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# Parametric Complexity Reduction of Discrete-Time Linear Systems Having a Slow Initial Onset or Delay

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**Abstract.** This paper is concerned with an optimal expansion of linear discrete time systems on Meixner functions. Many orthogonal functions have been widely used to reduce the model parameter number such as Laguerre functions, Kautz functions and orthogonal basis functions. However, when the system has a slow initial onset or delay, Meixner functions, which have a slow start, are more suitable in terms of providing a more accurate approximation to the system. The optimal approximation of Meixner model is ensured once the pole characterizing the Meixner functions is set to its optimal value. In this paper, a new recursive representation of Meixner model is proposed. Further we propose, from input/output measurements, an iterative pole optimization algorithm of the Meixner pole functions. The method consists in applying the Newton-Raphson's technique in which their elements are expressed analytically by using the derivative of the Meixner functions. Simulation results show the effectiveness of the proposed optimal modeling method.

**Keywords:** parametric complexity reduction; Meixner functions; pole optimization; recursive representation.

**AMS Subject Classification:** 42C05.

## 1 Introduction

Over the last decades a significant interest is given in developing models for estimation of Single Input Single Output (SISO) linear systems using the orthonormal basis functions. This interest is motivated essentially by the drastic reduction of the model parameter number. Among these functions, we note Laguerre functions [6, 11, 13, 14, 15, 16, 17, 19, 22, 23, 25, 27], Meixner functions [2, 4, 8, 20], Kautz functions [10, 26, 28] and generalized orthogonal bases functions [12]. Both Laguerre functions and Meixner functions are limited to the favorite case when the functions have only one real pole. The Meixner functions are an extension of Laguerre functions and are suitable when the system have a slow start and an exponential decay towards infinity [4, 8].

Some authors are interested for the use of the Meixner-like functions in estimation of Volterra kernels for nonlinear systems [2]. The Meixner functions depend on two parameters, the first is called the order of generalization that determines how fast the functions start and the second is the Meixner pole that appears in the error criterion in a nonlinear way and is usually hard to find. We notice that the problem of optimum choice of the free parameter of some orthogonal functions (Laguerre, Kautz) for the conventional system modeling has already been reported by many researchers [6, 7, 11, 15, 18, 24]. An optimum choice of Meixner pole has been addressed in [4, 9, 20, 24] using the impulse response.

In this paper, we propose a new method for obtaining models of discrete time systems based on Meixner functions. Mathematical background is given starting from the definition of the generalized Laguerre functions. Based on these functions Kulikovskikh et al. [20] have developed an appropriate Meixner functions which enable simpler and easier filter network realization. These filters are applied in finding the best approximation of system behavior in the sense of the Mean Square Error (MSE).

First, using Meixner filters, recursive representation is formed, capable to imitate transfer functions of real systems. Designed method for system modeling is based on setting optimal value for the Meixner functions in order to reduce the parameter number of the approximate Meixner model.

The optimal identification of the Meixner pole is achieved using the system input/output data and by exploiting the minimization of the MSE. For this criterion, the Meixner pole is given via the Newton-Raphson method in which we expressed the gradient and the Hessian analytically by using the derivative of the Meixner functions. This optimization problem ensures a parametric reduction which can be significant when the considered system is linear with a dominant first-order dynamic.

The main contributions of this article are mainly twofold. (1) We propose a new recursive representation for the input/output measurements of the linear discrete-time systems with a reduced parameter complexity model using the Meixner functions. (2) We propose an iterative algorithm via Newton-Raphson technique to optimize the Meixner pole based on analytical expressions of the gradient and the Hessian of the MSE criterion.

This article is organized as follows: in Section 2, we present the discrete time Meixner model. We develop the filter network of the Meixner model as well as its new recursive representation. In Section 3, we propose a method of the Meixner pole optimization by using the Newton-Raphson method. The gradient and the Hessian of which are expressed analytically by using the derivative of the Meixner functions. Section 4 evaluates, through simulation examples, the performances of the proposed optimization algorithm as well as the Meixner representation in term of approximation quality. Finally, some concluding remarks are made in Section 5.

# 2 Discrete-time Meixner model

## 2.1 Preliminaries

For each fixed pole $\sigma \in \varSigma$, where $\varSigma = [\sigma \in \mathbb{R} : \sigma > 0]$ and $a \in [a \in \mathbb{R} : a > -1]$, $n \in \mathbb{N}_0$ in the Hilbert space $L_2(\mathbb{R}^+)$, the generalized Laguerre functions $\lambda_n^{(a)}(\tau,\sigma)$ is given by [5]:

$$\lambda_n^{(a)}(\tau,\sigma) = e^{-\sigma\tau/2}\sqrt{\frac{\sigma\ n!}{\Gamma(n+a+1)}}\sum_{m=0}^{n}(-1)^m \left(\begin{array}{c} n+a \\ n-m \end{array}\right)\frac{(\sigma\tau)^{m+a/2}}{m!} \qquad (2.1)$$

are orthogonal with respect to the nonnegative weight function $\omega(\tau,a) = \tau^a$ over the interval $\tau \in \mathbb{R}^+$ and the norm $\left\|\lambda_n^{(a)}(\tau,\sigma)\right\|^2 = \Gamma(n+a+1)/(n!\ \sigma^{a+1})$, where $\Gamma(\cdot)$ is the Gamma function. After Laplace transform of (2.1), we obtain

$$\varLambda_n^{(a)}(s,\sigma) = \beta^{(a)}\left(\frac{1}{s+\sigma/2}\right)^{1+a/2}\sum_{m=0}^{n}\varPsi_{n,m}^{(a)}\left(\frac{\sigma}{s+\sigma/2}\right)^m.$$

Here $\beta^{(a)} = \sigma^{(1+a)/2}$ and $\varPsi_{n,m}^{(a)}$ is given by

$$\varPsi_{n,m}^{(a)} = \begin{cases} 0, & \text{if } n < m, \\ (-1)^m \left(\begin{array}{c} n \\ m \end{array}\right)\frac{\Gamma(m+a/2+1)}{\Gamma(m+a+1)}\sqrt{\frac{\Gamma(n+a+1)}{n!}}, & \text{if } n \geq m. \end{cases}$$

Therefore, the Laplace transform $\varLambda_n^{(a)}(s,\sigma)$ of $\lambda_n^{(a)}(\tau,\sigma)$ can be defined as [20]

$$\varLambda_n^{(a)}(s,\sigma) = \left(\frac{\sigma}{s+\sigma/2}\right)^{a+1}\left(\frac{s-\sigma/2}{s+\sigma/2}\right)^n. \qquad (2.2)$$

It is obvious that (2.2) differs from the Laplace transform neatly discussed in [3, 5, 8]. The filters $\varLambda_n^{(a)}(s,\xi)$ can be mapped onto a rational Z-transform $M_n^{(a)}(z,\xi)$ with real pole $|\xi| < 1$ using the following modified bilinear transformation [20]:

$$\begin{aligned} \varLambda_n^{(a)}(s,\sigma) &\longmapsto \frac{z}{z+1}\varLambda_n^{(a)}\left(a\frac{z-1}{z+1},\sigma\right) = \frac{(-1)^n 2a}{a+\sigma/2}M_n^{(a)}\left(z,\frac{a-\sigma/2}{a+\sigma/2}\right), \\ M_n^{(a)}(z,\xi) &\longmapsto \frac{2a}{a+s}M_n^{(a)}\left(\frac{a+s}{a-s},\xi\right) = \frac{(-1)^n}{1+\xi}\varLambda_n^{(a)}\left(s,2a\frac{1-\xi}{1+\xi}\right). \qquad (2.3) \end{aligned}$$

Applying (2.3) to (2.2), we can introduce the Meixner filters $M_n^{(a)}(z,\xi)$ as

$$M_n^{(a)}(z,\xi) = \frac{(1-\xi^2)\,z}{z-\xi}\left(\frac{(1-\xi)\,(z+1)}{z-\xi}\right)^a\left(\frac{1-\xi z}{z-\xi}\right)^n. \qquad (2.4)$$

It is interesting to note that, for each given parameter $a \in \{a \in \mathbb{R} : a > -1\}$, (2.2) and, accordingly, (2.4) can be appertained to the class of almost orthogonal systems [1] that are acceptable for the analysis of inaccurate systems consisting of parameters, values of which are not ideally precise.

The Meixner functions given by (2.4) are governed by two parameters: the Meixner pole $\xi$ and the order of generalization $a$. In the special case where $a = 0$ with dividing $M_n^{(a)}(z,\xi)$ by $\sqrt{1-\xi^2}$, the Laguerre functions are obtained.

Due to the completeness of the Meixner sequences, any desired linear discrete time system with specified transfer function $G(z)$ can be represented as

$$G(z) = \frac{Y(z)}{U(z)} = \sum_{n=0}^{\infty} g_n(\xi) M_n^{(a)}(z,\xi) \tag{2.5}$$

with $g_n(\xi)$ are known as Fourier coefficients; $U(z)$ and $Y(z)$ are the $Z$ transform of the system input and the system output respectively.

The relation (2.5) implies that

$$Y(z) = \sum_{n=0}^{\infty} g_n(\xi) M_n^{(a)}(z,\xi) U(z). \tag{2.6}$$

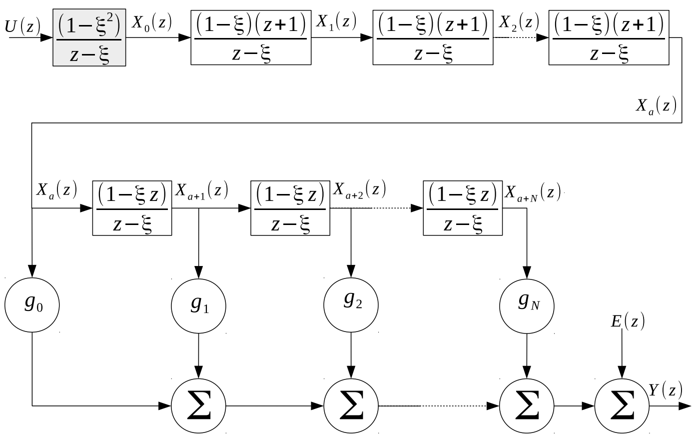



**Figure 1.** Meixner network

## 2.2 Meixner network

In physical applications for filter design, the infinite series (2.6) is truncated with limited number of Meixner stages, e.g. $N + 1$, with tolerating some truncation error $E(z)$.

$$Y(z) = \sum_{n=0}^{N} g_n(\xi) M_n^{(a)}(z,\xi) U(z) + E(z) = \sum_{n=0}^{N} g_n(\xi) X_n^{(a)}(z,\xi) + E(z) \tag{2.7}$$

with $X_n^{(a)}(z,\xi) = M_n^{(a)}(z,\xi)\, U(z)$.

The inverse Z-transform of the relation (2.7) is given by:

$$y(k) = \sum_{n=0}^{N} g_n x_n^{(a)}(k) + e(k),$$

where $x_n^{(a)}(k)$ and $e(k)$ are the Z-inverse transform of $X_n^{(a)}(z,\xi)$ and $E(z)$ respectively.

The filtering scheme given by Figure 1 using Meixner functions can be easily deduced from relation (2.7) in which a unit delay was incorporated in order to represent a strictly causal system.

### 2.3 Recursive representation of Meixner model

From Figure 1 and by considering $x_i(k)$ ($i = 0, \dots, a+N$) as state variables, we obtain the following recursive equations:

$$\begin{cases} x_0(k+1) = \xi\, x_0(k) + V\, u(k), \\ x_1(k+1) = \xi\, x_1(k) + V x_0(k) + (1-\xi) V\, u(k), \\ x_2(k+1) = \xi\, x_2(k) + V x_1(k) + (1-\xi)\, V x_0(k) + (1-\xi)^2 V\, u(k), \\ x_3(k+1) = \xi\, x_3(k) + V x_2(k) + (1-\xi)\, V x_1(k) + (1-\xi)^2\, V x_0(k) \\ \qquad +(1-\xi)^3 V\, u(k), \\ \dots \\ x_a(k+1) = \xi x_a(k) + \left[V \sum\limits_{j=1}^{a} (1-\xi)^{j-1}\, x_{a-j}(k)\right] + (1-\xi)^a V u(k), \\ x_{a+1}(k+1) = \xi x_{a+1}(k) + V x_a(k) - \left[\xi V \sum\limits_{j=1}^{a} (1-\xi)^{j-1} x_{a-j}(k)\right] \\ \qquad -\xi\,(1-\xi)^a\, V\, u(k), \\ \dots \\ x_{a+N}(k+1) = \xi x_{a+N}(k) \ + \ \left[V \sum\limits_{i=0}^{N-1} (-\xi)^{N-i-1}\ x_{a+i}(k)\ \right] \\ \qquad + \left[(-\xi)^N V\ \sum\limits_{j=1}^{a} (1-\xi)^{j-1} x_{a-j}(k)\right] + (-\xi)^N\ (1-\xi)^a\ V\, u(k) \end{cases} \tag{2.8}$$

with $V = 1-\xi^2$. The system (2.8) can be written with the recursive representation form as follow:

$$\begin{cases} X(k+1) = \Phi\, X(k) + B\, u(k), \\ y(k) = C^T\, \mathbb{X}_N^{(a)}(k) + e(k), \end{cases} \tag{2.9}$$

with

$$X(k) = \Big[\underbrace{x_0(k), \dots, x_{a-1}(k)}_{X_d^T(k)}, \underbrace{x_a(k),\ x_{a+1}(k), \dots, x_{a+N}(k)}_{[\mathbb{X}_N^{(a)}(k)]^T}\Big]^T,$$

$$X_d(k) = [x_0(k), \dots, x_{a-1}(k)]^T; \quad \mathbb{X}_N^{(a)}(k) = [x_a(k), \dots, x_{a+N}(k)]^T. \tag{2.10}$$

The vector $\mathbb{X}_N^{(a)}(k)$ of relation (2.10) will be noted as:

$$\mathbb{X}_N^{(a)}(k) = \left[x_0^{(a)}(k), \dots, x_N^{(a)}(k)\right]^T, \quad C = [g_0, \cdots, g_N]^T, \tag{2.11}$$

$$\Phi = \begin{bmatrix} \varphi_{0,0} & \cdots & \varphi_{0,a+N} \\ \vdots & \ddots & \vdots \\ \varphi_{a+N,0} & \cdots & \varphi_{a+N,a+N} \end{bmatrix},$$

$$\varphi_{i,j} = \begin{cases} 0, & & i < j, \\ \xi, & & i = j, \\ (1-\xi)^{i-j-1} V, & for & j < i \leq a, \\ (-\xi)^{i-a} (1-\xi)^{a-j-1} V, & & j < a < i, \\ (-\xi)^{i-j-1} V, & & a \leq j < i, \end{cases} \quad i,j=0,\ldots,N+a,$$

$$B = V[b_0, \cdots, b_a, b_{a+1}, \cdots, b_{a+N}]^T,$$

$$b_i = \begin{cases} (1-\xi)^i V, & & i \leq a, \\ & for & \\ (-\xi)^{i-a} (1-\xi)^a V, & & i > a, \end{cases} \qquad i = 0, \ldots, N+a.$$

The model given by (2.9) is linear with respect to the Fourier coefficients $g_n$ and therefore the classical linear identification methods can be applied. But this model is nonlinear with respect to the Meixner pole $\xi$ which requires the application of complicated estimation approaches.

## 3 Optimization of Meixner pole

### 3.1 Problem statement

From (2.7), the truncation error $E(z,\xi)$ can be written as:

$$E(z,\xi) = Y(z) - \sum_{n=0}^{N} g_n(\xi) X_n^{(a)}(z,\xi).$$

To ensure the reduction of the parameter number in the Meixner model, the pole characterizing the Meixner functions has to be optimized. To do so, the following cost function $J(\xi)$ is considered

$$J(\xi) = \frac{1}{2}\|E(z,\xi)\|^2 = \frac{1}{2} < E(z,\xi), E(z,\xi) > . \tag{3.1}$$

Since the criterion $J(\xi)$ is non-linear with respect to the Meixner pole, its solution can be transformed into an optimization problem as follow:

$$\min_{\xi} \{J(\xi)\}.$$

From relation (3.1), the gradient of $J(\xi)$ with respect to $\xi$ is:

$$\frac{\partial J(\xi)}{\partial \xi} =< \frac{\partial E(z,\xi)}{\partial \xi}, E(z,\xi) >= 0. \tag{3.2}$$

As mentioned in [4,22], the optimal Fourier coefficients $g_n$ $(n = 0, ..., N)$ follow from the well-known normal equations given by:

$$< X_n^{(a)}(z,\xi), E(z,\xi) >= 0. \tag{3.3}$$

The relation (3.3) will be particularly useful in order to simplify (3.2). As mentioned below, the optimization procedure of the Meixner pole is based on input/output measurement. Thus for $H$ input/output data, the criterion is given by:

$$J(\xi) = \frac{1}{2}\sum_{k=1}^{H}\left(y(k) - \left(\sum_{n=0}^{N} g_n\, x_n^{(a)}(k,\,\xi)\right)\right)^2. \tag{3.4}$$

In matrix notation the criterion $J(\xi)$ can be written as:

$$J(\xi) = \frac{1}{2}\left(\mathbb{Y} - \aleph_N^{(a)} C\right)^T \left(\mathbb{Y} - \aleph_N^{(a)} C\right) = \frac{1}{2}\mathbb{E}^T\mathbb{E} \tag{3.5}$$

with $\aleph_N^{(a)}$ a matrix that includes the state vector $\mathbb{X}_N^{(a)}(k)$ given by relation (2.11) for $k = 1, \ldots, H$;

$$\aleph_N^{(a)} = \left[\mathbb{X}_N^{(a)}(1), \ldots, \mathbb{X}_N^{(a)}(H)\right]^T = \begin{bmatrix} x_0^{(a)}(1) & \cdots & x_N^{(a)}(1) \\ \vdots & \ddots & \vdots \\ x_0^{(a)}(H) & \cdots & x_N^{(a)}(H) \end{bmatrix} \tag{3.6}$$

and $\mathbb{Y}$, $\mathbb{E}$ include the output and the truncation error respectively, for $k = 1, \ldots, H$;

$$\mathbb{Y} = \begin{bmatrix} y(1) & \cdots & y(H) \end{bmatrix}^{\mathrm{T}}; \quad \mathbb{E} = \begin{bmatrix} e(1) & \cdots & e(H) \end{bmatrix}^{\mathrm{T}}.$$

The optimal vector $C$ of the Fourier coefficients is identified by the Least Squares (LS) method as follow:

$$C = \left(\left\{\aleph_N^{(a)}\right\}^T \aleph_N^{(a)}\right)^{-1} \left\{\aleph_N^{(a)}\right\}^T \mathbb{Y}. \tag{3.7}$$

## 3.2 Proposed optimization technique

In this section, a new technique is applied in order to optimize the Meixner pole by minimizing the criterion $J(\xi)$ in (3.4). Many recursive optimization algorithms to minimize $J(\xi)$ are found in literature [15, 18, 21]. In this paper we opted for an iterative method based on the Newton-Raphson technique. This latter is based on the Taylor series development at the second order of the criterion $J(\xi)$. Thus the Meixner pole $\xi$ at the $(m+1)^{th}$ iteration noticed $\xi_{m+1}$ can be given by the following expression:

$$\xi_{m+1} = \xi_m - \mu \left|\frac{\partial^2 J(\xi)}{\partial \xi^2}\right|_{\xi_m}^{-1} \left(\frac{\partial J(\xi)}{\partial \xi}\right)_{\xi_m}, \tag{3.8}$$

where $\mu$ is a step-size. Large step-size provides the adaptive filter with the ability to learn fast during the initial learning phase or track fast in a non-stationary environment. However, in a stationary environment large step-size implies large asymptotical fluctuations of the adaptive parameter around its real solution. In other words, the choice of the step-size involves a trade-off between learning speed and accuracy.

The gradient of $J(\xi)$ with respect to $\xi$ is obtained by differentiating (3.5)

$$\frac{\partial J(\xi)}{\partial \xi} = -\mathbb{E}^{\mathrm{T}} \frac{\partial \aleph_N^{(a)}}{\partial \xi} C - \mathbb{E}^T \aleph_N^{(a)} \frac{\partial C}{\partial \xi}. \tag{3.9}$$

Using the discrete time form of the normal equations (3.3) given by [3, 15], the relation (3.9) can be simplified as:

$$\frac{\partial J(\xi)}{\partial \xi} = -\mathbb{E}^T \frac{\partial \aleph_N^{(a)}}{\partial \xi} C \tag{3.10}$$

and then the Hessian of $J(\xi)$ is obtain after differentiating (3.10)

$$\begin{aligned} \frac{\partial^2 J(\xi)}{\partial \xi^2} &= \left( C^{\mathrm{T}} \frac{\partial \left\{ \aleph_N^{(a)} \right\}^T}{\partial \xi} + \frac{\partial C^T}{\partial \xi} \left\{ \aleph_N^{(a)} \right\}^T \right) \frac{\partial \aleph_N^{(a)}}{\partial \xi} C \\ &- \mathbb{E}^{\mathrm{T}} \left( \frac{\partial^2 \aleph_N^{(a)}}{\partial \xi^2} C + \frac{\partial \aleph_N^{(a)}}{\partial \xi} \frac{\partial C}{\partial \xi} \right), \end{aligned} \tag{3.11}$$

where the gradient of $C$ with respect to $\xi$ is determined from relation (3.7) as:

$$\begin{aligned} \frac{\partial C}{\partial \xi} &= \left( \left\{ \aleph_N^{(a)} \right\}^T \aleph_N^{(a)} \right)^{-1} \frac{\partial \left\{ \aleph_N^{(a)} \right\}^T}{\partial \xi} \mathbb{Y} \\ &- \left( \left\{ \aleph_N^{(a)} \right\}^T \aleph_N^{(a)} \right)^{-1} \left( \frac{\partial \left\{ \aleph_N^{(a)} \right\}^T}{\partial \xi} \aleph_N^{(a)} + \left\{ \aleph_N^{(a)} \right\}^T \frac{\partial \aleph_N^{(a)}}{\partial \xi} \right) C. \end{aligned} \tag{3.12}$$

**Remark:** The calculation of the second derivative requires voluminous calculations which encourage to abandon the Newton-Raphson method in favor of the Gauss-Newton method [21] where the approximated Hessian is calculated by eliminating the second term of the second member in relation (3.11). The calculation of the approximated Hessian occurs, therefore, only from the knowledge of the first order sensitivity functions as:

$$\frac{\partial^2 J(\xi)}{\partial \xi^2} \cong \left( C^T \frac{\partial \left\{ \aleph_N^{(a)} \right\}^T}{\partial \xi} + \frac{\partial C^T}{\partial \xi} \left\{ \aleph_N^{(a)} \right\}^T \right) \frac{\partial \aleph_N^{(a)}}{\partial \xi} C. \tag{3.13}$$

The Meixner pole $\xi_{m+1}$ at iteration $m+1$ given by (3.8) can be computed by using the gradient and the Hessian of the criterion $J(\xi)$, given by (3.10) and (3.13). These quantities depend on the first and the second derivatives of $\aleph_N^{(a)}$ with respect to $\xi$.

**Lemma 1.** *The first and the second derivatives of the matrix $\aleph_N^{(a)}$ containing the Meixner state vector $\mathbb{X}_N^a(k)$ with respect to $\xi$ can be formulated by the following relations:*

- *The first derivative of $\aleph_N^{(a)}$ with respect to $\xi$ is:*

$$\frac{\partial \aleph_N^{(a)}}{\partial \xi} = \aleph_{N+1}^{(a)} \left\{ \mathbb{A}_{N+1}^{(a)} \right\}^T. \tag{3.14}$$

- *The second derivative of* $\aleph_N^{(a)}$ *with respect to* $\xi$ *is:*

$$\frac{\partial^2 \aleph_N^{(a)}}{\partial \xi^2} = \left[ \aleph_{N+1}^{(a)} \left\{ \frac{\partial \mathbb{A}_{N+1}^{(a)}}{\partial \xi} \right\}^T + \aleph_{N+2}^{(a)} \left\{ \mathbb{A}_{N+2}^{(a)} \right\}^T \left\{ \mathbb{A}_{N+1}^{(a)} \right\}^T \right], \tag{3.15}$$

*where* $\mathbb{A}_{\mathrm{N+1}}^{(a)}$ *is a tridiagonal matrix of dimension* $(N+1)\times(N+2)$ *with elements of* $j$*th row given by:*

$$\frac{1}{1-\xi^2}\big[-j,\ -(a+\xi),\ a+1\big], \quad j=0,1,\ldots,N, \tag{3.16}$$

$\mathbb{A}_{\mathrm{N+2}}^{(a)}$ *is a tridiagonal matrix of dimension* $(N+2)\times(N+3)$ *with elements of* $j$*th row given by:*

$$\frac{1}{1-\xi^2}\big[-j,\ -(a+\xi),\ a+1\big], \quad j=0,1,\ldots,N+1, \tag{3.17}$$

$\dfrac{\partial \mathbb{A}_{N+1}^{(a)}}{\partial \xi}$ *is a matrix of dimension* $(N+1)\times(N+2)$ *with elements of* $j$*th row given by:*

$$\frac{1}{(1-\xi^2)^2}\big[-2j\xi,\ -1-2a\xi-\xi^2),\ 2\xi(a+1)\big], \quad j=0,1,\ldots,N, \tag{3.18}$$

*Proof.* From relation (3.6), the gradient of $\aleph_N^{(a)}$ with respect to $\xi$ is given by:

$$\frac{\partial \aleph_N^{(a)}}{\partial \xi} = \left[ \begin{array}{ccc} \dfrac{\partial \mathbb{X}_N^{(a)}(1)}{\partial \xi} & \ldots & \dfrac{\partial \mathbb{X}_N^{(a)}(H)}{\partial \xi} \end{array} \right]^T,$$

where from (2.11) we have:

$$\frac{\partial \mathbb{X}_N^{(a)}(k)}{\partial \xi} = \left[ \begin{array}{ccc} \dfrac{\partial x_0^{(a)}(k)}{\partial \xi} & \ldots & \dfrac{\partial x_N^{(a)}(k)}{\partial \xi} \end{array} \right]^T. \tag{3.19}$$

It remains to find the analytical expression of the first and second derivative of $x_n^{(a)}(k)$ with respect to $\xi$ for $n=0,\ldots,\ N$. From subsection 2.2 we have:

$$x_n^{(a)}(k) \ = \ Z^{-1}\left\{X_n^{(a)}(z,\xi)\right\}; \quad X_n^{(a)}(z,\xi) = M_n^{(a)}(z,\xi)\ .\ U(z). \tag{3.20}$$

From relation (3.20), we obtain:

$$\frac{\partial X_n^{(a)}(z,\xi)}{\partial \xi} = \frac{\partial\, M_n^{(a)}\,(z,\xi)}{\partial \xi} U\,(z)\,. \tag{3.21}$$

From Appendix A the derivative of Meixner functions with respect to $\xi$ is written as:

$$\frac{\partial M_0^{(a)}\,(z,\xi)}{\partial \xi} = \frac{1}{1-\xi^2}\left[(a+1)M_1^a(z,\xi) - (a+\xi)M_0^a(z,\xi)\right] \tag{3.22}$$

and $\forall\ n = 1, \dots, N$

$$\frac{\partial M_n^{(a)}(z,\xi)}{\partial \xi} = \frac{1}{1-\xi^2}\left[(a+n+1)M_{n+1}^a(z,\xi) - (a+\xi)M_n^a(z,\xi) - nM_{n-1}^a(z,\xi)\right]. \tag{3.23}$$

In matrix notation the criterion (3.22) and (3.23) can be written as:

$$\begin{bmatrix} \frac{\partial M_0^{(a)}(z,\xi)}{\partial \xi} \\ \vdots \\ \frac{\partial M_N^{(a)}(z,\xi)}{\partial \xi} \end{bmatrix} = \mathbb{A}_{N+1}^{(a)} \begin{bmatrix} M_0^{(a)}(z,\xi) \\ \vdots \\ M_N^{(a)}(z,\xi) \\ M_{N+1}^{(a)}(z,\xi) \end{bmatrix}$$

with $\mathbb{A}_{N+1}^{(a)}$ is given by (3.16).

The substitution of (3.22) and (3.23) in (3.21) and the Z-transform inverse allow to write:

$$\frac{\partial x_0^{(a)}(k)}{\partial \xi} = \frac{1}{1-\xi^2}\left[(a+1)x_1^a(k) - (a+\xi)x_0^a(k)\right] \tag{3.24}$$

and $\forall\ n = 1, \dots, N$

$$\frac{\partial x_n^{(a)}(k)}{\partial \xi} = \frac{1}{1-\xi^2}\left[(a+n+1)x_{n+1}^a(k) - (a+\xi)x_n^a(k) - nx_{n-1}^a(k)\right]. \tag{3.25}$$

Using relations (3.24), (3.25) and notation introduced previously in (3.19), we can deduce the expressions of the first derivative of $\mathbb{X}_N^{(a)}(k)$ with respect to $\xi$:

$$\frac{\partial \mathbb{X}_N^{(a)}(k)}{\partial \xi} = \mathbb{A}_{N+1}^{(a)}\mathbb{X}_{N+1}^{(a)}(k), \tag{3.26}$$

with $\mathbb{X}_{N+1}^{(a)}(k) = \left[x_0^{(a)}(k), \dots, x_N^{(a)}(k),\, x_{N+1}^{(a)}(k)\right]^T = \left[\left\{\mathbb{X}_N^{(a)}(k)\right\}^T,\, x_{N+1}^{(a)}(k)\ \right]^T$ and the second derivative of $\mathbb{X}_N^{(a)}(k)$ with respect to $\xi$ is given by:

$$\begin{aligned} \frac{\partial^2 \mathbb{X}_N^{(a)}(k)}{\partial \xi^2} &= \frac{\partial \mathbb{A}_{N+1}^{(a)}}{\partial \xi}\mathbb{X}_{N+1}^{(a)}(k) + \mathbb{A}_{N+1}^{(a)}\frac{\partial \mathbb{X}_{N+1}^{(a)}(k)}{\partial \xi} \\ &= \frac{\partial \mathbb{A}_{N+1}^{(a)}}{\partial \xi}\mathbb{X}_{N+1}^{(a)}(k) + \mathbb{A}_{N+1}^{(a)}\mathbb{A}_{N+2}^{(a)}\mathbb{X}_{N+2}^{(a)}, \end{aligned}$$

$\mathbb{X}_{N+2}^{(a)}(k) = \left[x_0^{(a)}(k), \dots, x_{N+1}^{(a)}(k),\, x_{N+2}^{(a)}(k)\right]^T = \left[\left\{\mathbb{X}_{N+1}^{(a)}(k)\right\}^T, x_{N+1}^{(a)}(k)\ \right]^T$ and $\mathbb{A}_{N+1}^{(a)}$, $\mathbb{A}_{N+2}^{(a)}$ and $\dfrac{\partial \mathbb{A}_{N+1}^{(a)}}{\partial \xi}$ are given by (3.16), (3.17) and (3.18) respectively.

From relation (3.26), we can deduce the first derivative of $\aleph_N^{(a)}$ with respect to $\xi$ as:

$$\frac{\partial \aleph_N^{(a)}}{\partial \xi} = \aleph_{N+1}^{(a)}\ \left\{\mathbb{A}_{N+1}^{(a)}\right\}^T. \tag{3.27}$$

The second derivatives of $\aleph_N^{(a)}$ with respect to $\xi$ can be formulated from differentiating (3.27)

$$\frac{\partial^2 \aleph_N^{(a)}}{\partial \xi^2} = \left[ \aleph_{N+1}^{(a)} \left\{ \frac{\partial \mathbb{A}_{N+1}^{(a)}}{\partial \xi} \right\}^T + \aleph_{N+2}^{(a)} \left\{ \mathbb{A}_{N+2}^{(a)} \right\}^T \left\{ \mathbb{A}_{N+1}^{(a)} \right\}^T \right].$$

□

*Algorithm 1.* Meixner pole optimization

1. Choose the initial pole $\xi_0$ ($m = 0$), the truncating order $N$ and the order of generalization $a$.
2. Fix $\mu$ and a threshold $\epsilon$.
3. To compute the optimal pole, we apply the following steps:
   (a) Form the matrix $\aleph_N^{(a)}$ given by relation (3.6).
   (b) Compute the first and the second derivatives of $\aleph_N^{(a)}$ with respect to $\xi$ respectively given by relations (3.14) and (3.15).
   (c) Estimate the optimal vector $C$ of the Fourier coefficients from relation (3.7).
   (d) Calculate the gradient of the vector $C$ with respect to $\xi$ from relation (3.12).
   (e) Determine the gradient and the Hessian of $J(\xi)$ from relations (3.10) and (3.13) respectively.
   (f) Evaluate $\xi_{m+1}$ from relation (3.8).
   (g) If ($|\xi_{m+1}| > 1$),
      - adjust $\mu$ and go to (3),

      else,
      - if ($|\xi_{m+1} - \xi_m| > \varepsilon$) increment $m$ and go to (a),
      - else $\xi_{opt} = \xi_{m+1}$ (end of the algorithm).

## 4 Simulation results

In this section, we present the performance of the Meixner model via the developed recursive representation and the proposed optimization algorithm of the pole. This study is illustrated through a numerical simulation and the performance of the Meixner model is evaluated by the Normalized MSE (NMSE) criterion given by:

$$NMSE(dB) = 10 \times log_{10} \left\{ \sum_{k=1}^{H} [e(k)]^2 / \sum_{k=1}^{H} [y(k)]^2 \right\}.$$

We specify that in this paper our objective is the optimization of the Meixner pole $\xi$ and both the truncating order $(N)$ and the order of generalization $(a)$ can be fixed after several experiments and when a very low value of the MSE between the system output and the model output is obtained.

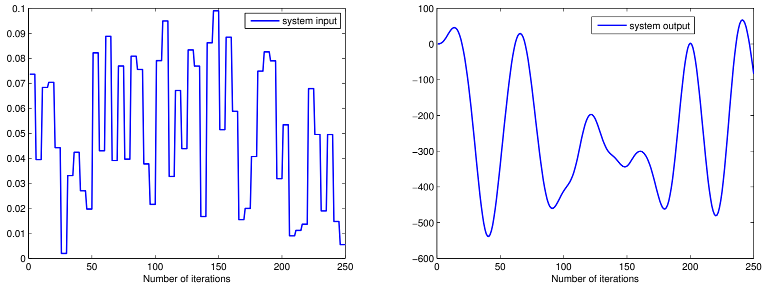
We consider the following transfer function having a slow start presented in [4]:

$$G(z^{-1}) = \left(\frac{z}{z-(0.9+0.1j)}\right)^4 + \left(\frac{z}{z-(0.9-0.1j)}\right)^4. \tag{4.1}$$

The equation (4.1) can be written as:

$$G(z^{-1}) = \big(1.215z^{-4} - 5.616z^{-3} + 9.6z^{-2} - 7.2z^{-1} + 2\big)/\big(0.4521z^{-8} - 3.97z^{-7} + 15.28z^{-6} - 33.65z^{-5} + 46.41z^{-4} - 41.04z^{-3} + 22.72z^{-2} - 7.2z^{-1} + 1\big).$$

The optimization procedure of the Meixner pole needs only the response of the system to a persistently exciting input signal. Thus we consider a multilevel pseudo-random sequence for the input system as illustrated in Figure 2(a) and the corresponding system output is given by Figure 2(b). The optimization algorithm needs the step-size which is chosen as $\mu = 0.5$, the convergence thresholds which is chosen equal to $\varepsilon = 10^{-3}$ and initial value of the Meixner pole. After several experiments, we have adopted a truncating order $N = 5$ and a generalization order $a = 4$ for the Meixner model.

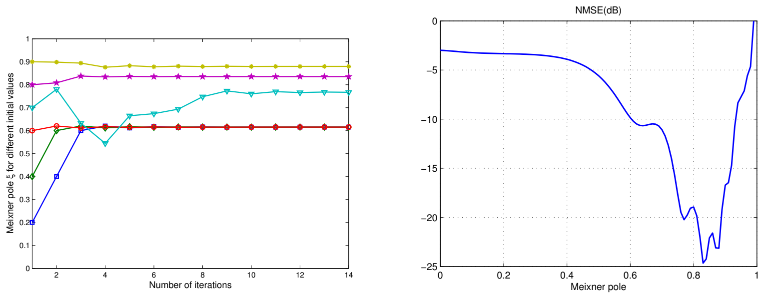



(a) System input (b) System output

**Figure 2.** System input/output

It is obviously an inherent feature of the iterative search routines that only convergence to a local solution can be guaranteed. This can be illustrated by Figure 3(a) where the proposed algorithm converges to a local solution for different Meixner pole values initialization. Figure 3(b) shows the concordance between the minimums of the NMSE criterion and the obtained Meixner poles.

To find the global solution, there is usually no other way than to start the iterative minimization algorithm at different feasible initial values $\xi_0$ and to compare the results. The optimal value of Meixner pole obtained is equal to 0.835 which is similar to that given in [8]. It can be mentioned that thanks to

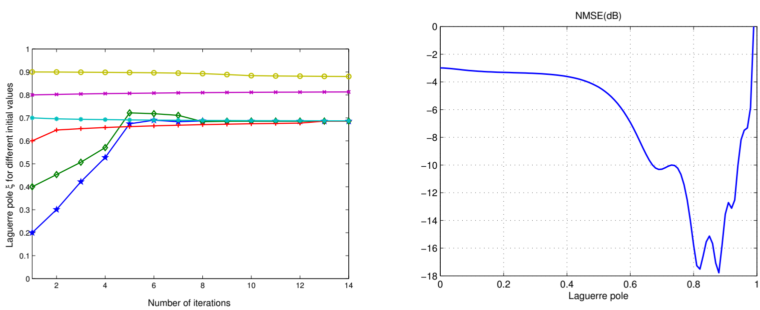



(a) Estimation of Meixner pole for different initial values (b) Evolution of the NMSE Criterion

**Figure 3.** Identification of Meixner pole

the Newton Raphson's algorithm, optimal poles converge towards its optimal values in relatively few iterations.

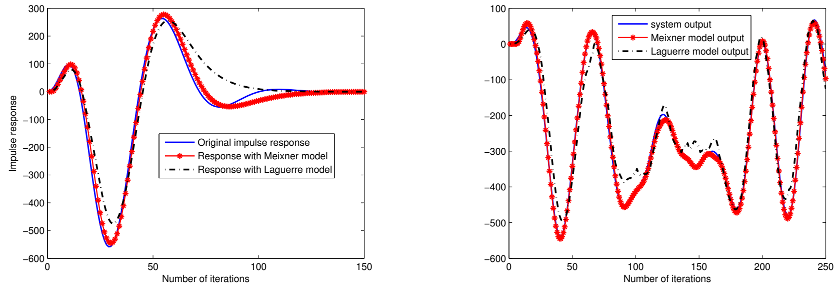
To confirm the deficiency of the Laguerre model for the approximation of the transfer function (4.1), we estimate the Laguerre pole as illustrated in Figure 4(a) with the proposed optimization algorithm for $a = 0$ and by taking $V = \sqrt{1-\xi^2}$ in the recursive representation. With the same way given for the Meixner pole, it can be shown that the algorithm converge to local solutions of the NMSE criterion given by Figure 4(b).




(a) Estimation of Laguerre pole for different initial values (b) Evolution of the NMSE Criterion

**Figure 4.** Identification of Laguerre pole

From Figure 5, we deduce that the Meixner model has a close approximation compared to the Laguerre model. That is provided by a NMSE= -24.81 dB for the Meixner model and a NMSE= -17.89 dB for the Laguerre model. Thus, the model based on Meixner functions have a good ability to identify the behavior of discrete-time system with slow initial onset. Note that the estimation of Meixner pole requires nonlinear optimization, whereas the identification of expansion coefficients is a convex optimization problem. The accuracy of the model is directly linked to the truncating order and converges monotonically toward the best approximation. Moreover, the truncating order of the Meixner filters can be optimized by an adequate choice of the pole which can be obtained by the proposed algorithm. We notice that both the truncating order $N$ and the order of generalization $a$ should be selected during an off-line identification experiment. However, we can use the proposed recursive representation for an adaptive filtering by optimizing adaptively the Fourier coefficients using algorithms such as Least Mean Squares (LMS) or Recursive Least Squares (RLS) algorithms and adjusting the Meixner pole with our approach.




(a) The impulse response function and its approximations with Meixner and Laguerre models

(b) The system output and its approximations with Meixner and Laguerre models

**Figure 5.** Validation of the Meixner and Laguerre models

# Conclusions

In this paper, a new recursive representation of linear discrete-time system has been proposed. It has been provided by filtering the input system with orthogonal Meixner functions. To identify the coefficients of the Meixner model and the pole, we propose an approach based on the minimization of the MSE criterion that is formulated as an optimization problem in which the Meixner pole can be optimized using an iterative algorithm based on the Newton-Raphson method. This identification approach requires the use of a set of input/output observations collected on the system. We have proved theoretically by exploiting the derivative of the Meixner functions that the gradient and the Hessian of the considered criterion are expressed analytically. The theoretical study is validated in numerical simulation which confirms the efficiency of the proposed recursive representation as well as the optimization approach of the Meixner pole.

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## Appendix A: Derivative of Meixner functions

The derivative of the Z-transform of the Meixner function with respect to $\xi$ is given by: for $n=0$

$$
\begin{aligned}
\frac{\partial M_0^a(\xi,z)}{\partial \xi} &= \frac{z}{(z-\xi)^2}\left(\frac{(1-\xi)(z+1)}{z-\xi}\right)^a \left(a+a\xi-az-2\xi z+\xi^2-a\xi z+1\right)\\
&= \frac{z}{(z-\xi)^2}\left(\frac{(1-\xi)(z+1)}{z-\xi}\right)^a (a(1-\xi z)+a(\xi-z)+(1-\xi z)+\xi(\xi-z))\\
&= \frac{z}{(z-\xi)^2}\left(\frac{(1-\xi)(z+1)}{z-\xi}\right)^a ((a+1)(1-\xi z)+(a+\xi)(\xi-z))\\
&= \frac{(1-\xi^2)z}{(z-\xi)}\left(\frac{(1-\xi)(z+1)}{z-\xi}\right)^a \left(\frac{(a+1)(1-\xi z)+(a+\xi)(\xi-z)}{(1-\xi^2)(z-\xi)}\right)\\
&= \frac{1}{1-\xi^2}\left[(a+1)M_1^a-(a+\xi)M_0^a\right]
\end{aligned}
$$

and $\forall\ n=1,\ldots,N$

$$
\frac{\partial M_n^a(\xi,z)}{\partial \xi} = \frac{z}{(1-\xi z)(z-\xi)^2}\left(\frac{(1-\xi)(z+1)}{z-\xi}\right)^a\left(\frac{1-\xi z}{z-\xi}\right)^n F,
$$

where $F=\alpha+\beta+\delta$,

$$
\begin{aligned}
&\alpha = a+a\xi-az+a\xi z^2-a\xi^2 z+a\xi^2 z^2-2a\xi z,\\
&\beta = 2\xi^2 z^2-3\xi z-\xi^3 z+\xi^2+1,\quad \delta = n-n\xi^2-nz^2+n\xi^2 z^2.
\end{aligned}
$$

The expression of $F$ is simplified as:

$$
F \;=\; (1-\xi z)^2(a+n+1)+(\xi-z)(1-\xi z)(a+\xi)-n(z-\xi)^2.
$$

Thus $\dfrac{\partial M_n^a(\xi,z)}{\partial \xi}$ becomes

$$
\begin{aligned}
\frac{\partial M_n^a(\xi,z)}{\partial \xi} &= \frac{z((1-\xi)(z+1)/(z-\xi))^a}{(1-\xi z)(z-\xi)^2}\left(\frac{1-\xi z}{z-\xi}\right)^n\\
&\quad\times\left((1-\xi z)^2(a+n+1)+(\xi-z)(1-\xi z)(a+\xi)-n(z-\xi)^2\right)\\
&= \frac{(1-\xi^2)z}{(z-\xi)}\left(\frac{(1-\xi)(z+1)}{z-\xi}\right)^a\left(\frac{1-\xi z}{z-\xi}\right)^n\\
&\quad\times\left(\frac{(1-\xi z)^2(a+n+1)+(\xi-z)(1-\xi z)(a+\xi)-n(z-\xi)^2}{(1-\xi^2)(1-\xi z)(z-\xi)}\right)\\
&= \frac{1}{1-\xi^2}\left[(a+n+1)M_{n+1}^a-(a+\xi)M_n^a-nM_{n-1}^a\right].
\end{aligned}
$$