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An Ostrowski Type Inequality for Twice Differentiable Mappings and Applications

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Abstract. We establish an Ostrowski type inequality for mappings whose second derivatives are bounded, then some results of this inequality that are related to previous works are given. Finally, some applications of these inequalities in numerical integration and for special means are provided.

Keywords: Ostrowski inequality, numerical integration, special means.

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1 Introduction

In 1938, Ostrowski established the integral inequality which is one of the fundamental inequalities of mathematic as follows (see, [13]).

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'}:(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, the inequality holds:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\frac{(x-(a+b)/2)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

Inequality (1.1) has wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoid and Simpson rules and quadrature rules, etc. Hence, inequality (1.1) has attracted considerable attention and interest from mathematicians and researchers. In addition, the current approach of obtaining the bounds, for a particular quadrature rule, have depended on the use of Peano kernel. The general approach in the past has involved the assumption of bounded derivatives.

In [3], the following inequality was proved by Cerone, Dragomir and Roumeliotis.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping such that $f'':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f''\|_\infty=\sup_{t\in(a,b)}|f''(t)|<\infty$. Then we have the inequality*

$$\left|f(x)-\left(x-\frac{a+b}{2}\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \tag{1.2}$$
$$\leq\left[\frac{(b-a)^2}{24}+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right]\|f''\|_\infty\leq\frac{(b-a)^2}{6}\|f''\|_\infty$$

for all $x\in[a,b]$.

In [6], Dragomir and Barnett proved the following inequality.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping such that $f'':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f''\|_\infty=\sup_{t\in(a,b)}|f''(t)|<\infty$. Then we have the inequality*

$$\left|f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\frac{(b-a)^2}{2}\left\{\left[\left(\frac{x-(a+b)/2}{b-a}\right)^2+\frac{1}{4}\right]+\frac{1}{12}\right\}\|f''\|_\infty\leq\frac{(b-a)^2}{6}\|f''\|_\infty$$

for all $x\in[a,b]$.

In recent years, researchers have studied Qstrowski type inequalities for various convex functions and mappings whose derivatives are bounded. You can check ([1], [2], [4], [5], [6], [7], [8], [9], [10], [12], [14], [15], [16], [17], [18], [19], [20], [21]) and the references included there.

In this study, we derive a new inequality that is connected with the celebrated Ostrowski type integral inequalities using functions whose second derivatives are bounded. We give Trapezoid and Midpoint inequality for twice differentiable mappings by using this inequality. The results presented here would provide extensions of those given in earlier works.

2 Main Results

In order to prove our main results we need the following lemma:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° (I° is the interior of I), and let $a, b \in I^{\circ}$ with $a < b$. If $f'' \in L[a,b]$, then the following identity holds:*

$$\frac{1}{2(b-a)} \int_a^b P_h(x,t) f''(t)\, dt = \frac{h-2}{2} \left(x - \frac{a+b}{2} \right) f'(x) + f(x) \tag{2.1}$$
$$- \frac{f(b) - f(a)}{2(b-a)} m_h(x) - \frac{1}{b-a} \int_a^b f(t)\, dt =: S_{x,h}(f)$$

for

$$P_h(x,t) := \begin{cases} (a-t)(t-a-m_h(x)), & a \le t < x, \\ (b-t)(t-b-m_h(x)), & x \le t \le b, \end{cases}$$

where $m_h(x) = h\left(x - \frac{a+b}{2}\right)$, $h \in [0,2]$ and $x \in [a,b]$.

Proof. Integrating by parts twice, we have

$$\begin{aligned}
&\int_a^b P_h(x,t) f''(t)\, dt \\
&= \int_a^x (a-t)(t-a-m_h(x)) f''(t)\, dt + \int_x^b (b-t)(t-b-m_h(x)) f''(t)\, dt \\
&= (a-x)(x-a-m_h(x)) f'(x) - \int_a^x (-2t + 2a + m_h(x)) f'(t)\, dt \\
&- (b-x)(x-b-m_h(x)) f'(x) - \int_x^b (-2t + 2b + m_h(x)) f'(t)\, dt \\
&= [(a-x)(x-a-m_h(x)) - (b-x)(x-b-m_h(x))] f'(x) \\
&+ 2(b-a) f(x) - [f(b) - f(a)] m_h(x) - 2 \int_a^b f(t)\, dt.
\end{aligned}$$

From which we get the identity (2.1) which completes the proof. $\square$

Now, we establish our theorem and also give some results related to this theorem.

Theorem 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° (I° is the interior of I), and let $a, b \in I^{\circ}$ with $a < b$. If $f'' : (a,b) \to \mathbb{R}$ is bounded on (a,b), denote $\|f''\|_{\infty} = \sup_{t \in (a,b)} |f''(t)| < \infty$, then the following inequalities hold:*

$$\begin{aligned}
&|S_{x,h}(f)| \\
&\le \frac{1}{2} \left\{ (b-a)^2 \left(\frac{1}{12} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) - h\left(x - \frac{a+b}{2} \right)^2 - \frac{[m_h(x)]^3}{3(b-a)} \right\} \|f''\|_{\infty}
\end{aligned} \tag{2.2}$$

for all $a \le x \le \frac{a+b}{2}$ with $h \in [0,2]$ and

$$\begin{aligned}
&|S_{x,h}(f)| \\
&\le \frac{1}{2} \left\{ (b-a)^2 \left(\frac{1}{12} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) - h\left(x - \frac{a+b}{2} \right)^2 + \frac{[m_h(x)]^3}{3(b-a)} \right\} \|f''\|_{\infty}
\end{aligned} \tag{2.3}$$

for all $\frac{a+b}{2} \le x \le b$ *with* $h \in [0,2]$, *where* $m_h(x) = h\left(x - \frac{a+b}{2}\right)$.

Proof. From (2.1) and under the assumptions of theorem, we have

$$\begin{aligned} |S_{x,h}(f)| &\le \frac{1}{2(b-a)} \int_a^b |P_h(x,t)| \, |f''(t)| \, dt \\ &\le \frac{\|f''\|_\infty}{2(b-a)} \int_a^b |P_h(x,t)| \, dt = \frac{\|f''\|_\infty}{2(b-a)} L, \end{aligned}$$

where

$$L = \int_a^x |a-t| \, |t-a-m_h(x)| \, dt + \int_x^b |b-t| \, |t-b-m_h(x)| \, dt.$$

Now, let us consider that

$$\begin{aligned} \int_p^r |t-p| \, |t-q| \, dt &= \int_p^q (t-p)(q-t) \, dt + \int_q^r (t-p)(t-q) \, dt \\ &= \frac{(q-p)^3}{3} + \frac{(r-p)^3}{3} - \frac{(q-p)(r-p)^2}{2} \end{aligned} \tag{2.4}$$

for all r, p, q such that $p \le q \le r$.

We calculate the integral L for the intervals $a \le x \le \frac{a+b}{2}$ and $\frac{a+b}{2} \le x \le b$. For $a \le x \le \frac{a+b}{2}$ we have

$$\begin{aligned} \int_a^x |a-t| \, |t-a-m_h(x)| \, dt &= \int_a^x (t-a)(t-a-m_h(x)) \, dt \\ = \int_0^{x-a} u(u-m_h(x)) du &= \frac{(x-a)^3}{3} - \frac{(x-a)^2}{2} m_h(x). \end{aligned}$$

Using the equality (2.4), we get

$$\int_x^b |b-t| \, |t-b-m_h(x)| \, dt = -\frac{[m_h(x)]^3}{3} + \frac{(b-x)^3}{3} + \frac{(b-x)^2}{2} m_h(x).$$

For $\frac{a+b}{2} \le x \le b$ using the equality (2.4) again, we obtain

$$\int_a^x |a-t| \, |t-a-m_h(x)| \, dt = \frac{[m_h(x)]^3}{3} + \frac{(x-a)^3}{3} - \frac{(x-a)^2}{2} m_h(x).$$

Also, we get

$$\int_x^b |b-t| \, |t-b-m_h(x)| \, dt = \frac{(b-x)^3}{3} + \frac{(b-x)^2}{2} m_h(x).$$

Then, we have

$$L = \frac{(b-x)^3 + (x-a)^3}{3} - h(b-a)\left(x - \frac{a+b}{2}\right)^2 - \frac{[m_h(x)]^3}{3} \tag{2.5}$$

for $a \le x \le \frac{a+b}{2}$ and

$$L = \frac{(b-x)^3 + (x-a)^3}{3} - h(b-a)\left(x - \frac{a+b}{2}\right)^2 + \frac{[m_h(x)]^3}{3} \tag{2.6}$$

for $\frac{a+b}{2} < x \le b$. From (2.5) and (2.6), we obtain desired results. The proof is thus completed. $\square$

Remark 1. If we choose $x = \frac{a+b}{2}$ in Theorem 3, then we have the mid-point inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{(b-a)^2}{24} \|f''\|_\infty$$

which was given by Cerone et al. in [3].

Remark 2. If we choose $h = 0$ in Theorem 3, then the inequalities (2.2) and (2.3) reduce to (1.2).

Corollary 1. Let us substitute $x = a$ and $x = b$ in Theorem 3. Subsequently, if we add the obtained results and use the triangle inequality for the modulus, we get the inequality

$$\begin{aligned} &\left| \frac{h-2}{2}\frac{b-a}{4}\left(f'(b) - f'(a)\right) + \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\le \frac{(b-a)^2}{2}\left[\frac{1}{3} - \frac{h}{4} + \frac{h^3}{24}\right] \|f''\|_\infty . \end{aligned}$$

Remark 3. If we take $h = 0$ in Corollary 1, then we obtain

$$\left| \frac{f(a)+f(b)}{2} - \frac{b-a}{4}\left(f'(b) - f'(a)\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{(b-a)^2}{6} \|f''\|_\infty ,$$

which was given by Cerone et al. in [3].

Remark 4. If we take $h = 2$ in Corollary 1, then we have the trapezoid inequality

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{(b-a)^2}{12} \|f''\|_\infty , \tag{2.7}$$

which was given by Liu in [11].

Corollary 2. Under the same assumptions of Theorem 3 with $h = 2$, we get the following inequalities

$$\begin{aligned} &\left| f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\le \frac{1}{2}\left\{ (b-a)^2 \left(\frac{1}{12} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) - 2\left(x - \frac{a+b}{2}\right)^2 - \frac{8}{3}\frac{\left(x - \frac{a+b}{2}\right)^3}{(b-a)} \right\} \|f''\|_\infty \end{aligned}$$

for all $a \leq x \leq \frac{a+b}{2}$ with $h \in [0,2]$ and

$$\left| f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$

$$\leq \frac{1}{2}\left\{ (b-a)^2 \left(\frac{1}{12} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right) -2\left(x - \frac{a+b}{2}\right)^2 + \frac{8}{3}\frac{\left(x-\frac{a+b}{2}\right)^3}{(b-a)} \right\} \|f''\|_\infty$$

for all $\frac{a+b}{2} \leq x \leq b$ with $h \in [0,2]$.

3 Applications to Numerical Integration

We now consider applications of the integral inequalities developed in the previous section, to obtain estimates of composite quadrature rules which, it turns out have a markedly smaller error than that which may be obtained by the classical results.

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$. Define the quadrature

$$S(f,f',\xi,I_n) := \frac{h-2}{2}\sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right) k_i f'(\xi_i) + \sum_{i=0}^{n-1} k_i f(\xi_i)$$
$$- h\sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)\frac{f(x_{i+1})-f(x_i)}{2}, \tag{3.1}$$

where $k_i = x_{i+1} - x_i$, $i = 0, ..., n-1$.

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, the interior of the interval I, where $a, b \in I^\circ$ with $a < b$. If $f'' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f''\|_\infty < \infty$, then we have the representation*

$$\int_a^b f(x)dx = S(f,f',\xi,I_n) + R(f,f',\xi,I_n),$$

where $S(f,f',\xi,I_n)$ is as defined in (3.1) and the remainder satisfies the estimations:

$$|R(f,f',\xi,I_n)| \leq \frac{1}{2}\left\{ \sum_{i=0}^{n-1} k_i^3 \left(\frac{1}{12} + \frac{(\xi_i - (x_i+x_{i+1})/2)^2}{k_i^2} \right) \right. \tag{3.2}$$
$$\left. - h\sum_{i=0}^{n-1} k_i \left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2 - \frac{h^3}{3}\sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^3 \right\} \|f''\|_\infty$$

for $x_i \leq \xi_i \leq \frac{x_i+x_{i+1}}{2}$ with $h \in [0,2]$ and

$$|R(f,f',\xi,I_n)| \leq \frac{1}{2}\left\{ \sum_{i=0}^{n-1} k_i^3 \left(\frac{1}{12} + \frac{(\xi_i - (x_i+x_{i+1})/2)^2}{k_i^2} \right) \right. \tag{3.3}$$
$$\left. - h\sum_{i=0}^{n-1} k_i \left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2 + \frac{h^3}{3}\sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^3 \right\} \|f''\|_\infty$$

for $\frac{x_i+x_{i+1}}{2} \leq \xi_i \leq x_{i+1}$ *with* $h \in [0,2]$, $i = 0, ..., n-1$.

Proof. Applying Theorem 3 on the interval $[x_i, x_{i+1}]$, $i = 0, ..., n-1$, we obtain

$$\begin{aligned}
&\left| \frac{h-2}{2} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) k_i f'(\xi_i) + k_i f(\xi_i) \right. \\
&\left. -h \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) \frac{f(x_{i+1}) - f(x_i)}{2} - \int_{x_i}^{x_{i+1}} f(x)dx \right| \\
&\leq \frac{1}{2} \left\{ k_i^3 \left(\frac{1}{12} + \frac{(\xi_i - (x_i + x_{i+1})/2)^2}{k_i^2} \right) - hk_i \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right. \\
&\left. - \frac{h^3}{3} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^3 \right\} \|f''\|_\infty
\end{aligned}$$

for $x_i \leq \xi_i \leq \frac{x_i+x_{i+1}}{2}$ with $h \in [0,2]$ and

$$\begin{aligned}
&\left| \frac{h-2}{2} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) k_i f'(\xi_i) + k_i f(\xi_i) \right. \\
&\left. -h \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) \frac{f(x_{i+1}) - f(x_i)}{2} - \int_{x_i}^{x_{i+1}} f(x)dx \right| \\
&\leq \frac{1}{2} \left\{ k_i^3 \left(\frac{1}{12} + \frac{(\xi_i - (x_i + x_{i+1})/2)^2}{k_i^2} \right) - hk_i \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right. \\
&\left. + \frac{h^3}{3} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^3 \right\} \|f''\|_\infty
\end{aligned}$$

for $\frac{x_i+x_{i+1}}{2} \leq \xi_i \leq x_{i+1}$ with $h \in [0,2]$, $i = 0, ..., n-1$. Summing over i from 0 to $n-1$ and using the triangle inequality we obtain the estimations (3.2) and (3.3). □

It is clear that inequalities (3.2) and (3.3) are much better than the classical averages of the remainders of the Midpoint and Trapezoidal quadratures.

Remark 5. If we choose $\xi_i = \frac{x_i+x_{i+1}}{2}$ in Theorem 4, then we recapture the midpoint quadrature formula

$$\int_a^b f(x)dx = A_M(f, I_n) + R_M(f, I_n),$$

where the remainder $R_M(f, I_n)$ satisfies the estimation

$$|R_M(f, I_n)| \leq \frac{\|f''\|_\infty}{24} \sum_{i=0}^{n-1} k_i^3.$$

Also, if we consider the inequality (2.7), then we recapture the trapezoidal quadrature formula

$$\int_a^b f(x)dx = A_T(f, I_n) + R_T(f, I_n),$$

where the remainder $R_T(f, I_n)$ satisfies the estimation

$$|R_T(f, I_n)| \leq \frac{\|f''\|_\infty}{12} \sum_{i=0}^{n-1} k_i^3.$$

4 Applications to Some Special Means

Let us recall the following means:
(a) The *Arithmatic mean:*

$$A = A(a,b) := \tfrac{a+b}{2}, \qquad a, b \geq 0.$$

(b) The *Geometric mean*:

$$G = G(a,b) := \sqrt{ab}, \qquad a, b \geq 0.$$

(c) The *Harmonic mean*:

$$H = H(a,b) := \tfrac{2}{1/a+1/b}, \qquad a, b > 0.$$

(d) The *Logarithmic mean*:

$$L = L(a,b) := \begin{cases} a, & \text{if } \ a = b \\ \frac{b-a}{\ln b - \ln a}, & \text{if } \ a \neq b. \end{cases}, \qquad a, b > 0.$$

(e) The *Identric mean*:

$$I = L(a,b) := \begin{cases} a, & \text{if } \ a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } \ a \neq b \end{cases}, \qquad a, b > 0.$$

(f) The $p-$*logarithmic mean*:

$$L_p = L_p(a,b) := \begin{cases} a, & \text{if } \ a = b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } \ a \neq b, \end{cases}, \qquad a, b > 0,$$

where $p \in \mathbb{R} \backslash \{-1, 0\}$.

The following simple relationships are known in literature

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing in $p \in \mathbb{R}$ with $L_0 = I$ and $L_{-1} = L$.

(1) Consider the mapping $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^p$, $p \in \mathbb{R} \backslash \{-1, 0\}$. Then, we have, for $0 < a < b$,

$$\frac{1}{b-a} \int_a^b f(t)\, dt = L_p^p$$

and

$$\|f''\|_\infty = |p(p-1)|\, \delta_p(a,b), \qquad p \in \mathbb{R} \backslash \{-1, 0\},$$

where

$$\delta_p(a,b)=\begin{cases} b^{p-1}, & \text{if } p\in(1,\infty) \\ a^{p-1}, & \text{if } p\in(-\infty,1)\setminus\{-1,0\}. \end{cases}$$

Using the inequalities (2.2) and (2.3) we have the results:

$$\left|\frac{p\,(h-2)}{2}(x-A)\,x^{p-1}+x^p-\frac{p.h}{2}L_{p-1}^{p-1}(x-A)-L_p^p\right|\leq\frac{|p(p-1)|}{2} \tag{4.1}$$
$$\times\left\{(b-a)^2\left(\frac{1}{12}+\frac{(x-A)^2}{(b-a)^2}\right)-h\,(x-A)^2-\frac{h^3\,(x-A)^3}{3\,(b-a)}\right\}\delta_p(a,b)$$

for all $a\leq x\leq A$ and

$$\left|\frac{p\,(h-2)}{2}(x-A)\,x^{p-1}+x^p-\frac{p.h}{2}L_{p-1}^{p-1}(x-A)-L_p^p\right|\leq\frac{|p(p-1)|}{2} \tag{4.2}$$
$$\times\left\{(b-a)^2\left(\frac{1}{12}+\frac{(x-A)^2}{(b-a)^2}\right)-h\,(x-A)^2+\frac{h^3\,(x-A)^3}{3\,(b-a)}\right\}\delta_p(a,b)$$

for all $A\leq x\leq b$.

If we choose $h=0$ in (4.1) and (4.2), we have the inequality

$$\left|x^p-p\,(x-A)\,x^{p-1}-L_p^p\right|\leq\frac{|p(p-1)|}{6}\left\{\frac{(b-a)^2}{4}+3\,(x-A)^2\right\}\delta_p(a,b)$$
$$\leq\frac{|p(p-1)|\,(b-a)^2}{6}\delta_p(a,b),$$

which was given by Cerone et al. in [3].

(2) Consider the mapping $f(x)=\frac{1}{x}$, $x\in[a,b]\subset(0,\infty)$. Then, we have

$$\frac{1}{b-a}\int_a^b f\,(t)\,dt=L_{-1}^{-1}=\frac{1}{L},\quad \|f''\|_\infty=\frac{2}{a^3}.$$

Using the inequalities (2.2) and (2.3) we have the results:

$$\left|\frac{h\,(x-A)}{2ab}+\frac{1}{x}-\frac{h-2}{2x^2}(x-A)-\frac{1}{L}\right| \tag{4.3}$$
$$\leq\frac{1}{a^3}\left\{(b-a)^2\left(\frac{1}{12}+\frac{(x-A)^2}{(b-a)^2}\right)-h\,(x-A)^2-\frac{h^3\,(x-A)^3}{3\,(b-a)}\right\}$$

for all $a\leq x\leq A$ and

$$\left|\frac{h\,(x-A)}{2ab}+\frac{1}{x}-\frac{h-2}{2x^2}(x-A)-\frac{1}{L}\right| \tag{4.4}$$
$$\leq\frac{1}{a^3}\left\{(b-a)^2\left(\frac{1}{12}+\frac{(x-A)^2}{(b-a)^2}\right)-h\,(x-A)^2-\frac{h^3\,(x-A)^3}{3\,(b-a)}\right\}$$

for all $A \leq x \leq b$.

If we take $h = 0$ in (4.3) and (4.4), we have the inequality

$$\left|\frac{1}{x} + \frac{x-A}{x^2} - \frac{1}{L}\right| \leq \frac{1}{3a^3}\left[\frac{(b-a)^2}{4} + 3(x-A)^2\right] \leq \frac{(b-a)^2}{3a^3},$$

which was given by Cerone et al. in [3].

(3) Consider the mapping $f(x) = \ln x$, $x \in [a,b] \subset (0,\infty)$. Then, we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt = \ln I(a,b), \quad \|f''\|_\infty = \frac{1}{a^2}.$$

Using the inequalities (2.2) and (2.3) we have the results:

$$\begin{aligned} &\left|\frac{(h-2)(x-A)}{2x} + \ln x - \frac{h(x-A)}{2L} - \ln I\right| \qquad (4.5) \\ \leq\ & \frac{1}{2a^2}\left\{(b-a)^2\left(\frac{1}{12} + \frac{(x-A)^2}{(b-a)^2}\right) - h(x-A)^2 - \frac{h^3(x-A)^3}{3(b-a)}\right\} \end{aligned}$$

for all $a \leq x \leq A$ and

$$\begin{aligned} &\left|\frac{(h-2)(x-A)}{2x} + \ln x - \frac{h(x-A)}{2L} - \ln I\right| \qquad (4.6) \\ \leq\ & \frac{1}{2a^2}\left\{(b-a)^2\left(\frac{1}{12} + \frac{(x-A)^2}{(b-a)^2}\right) - h(x-A)^2 + \frac{h^3(x-A)^3}{3(b-a)}\right\} \end{aligned}$$

for all $A \leq x \leq b$.

If we choose $h = 0$ in (4.5) and (4.6), we have the inequality

$$\left|\ln x - \frac{(x-A)}{x} - \ln I\right| \leq \frac{1}{6a^2}\left[\frac{(b-a)^2}{4} + 3(x-A)^2\right] \leq \frac{(b-a)^2}{6a^2},$$

which was given by Cerone et al. in [3].

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