \mathbb{C} : \sigma > \alpha\}$ satisfying the following conditions:

1^0 *As $|t| \to \infty$,*

$$\frac{1}{\# A_m} \sum_{\omega \in A_m} |X_j(\sigma + it, \omega)|^2 = \mathcal{O}\left(|t|^A\right), \quad A > 0$$

uniformly for $m \in \mathbb{N}$ and σ in compact subsets of the interval (α, ∞);

2^0

$$\sum_{\omega \in A_m} |X_j(\sigma, \omega)|^2 = \mathcal{O}(\# A_m)$$

as $m \to \infty$, uniformly for s on compact subsets of D_α;

Moreover, suppose that

$$\frac{\#\{A \cap A_m\}}{\# A_m}, \quad A \in \mathcal{B}(\Omega)$$

converges weakly to the Haar measure m_H. Then

$$\frac{1}{\# A_m} \# \left\{\omega \in A_m : (X_1(s,\omega), ..., X_r(s,\omega)) \in A\right\},$$

where $A \in H^r(D_\alpha)$, converges weakly to the distribution of the random element $(X_1(s,\omega), ..., X_r(s,\omega))$ as $m \to \infty$.

Proof. A way of the proof is completely analogical to that in one-dimensional case presented in [2], Proposition 4.4.1. In our case, the metric in $H^r(D_\alpha)$ inducing its topology of uniform convergence on compacta is applied, and the joint case is reduced to the one-dimensional case. □

The next lemma is devoted to checking the hypotheses of Lemma 1, and contains an approximate functional equation of L-functions of cusp forms of weight 2 and level N. Let $F(z)$ be a new form of weight 2 and level N with

Fourier coefficients $c(m)$. Moreover, let $\Gamma(s)$, as usual, denote the gamma-function, and let $\Gamma(s,z)$ be the incomplete gamma-function,

$$\Gamma(s,z)=\int_z^\infty \mathrm{e}^{-t}t^{s-1}\mathrm{d}t, \quad \sigma>0,\ z\in\mathbb{R}.$$

Lemma 2. [1]. *Suppose that $L(s,F)$, $s=\sigma+it$, is the L-function associated to the form F, $\frac12\leqslant\sigma\leqslant\frac32$, $M>\frac{t\sqrt N}{4}$, $r=\mathrm{e}^{i(\frac{\pi}{2}-\delta(t))}$ with $0<\delta(t)\leqslant\frac{\pi}{2}$. Then*

$$\begin{aligned}L(s,F)={}&\frac{1}{\Gamma(s)}\sum_{m\leqslant M}\frac{c(m)}{m^s}\Gamma\left(s,\frac{2\pi mr}{\sqrt N}\right)\\&-\frac{\mu N^{1-s}(2\pi)^{2(s-1)}}{\Gamma(s)}\sum_{m\leqslant M}\frac{c(m)}{m^{2-s}}\Gamma\left(2-s,\frac{2\pi m}{\sqrt N r}\right)+\frac{(2\pi)^s}{\Gamma(s)}R,\end{aligned}$$

where

$$\begin{aligned}|R|<{}&\mathrm{e}^{-\frac{\pi t}{2}}\mathrm{e}^{\delta(t)\left(t-\frac{4M}{\sqrt N}\right)}N^{\frac{1-\sigma}{2}}\sqrt M\delta^{-1}(t)\\&\times\left(1+\frac{\log M+\sigma+1}{2t\delta(t)}+\frac{(\sigma-1)(\log M+2)}{4(t\delta(t))^2}\right).\end{aligned}$$

Let G be a compact Abelian group. Then, on $(G,\mathcal{B}(G))$, the probability Haar measure μ can be defined. We recall that a sequence $\{x_m: m\in\mathbb{N}\}\subset G$ is said to be uniformly distributed if

$$\lim_{n\to\infty}\frac1n\sum_{m=1}^n f(x_m)=\int_G f\mathrm{d}\mu$$

for any real bounded Borel measurable function f.

The next lemma is a criterion of uniform distribution for sequences in G.

Lemma 3. *The sequence $\{x_m: m\in\mathbb{N}\}\subset G$ is uniformly distributed in G if and only if, for any nontrivial character χ_G, the equality*

$$\lim_{n\to\infty}\frac1n\sum_{m=1}^n \chi_G(x_m)=0$$

holds.

The proof of the lemma can be found in [11], Chapter 4, Corollary 1.2.

3 Proof of Theorem 2

In the notation of Lemma 1, we have that $a_{mj}=\lambda_j(m)\chi(m)$. Since $\lambda_j(m)$ coincides with Fourier coefficients of a new cusp form, we have, by the estimate (14.53) from [10], that

$$\sum_{m\leqslant n}|a_{mj}|^2=\mathrm{O}(n^2).$$

Therefore, $\alpha = 1$ in Lemma 1.

Denote by $\mathbb{P}$ the set of all prime numbers. Let χ be a Dirichlet character modulo $q \in \mathbb{P}$, and

$$\hat{\chi}(p) = \begin{cases} \chi(p), & \text{if } p \in \mathbb{P} \setminus \{q\}, \\ 1, & \text{if } p = q. \end{cases}$$

Then $\{\hat{\chi}(p) : p \in \mathbb{P}\}$ is an element of the torus Ω. Putting

$$l_{E_j,q}(s) = \begin{cases} 1 - \frac{\lambda_j(q)}{q^s}, & \text{if } q \mid \Delta, \\ 1 - \frac{\lambda_j(q)}{q^s} + \frac{1}{q^{2s-1}}, & \text{if } q \nmid \Delta, \end{cases}$$

we have that

$$L_{E_j}(s,\chi) = l_{E_j,q}(s) L_{E_j}(s,\hat{\chi}). \tag{3.1}$$

Denote by p_m the m th prime number, and define $A_m = \{\chi(\text{mod} p_m) : \chi \neq \chi_0\}$. Obviously, $\#A_m = p_m - 2$. Define one more set $\hat{A}_m = \{\hat{\chi} : \chi \in A_m\}$. On a certain probability space $(\hat{\Omega}, \mathcal{A}, P)$, define the $H^r(D)$-valued random elements $\underline{X}_m(s)$ and $\underline{\hat{X}}_m(s)$ by the formulae

$$P\left(\underline{X}_m(s) = L_{\underline{E}}(s,\chi)\right) = \frac{1}{p_m - 2}, \quad \chi \in A_m,$$
$$P\left(\underline{\hat{X}}_m(s) = L_{\underline{E}}(s,\hat{\chi})\right) = \frac{1}{p_m - 2}, \quad \hat{\chi} \in \hat{A}_m.$$

By the definition of $l_{E_j,p_m}(s)$, we see that

$$l_{E_j,p_m}(s) \to 1 \tag{3.2}$$

in the space $H(D)$ as $m \to \infty$. Hence, $\underline{X}_m(s)$ converges in distribution to $\hat{X}_m$ as $m \to \infty$. Therefore, for the proof that

$$Q_q(A) \stackrel{\text{def}}{=} \frac{1}{q-2} \#\{\chi(\text{mod} q), \chi \neq \chi_0 : \underline{L}(s,\chi) \in A\}, \quad A \in \mathcal{B}(H^r(D))$$

converges weakly to $P_{\underline{L}}$ as $q \to \infty$ it suffices to obtain that the random element $\underline{\hat{X}}_m$ converges in distribution to $P_{\underline{L}_{\underline{E}}}$. Thus, the sequence $\{\hat{A}_m : m \in \mathbb{N}\}$ corresponds the sequence $\{A_m : m \in \mathbb{N}\}$ in Lemma 1, and $\underline{X}(s,\omega) = \underline{L}_{\underline{E}}(s,\hat{\chi})$. Moreover, in virtue of (3.1) and (3.2), $\underline{L}_{\underline{E}}(s,\hat{\chi})$ can be replaced by $\underline{L}_{\underline{E}}(s,\chi)$.

It remains to check other hypotheses of Lemma 1. Let K be a compact subset of $(1,\infty)$. We continue with estimate for

$$D_q(\sigma,t) \stackrel{\text{def}}{=} \frac{1}{q-2} \sum_{\chi(\text{mod} q)} |L_{E_j}(\sigma + it, \chi)|^2, \quad t > 0,$$

when q runs prime numbers. For this, we apply Lemma 2 with $M = ct\sqrt{N_j} \times q \log^2 q$, $c > 0$, and $\delta = t^{-1}$. We have

$$D_q(\sigma,t) \ll_K D_{1,q}(\sigma,t) + D_{2,q}(\sigma,t) + D_{3,q}(\sigma,t), \tag{3.3}$$

where

$$D_{1,q}(\sigma,t)=\frac{1}{q}\sum_{\chi(\mathrm{mod}q)}\frac{1}{|\Gamma(s)|^2}\left|\sum_{m\leqslant M}\frac{\lambda(m)\chi(m)}{m^{\sigma+it}}\Gamma\left(\sigma+it,\frac{2\pi mr}{\sqrt{N_j}q}\right)\right|^2$$
$$\ll\frac{1}{|\Gamma(s)|^2}\sum_{m\leqslant M}\frac{|\lambda(m)|^2}{m^{2\sigma}}\left|\Gamma\left(\sigma+it,\frac{2\pi mr}{\sqrt{N_j}q}\right)\right|^2\frac{1}{q}\sum_{\chi(\mathrm{mod}q)}|\chi(m)|^2$$
$$+\frac{1}{|\Gamma(s)|^2}\sum_{m\leqslant M}\sum_{\substack{n\leqslant M\\ m\neq n}}\frac{|\lambda(m)||\lambda(n)|}{m^{\sigma}n^{\sigma}}\left|\Gamma\left(\sigma+it,\frac{2\pi mr}{\sqrt{N_j}q}\right)\right|$$
$$\times\left|\Gamma\left(\sigma+it,\frac{2\pi nr}{\sqrt{N_j}q}\right)\right|\left|\frac{1}{q}\sum_{\chi(\mathrm{mod}q)}\chi(m)\overline{\chi(n)}\right|, \qquad (3.4)$$

$$D_{2,q}(\sigma,t)=\frac{1}{q}\sum_{\chi(\mathrm{mod}q)}\frac{N_j^{2-2\sigma}q^{4-4\sigma}}{|\Gamma(s)|^2}\left|\sum_{m\leqslant M}\frac{\lambda(m)\chi(m)}{m^{2-\sigma-it}}\Gamma\left(2-\sigma-it,\frac{2\pi m}{\sqrt{N_j}qr}\right)\right|^2$$
$$=\frac{N_j^{2-2\sigma}q^{4-4\sigma}}{|\Gamma(s)|^2}\sum_{m\leqslant M}\frac{|\lambda(m)|^2}{m^{4-2\sigma}}\left|\Gamma\left(2-\sigma-it,\frac{2\pi m}{\sqrt{N_j}qr}\right)\right|^2\frac{1}{q}\sum_{\chi(\mathrm{mod}q)}|\chi(m)|^2$$
$$+\frac{N_j^{2-2\sigma}q^{4-4\sigma}}{|\Gamma(s)|^2}\sum_{m\leqslant M}\sum_{\substack{n\leqslant M\\ m\neq n}}\frac{|\lambda(m)||\lambda(n)|}{m^{2-\sigma}n^{2-\sigma}}\left|\Gamma\left(2-\sigma-it,\frac{2\pi m}{\sqrt{N_j}qr}\right)\right|$$
$$\times\left|\Gamma\left(2-\sigma-it,\frac{2\pi n}{\sqrt{N_j}qr}\right)\right|\left|\frac{1}{q}\sum_{\chi(\mathrm{mod}q)}\chi(m)\overline{\chi(n)}\right|, \qquad (3.5)$$

$$D_{3,q}(\sigma,t)=\frac{1}{|\Gamma(s)|^2}R^2. \qquad (3.6)$$

Let $d(m)=\sum_{k|m}1$ be the divisor function. Using the well - known bounds

$$|\lambda(m)|\leqslant\sqrt{m}d(m),\qquad\sum_{m\leqslant x}d^2(m)\ll x\log^4 x$$

as well as [1]

$$|\Gamma(\sigma+it,Ar)|\ll A^{\sigma}\mathrm{e}^{-\left(\frac{\pi}{2}-\delta(t)\right)t},$$

and the properties of the gamma-function, we find that the first term in the right-hand side of (3.4) is estimated as

$$\ll_K q^{2-2\sigma}(\log q)^{c_1}t^{A_1}\ll_K t^{A_1},\ c_1>0,\ A_1>0 \qquad (3.7)$$

uniformly in $\sigma\in K$ and q. Moreover, in view of the equalities

$$\sum_{\chi(\mathrm{mod}q)}\chi(m)\overline{\chi(n)}=\begin{cases}q-1, & \text{if } m\equiv n(\mathrm{mod}q),\\ 0, & \text{if } m\not\equiv n(\mathrm{mod}q)\end{cases} \qquad (3.8)$$

and the bound $d(m) \ll m^{\varepsilon}$, $\varepsilon > 0$, we obtain that the second term in the right-hand side of (3.4) has a bound

$$\ll \sum_{\substack{m\leqslant M \\ m\equiv n (\mathrm{mod} q)}} \sum_{n\leqslant M} \frac{|\lambda(m)||\lambda(n)|}{m^{\sigma}n^{\sigma}} \left(\frac{m}{q}\right)^{\sigma} \left(\frac{n}{q}\right)^{\sigma}$$
$$\ll \frac{1}{q^{2\sigma}} \sum_{k\leqslant M/q} \sum_{m\leqslant M} \sqrt{m} d(m) \sqrt{m+kq}\, d(m+kq)$$
$$\ll q^{2-2\sigma+\varepsilon} t^{A_2} \ll_K t^{A_2}, \quad A_2 > 0$$

uniformly in $\sigma \in K$ and q. This together with (3.7) shows that

$$D_{1,q}(\sigma,t) \ll_K t^{A_3}, \quad A_3 > 0 \tag{3.9}$$

uniformly in $\sigma \in K$ and q.

In a similar way, we obtain that

$$D_{2,q}(\sigma,t) \ll_K t^{A_4}, \quad A_4 > 0 \tag{3.10}$$

uniformly in $\sigma \in K$ and q. The definition of R and the choice of M and $\delta(t)$ imply the estimate

$$D_{3,q}(\sigma,t) \ll_K \mathrm{e}^{8c\log^2 q} q^{1-\sigma} \sqrt{q} (\log q)^{c_3} t^{A_5} \ll_K t^{A_5}, \quad A_5 > 0$$

uniformly in $\sigma \in K$ and q. From this, (3.9), (3.10), and (3.3) we have that

$$D_q(\sigma,t) \ll_K t^A, \quad A > 0$$

uniformly in $\sigma \in K$ and q. Obviously, then

$$\frac{1}{q-2} \sum_{\substack{\chi (\mathrm{mod} q) \\ \chi\neq\chi_0}} \left|L_{E_j}(\sigma+it,\chi)\right|^2 \ll_K |t|^A$$

uniformly in $\sigma \in K$ and q.

In a similar manner, we obtain that

$$\sum_{\substack{\chi (\mathrm{mod} q) \\ \chi\neq\chi_0}} \left|L_{E_j}(\sigma,\chi)\right|^2 = \mathcal{O}(q)$$

as $q \to \infty$, uniformly for s on compact subsets of the half-plane D.

Next we will consider the sequence $\{\hat{A}_m : m \in \mathbb{N}\}$, and will prove that it is uniformly distributed. Let χ_{Ω} be a character of the group Ω. Then it is well known that

$$\chi_{\Omega}(\omega) = \prod_p \omega^{k_p}(p), \quad \omega \in \Omega,$$

where only a finite number of integers k_p are distinct from zero. Hence,

$$\chi_\Omega(\omega) = \omega(m_1)\,\overline{\omega(m_2)}$$

with $(m_1, m_2) = 1$. Then we have

$$\begin{aligned}\frac{1}{\#\hat{A}_m}\sum_{\omega\in\hat{A}_m}\chi_\Omega(\omega) &= \frac{1}{p_m-2}\sum_{\omega\in\hat{A}_m}\omega(m_1)\,\overline{\omega(m_2)}\\ &= \frac{1}{p_m-2}\sum_{\chi(\mathrm{mod}p_m),\,\chi\neq\chi_0}\hat{\chi}(m_1)\overline{\hat{\chi}(m_2)}. \qquad (3.11)\end{aligned}$$

The numbers $m_1, m_2 \in \mathbb{N}$ are fixed. Therefore, for sufficiently large m,

$$p_m \nmid m_1,\ p_m \nmid m_2,\ \text{and } p_m \nmid (m_1 - m_2).$$

Thus, taking into account (3.11) and (3.8), we find that

$$\begin{aligned}\frac{1}{\#\hat{A}_m}\sum_{\omega\in\hat{A}_m}\chi_\Omega(\omega) &= -\frac{1}{p_m-2} + \frac{1}{p_m-2}\sum_{\chi(\mathrm{mod}p_m)}\chi(m_1)\overline{\chi(m_2)}\\ &= -1/(p_m-2) \ \to\ 0\end{aligned}$$

as $m \to \infty$. Therefore, by Lemma 3, the sequence $\{\hat{A}_m : m \in \mathbb{N}\}$ is uniformly distributed. Thus, all hypotheses of Lemma 1 are fulfilled, and we have that

$$\frac{1}{q-2}\#\left\{\chi(\mathrm{mod}q), \chi \neq \chi_0 : \underline{L}(s, \hat{\chi}) \in A\right\}, \quad A \in \mathcal{B}(H^r(D))$$

converges weakly to $P_{\underline{L}_{\underline{E}}}$ as $q \to \infty$, and this is true for Q_q as well.

From the weak convergence of Q_q to $P_{\underline{L}_{\underline{E}}}$ as $q \to \infty$, it follows that of $P_{Q,\underline{E}}$ as $Q \to \infty$, see [12].

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