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Convergence of the Solutions on the Generalized Korteweg–de Vries Equation*

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Abstract. We consider the generalized Korteweg-de Vries equation, which contains nonlinear dispersive effects. We prove that as the diffusion parameter tends to zero, the solutions of the dispersive equation converge to discontinuous weak solutions of the scalar conservation law. The proof relies on deriving suitable a priori estimates together with an application of the compensated compactness method in the L^p setting.

Keywords: singular limit, compensated compactness, connected compactness, Korteweg-de Vries equation.

AMS Subject Classification: 35G25; 35L65; 35L05.

1 Introduction

The evolution equation

$$\partial_t u + \alpha\partial_x(u^n) + a\beta\partial_{xxx}^3 u = 0, \quad \alpha,\, a \in \{-1, 1\}, \quad n \in \mathbb{N} \setminus \{0\} \tag{1.1}$$

is known as the generalized Korteweg-de Vries equation (see [1,9]).

When $n = 2$, $\alpha = 1$, $a = 1$, (1.1) reads

$$\partial_t u + \partial_x(u^2) + \beta\partial_{xxx}^3 u = 0, \tag{1.2}$$

which is Korteweg-de Vries equation (see [12]). (1.2) is a mathematical model of waves on shallow water surfaces. Existence and regularity of solutions to (1.2) has been studied in [8].

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Observe that if $\beta = 0$, (1.2) reads

$$\partial_t u + \partial_x(u^2) = 0, \tag{1.3}$$

which is the Burgers equation.

Therefore, an interesting problem is to prove that the solution of (1.2) converges to the solution of (1.3). This result is proven in [21] using a compensated compactness argument based on L^2 and L^4 estimates. In other to do this, the author chose the following approximation of (1.2)

$$\partial_t u + \partial_x(u^2) + \beta\partial^3_{xxx}u = \varepsilon\partial^2_{xx}u. \tag{1.4}$$

Under the assumption,

$$u(0,\cdot) = u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}), \quad \beta = \mathcal{O}\left(\varepsilon^2\right), \tag{1.5}$$

she proved that the solution of (1.2) converges to the distributional solution of (1.3). This result holds also when $\alpha = -1$. From the mathematical point of view, to prove L^4- estimate, an $L^\infty-$ estimate is needed (see [21, pages 986 - 988]). If (1.5) does not hold we may have the convergence to nonclassical solutions [16].

The convergence of the solutions of (1.4) to the entropy solutions of (1.3) is proven in [17], under the assumption

$$u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}), \quad \beta = o\left(\varepsilon^2\right).$$

[3, Appendixes A and B] show that it is possible to obtain the same convergence result, under the following assumptions

$$u_0 \in L^2(\mathbb{R}), \quad -\infty < \int_{\mathbb{R}} u_0(x)dx < \infty, \quad \beta = o\left(\varepsilon^3\right),$$

or

$$u_0 \in L^2(\mathbb{R}), \quad \beta = o\left(\varepsilon^4\right).$$

Assuming $\alpha = -1$, $n = 2p+1$, $a = 1$, (1.1) reads

$$\partial_t u - \partial_x(u^{2p+1}) + \beta\partial^3_{xxx}u = 0. \tag{1.6}$$

If $\beta = 0$, we have the following scalar conservation law

$$\partial_t u - \partial_x(u^{2p+1}) = 0. \tag{1.7}$$

Using the following approximation

$$\partial_t u - \partial_x(u^{2p+1}) + \beta\partial^3_{xxx}u = \varepsilon\partial^2_{xx}u, \tag{1.8}$$

in [21], the convergence of the solution of (1.6) to the distributional solution of (1.7) is proven, under the assumption

$$u_0 \in L^2(\mathbb{R}) \cap L^{2p+2}(\mathbb{R}), \quad \beta = \mathcal{O}\left(\varepsilon^2\right).$$

From the mathematical point of view, to obtain the convergence, the author used the following conserved quantity

$$t \to \int_{\mathbb{R}} \left(\frac{u^{2n+2}}{2n+2} + \frac{\beta(\partial_x u)^2}{2} \right) dx.$$

The convergence of the solution of (1.6) to the unique entropy solution of (1.7) is proven in [17], under the assumption

$$u_0 \in L^2(\mathbb{R}) \cap L^{2p+2}(\mathbb{R}), \quad \beta = o\left(\varepsilon^2\right)$$

and using the approximation (1.8).

In this paper we consider (1.1). If we send $\beta \to 0$, we pass in (1.1), to the scalar conservation law

$$\partial_t u + \alpha \partial_x \left(u^n\right) = 0. \tag{1.9}$$

Our goal is to prove the convergence of the solution of (1.1) to (1.9). In other to do this, we use the following approximation

$$\partial_t u + \alpha \partial_x (u^n) + a\beta \partial^3_{xxx} u = \varepsilon \partial^2_{xx} u - \beta\varepsilon \partial^4_{xxxx} u. \tag{1.10}$$

Hence, we use a fourth order approximation for a third order equation. This idea is motivated by [4, 7], where the authors used a fourth order approximation for the Camassa-Holm equation, which is a third order one. The main differences between (1.10) and the one considered in [4, 7] are in the flux (the one in [4, 7] corresponds to $n = 2$) and the third order terms (the ones in [4, 7] are nonlinear).

Observe that $\alpha = a = 1$, $n = 2$, $\varepsilon = 1$, (1.10) reads

$$\partial_t u + \partial_x (u^2) + \beta \partial^3_{xxx} u = \partial^2_{xx} u - \beta \partial^4_{xxxx} u, \tag{1.11}$$

which is the Kuramoto-Sinelshchikov equation.

(1.11) was derived independently by Kuramoto [13, 14, 15] as a model for phase turbulence in reaction-diffusion systems and by Sivashinsky [22] as a model for plane flame propagation, describing the combined influence of diffusion and thermal conduction of the gas on stability of a plane flame front. The well-posedness and dynamical properties of (1.11) have been investigated in [11, 13, 19, 20]. Moreover, in [3], the convergence of the solution of (1.4) to the unique entropy solution of (1.3) is proven.

Observe that from the mathematical point of view, using the approximation (1.10) for (1.1), the sign of the flow can be both positive and negative and the power of the flow can be both even and odd. To prove the L^p- estimate, the $L^\infty-$ estimate is not needed. Moreover, if $\beta = \mathcal{O}\left(\varepsilon^2\right)$, we prove that the solution of (1.10) converges to the distributional solution of (1.9) and, following [10], we also prove the dissipation of energy.

Definition 1. *Let $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a function. We say that u is a distributional solution of (1.9) if for every test function $\varphi \in C^\infty(\mathbb{R}^2)$ with compact support we have that*

$$\int_0^\infty \int_{\mathbb{R}} (u\partial_t \varphi + \alpha u^n \partial_x \varphi) dt dx + \int_{\mathbb{R}} u_0(x) \varphi(0, x) dx = 0.$$

We say that u is an entropy solution of (1.9) *if for every nonnegative test function $\varphi \in C^\infty(\mathbb{R}^2)$ with compact support and every convex entropy $\eta \in C^2(\mathbb{R})$ with flux $q \in C^2(\mathbb{R})$ satisfying $q'(u) = \alpha n u^{n-1}\eta'(u)$ we have that*

$$\int_0^\infty\int_{\mathbb{R}} (\eta(u)\partial_t\varphi + q(u)\partial_x\varphi)dtdx + \int_{\mathbb{R}} \eta(u_0(x))\varphi(0,x)dx \geq 0.$$

The main result of this paper is the following theorem.

Theorem 1. *Let $\alpha,\, a \in \{-1, 1\}$, $u_0 : \mathbb{R} \to \mathbb{R}$ and $n \in \{4, 5, 6\}$, $m,\, \mu \in \mathbb{N}\setminus\{0\}$. m and μ are the smallest integers such that $2\,(n-1) < 4m$, $2\,(n-1) < 4\mu$. If*

$$\beta = \mathcal{O}\left(\varepsilon^2\right), \quad u_0 \in L^2(\mathbb{R}) \cap L^{4m}(\mathbb{R}), \tag{1.12}$$

then, there exist two sequences $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\{\beta_k\}_{k\in\mathbb{N}}$, with $\varepsilon_k, \beta_k \to 0$ and a limit function $u \in L^\infty((0,\infty); L^2(\mathbb{R}) \cap L^{4m}(\mathbb{R}))$, such that

i) $u_{\varepsilon_k,\beta_k} \to u$ strongly in $L^p_{loc}((0,\infty)\times\mathbb{R})$, for each $1 \leq p < 4m$,

ii) u is a distributional solution of (1.9).

In particular, we have

iii) dissipation of energy, that is

$$\partial_t\left(\frac{u^2}{2}\right) + \frac{\alpha}{n+1}\partial_x\left(u^{n+1}\right) \leq 0, \quad \textit{in weak sense on } (0,\infty)\times\mathbb{R}.$$

Moreover, if

$$\beta = o\left(\varepsilon^2\right), \quad u_0 \in L^2(\mathbb{R}) \cap L^{4\mu}(\mathbb{R}), \tag{1.13}$$

then,

iv) $u \in L^\infty((0,\infty); L^2(\mathbb{R}) \cap L^{4\mu}(\mathbb{R}))$,

v) $u_{\varepsilon_k,\beta_k} \to u$ strongly in $L^p_{loc}((0,\infty)\times\mathbb{R})$, for each $1 \leq p < 4\mu$,

vi) u is the unique entropy solution of (1.9).

The main technical tool of our argument is the following lemma [18].

Lemma 1. *Let Ω be a bounded open subset of $\mathbb{R}^2$. Suppose that the sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ lies in a bounded subset of $\mathcal{M}_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$.

Instead of (1.12), or (1.13), we assume directly

$$\beta \leq \frac{\varepsilon^2}{3}. \tag{1.14}$$

As in [3], (1.14) dot not depend on the initial datum. We dot not have a similar condition in [5,6].

Observe that $m \le \mu$. For example, if $n = 4$, then $m = \mu = 2$. Instead, if $n = 5, 6$, then $m = 2$ and $\mu = 3$.

In this paper paper, we analyze the cases $n = 4$ (see Section 2), $n = 6$ (see Section 3) and $n = 5$ (see Section 4). We consider the case $n = 6$ before the to $n = 5$ one, because, in the case $n = 5$, we use the same estimates for the case $n = 6$. We believe that similar arguments can be applied for every n.

The paper is organized in four sections. In Sections 2, 3, 4, we prove Theorem 1, when $n = 4$, $n = 6$ and $n = 5$, respectively.

2 The Korteweg-de Vries equation with flux u^4.

In this section, we prove Theorem 1, when $n = 4$. In this case, (1.1) reads

$$\partial_t u + \alpha\partial_x(u^4) + a\beta\partial^3_{xxx}u = 0, \tag{2.1}$$

while (1.9) reads

$$\partial_t u + \alpha\partial_x(u^4) = 0. \tag{2.2}$$

We augment (2.1) with the initial condition

$$u(0, x) = u_0(x),$$

on which we assume that

$$u_0 \in L^2(\mathbb{R}) \cap L^8(\mathbb{R}). \tag{2.3}$$

We study the dispersion-diffusion limit for (2.1). Therefore, we fix two small numbers ε, β and consider the following fourth order approximation

$$\begin{cases} \partial_t u_{\varepsilon,\beta} + \alpha\partial_x u^4_{\varepsilon,\beta} + a\beta\partial^3_{xxx}u_{\varepsilon,\beta} = \varepsilon\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^4_{xxxx}u_{\varepsilon,\beta}, & t > 0,\ x \in \mathbb{R}, \\ u_{\varepsilon,\beta}(0, x) = u_{\varepsilon,\beta,0}(x), & x \in \mathbb{R}, \end{cases} \tag{2.4}$$

where $u_{\varepsilon,\beta,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned} & u_{\varepsilon,\,\beta,\,0} \to u_0 \quad \text{in } L^p_{loc}(\mathbb{R}),\ 1 \le p < 8, \text{ as } \varepsilon,\ \beta \to 0, \\ & \|u_{\varepsilon,\beta,0}\|^8_{L^8(\mathbb{R})} + \|u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta\,\|\partial_x u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \le C_0, \quad \varepsilon, \beta > 0, \end{aligned} \tag{2.5}$$

and C_0 is a constant independent on ε and β.

With (2.5), we are able to prove *i*), *ii*), *iii*) of Theorem 1. To prove *iv*), *v*), *vi*), we need the following assumption

$$\varepsilon^2\,\|\partial_x u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \le C_0, \quad \varepsilon, \beta > 0. \tag{2.6}$$

Let us prove some a priori estimates on $u_{\varepsilon,\beta}$, denoting with C_0 the constants which depend only on the initial data.

Lemma 2. *For each $t > 0$,*

$$\begin{aligned} \|u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ 2\varepsilon\int_0^t \|\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds \\ &+ 2\beta\varepsilon\int_0^t \left\|\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le C_0. \end{aligned} \tag{2.7}$$

Proof. Multiplying (2.4) by $2u_{\varepsilon,\beta}$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\|u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx\\ =& -8\alpha\int_{\mathbb{R}} u^4_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx - 2a\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}dx\\ &+ 2\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta} - 2\beta\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta}dx\\ =& -2\varepsilon\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\beta^{\frac{3}{2}}\varepsilon\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\end{aligned}$$

that is

$$\frac{d}{dt}\|u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta\varepsilon\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = 0. \quad (2.8)$$

Integrating (2.8) on $(0,t)$, from (2.5), we have (2.7). □

Lemma 3. *Assume* (1.14). *For each* $t>0$,

$$\begin{aligned}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} &+ \frac{3\varepsilon}{2}\int_0^t \|u_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\ &+ \frac{5\beta\varepsilon}{2}\int_0^t \left\|u_{\varepsilon,\beta}(s,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le C_0.\end{aligned} \quad (2.9)$$

Proof. Multiplying (2.4) by $u^3_{\varepsilon,\beta}$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} &+ 3\varepsilon\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &+ 3\beta\varepsilon\left\|u^2_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ =& 3a\beta\int_{\mathbb{R}} u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}dx + 2\beta\varepsilon\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx.\end{aligned} \quad (2.10)$$

Due to (1.1), (1.14) and the Young inequality,

$$\begin{aligned}3|a|\beta\int_{\mathbb{R}} u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial^2_{xx}u_{\varepsilon,\beta}|dx &= \beta\int_{\mathbb{R}}\left|\frac{3u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}}{\varepsilon^{\frac{1}{2}}}\right|\left|\varepsilon^{\frac{1}{2}}u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}\right|dx\\ &\le \frac{9\beta}{2\varepsilon}\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta\varepsilon}{2}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\le \frac{\varepsilon}{2}\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta\varepsilon}{2}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, from (2.10), we get

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} &+ \frac{3\varepsilon}{2}\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &+ \frac{5\beta\varepsilon}{2}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le 2\beta\varepsilon\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx.\end{aligned} \quad (2.11)$$

To estimate the second term of (2.11), we use the following inequality (see [7, Lemma 4.2])

$$\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx \leq c_1 \left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \tag{2.12}$$

for some constant $c_1 > 0$. Due to (2.8) and (2.12), we have

$$\begin{aligned}\beta\varepsilon\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx \leq&\beta\varepsilon c_1 \left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leq& C_0\beta\varepsilon\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (2.11) and (2.11) that

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{3\varepsilon}{2}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad+\frac{5\beta\varepsilon}{2}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C_0\beta\varepsilon\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

(2.5), (2.7) and an integration on $(0,t)$ gives (2.9). □

Lemma 4. *For each $t > 0$,*

$$\begin{aligned}\frac{1}{8}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^8_{L^8(\mathbb{R})} &+ \frac{35\varepsilon}{6}\int_0^t\left\|u^3_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &+\frac{7\beta\varepsilon}{2}\int_0^t\left\|u^3_{\varepsilon,\beta}(s,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \leq C_0.\end{aligned} \tag{2.13}$$

The proof of the previous lemma is based on the regularity of the functions $u_{\varepsilon,\beta}$ and the following result.

Lemma 5 [Connected Inequality]. *We have that*

$$\begin{aligned}\int_{\mathbb{R}}u^4_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^4dx \leq&\frac{c_1^2}{2}\int_{\mathbb{R}}u^4_{\varepsilon,\beta}dx\int_{\mathbb{R}}u^2_{\varepsilon,\beta}dx\int_{\mathbb{R}}(\partial^2_{xx}u_{\varepsilon,\beta})^2dx\\ &+\frac{c_1}{2}\int_{\mathbb{R}}u^4_{\varepsilon,\beta}dx\int_{\mathbb{R}}u^2_{\varepsilon,\beta}(\partial^2_{xx}u_{\varepsilon,\beta})^2dx.\end{aligned} \tag{2.14}$$

Proof. Consider the auxiliary function

$$\begin{aligned}&v=\frac{1}{2}u^2_{\varepsilon,\beta}, \qquad \partial_x v = u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta},\\ &\partial^2_{xx}v = (\partial_x u_{\varepsilon,\beta})^2 + u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}.\end{aligned} \tag{2.15}$$

It follows from (2.12) that

$$\int_{\mathbb{R}}(\partial_x v)^4dx \leq c_1\int_{\mathbb{R}}v^2dx\int_{\mathbb{R}}(\partial^2_{xx}v)^2dx. \tag{2.16}$$

Therefore, from (2.15) and (2.16),

$$\int_{\mathbb{R}}u^4_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^4dx \leq \frac{c_1}{4}\int_{\mathbb{R}}u^4_{\varepsilon,\beta}dx\int_{\mathbb{R}}\left[(\partial_x u_{\varepsilon,\beta})^2 + u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}\right]^2dx. \tag{2.17}$$

Due to the Young inequality,

$$\left[(\partial_x u_{\varepsilon,\beta})^2 + u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}\right]^2 \leq 2(\partial_x u_{\varepsilon,\beta})^4 + 2u_{\varepsilon,\beta}^2(\partial_{xx}^2 u_{\varepsilon,\beta})^2.$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}} &\left[(\partial_x u_{\varepsilon,\beta})^2 + u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}\right]^2 dx \\ &\leq 2\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4 + 2\int_{\mathbb{R}} u_{\varepsilon,\beta}^2(\partial_{xx}^2 u_{\varepsilon,\beta})^2 dx.\end{aligned} \tag{2.18}$$

(2.17) and (2.18) give

$$\begin{aligned}\int_{\mathbb{R}} u_{\varepsilon,\beta}^4(\partial_x u_{\varepsilon,\beta})^4 dx \leq &\frac{c_1}{2}\int_{\mathbb{R}} u_{\varepsilon,\beta}^4 dx \int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4 dx \\ &+ \frac{c_1}{2}\int_{\mathbb{R}} u_{\varepsilon,\beta}^4 dx \int_{\mathbb{R}} u_{\varepsilon,\beta}^2(\partial_{xx}^2 u_{\varepsilon,\beta})^2 dx.\end{aligned} \tag{2.19}$$

(2.14) follows from (2.12) and (2.19). □

Proof. [Proof of Lemma 4] Multiplying (2.4) by $u_{\varepsilon,\beta}^7$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{1}{8}\frac{d}{dt}&\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^8(\mathbb{R})}^8 + 7\varepsilon\left\|u_{\varepsilon,\beta}^3(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \\ &+ 7\beta\varepsilon\left\|u_{\varepsilon,\beta}^3(t,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \\ =&7a\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}^6\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta} dx + 56\beta\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}^4(\partial_x u_{\varepsilon,\beta})^4 dx.\end{aligned} \tag{2.20}$$

Due to (1.1), (1.14), and the Young inequality

$$\begin{aligned}7|a|\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}^6|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx &= 7\beta\int_{\mathbb{R}}\left|\frac{u_{\varepsilon,\beta}^3\partial_x u_{\varepsilon,\beta}}{\varepsilon^{1/2}}\right|\left|\varepsilon^{\frac{1}{2}} u_{\varepsilon,\beta}^3\partial_{xx}^2 u_{\varepsilon,\beta}\right| dx \\ &\leq \frac{7\beta}{2\varepsilon}\left\|u_{\varepsilon,\beta}^3(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{7\beta\varepsilon}{2}\left\|u_{\varepsilon,\beta}^3(t,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{7\varepsilon}{6}\left\|u_{\varepsilon,\beta}^3(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{7\beta\varepsilon}{2}\left\|u_{\varepsilon,\beta}^3(t,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

Due to (2.7), (2.9) and (2.14),

$$\begin{aligned}56\beta\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}^4(\partial_x u_{\varepsilon,\beta})^4 dx \leq &\beta\varepsilon C_0\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \\ &+ C_0\beta\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

Hence, from (2.20), we gain

$$\begin{aligned}\frac{1}{8}\frac{d}{dt}&\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^8(\mathbb{R})}^8 + \frac{35\varepsilon}{6}\left\|u_{\varepsilon,\beta}^3(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \\ &+ \frac{7\beta\varepsilon}{2}\left\|u_{\varepsilon,\beta}^3(t,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \\ \leq&\beta\varepsilon C_0\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + C_0\beta\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

(2.13) follows from (2.5), (2.7), (2.9) and an integration on $(0,t)$. □

Lemma 6. *Assume* (2.5). *For each* $t > 0$,

$$\begin{aligned}\beta \|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3\beta\varepsilon}{2}\int_0^t \left\|\partial^2_{xx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\ + 2\beta^2\varepsilon \int_0^t \left\|\partial^3_{xxx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C_0.\end{aligned} \tag{2.21}$$

Proof. Multiplying (2.4) by $-2\beta\partial^2_{xx}u_{\varepsilon,\beta}$, integrating on $\mathbb{R}$, we have that

$$\begin{aligned}\beta\frac{d}{dt} \|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta\varepsilon \left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ + 2\beta^2\varepsilon \left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = 8\alpha\beta\int_{\mathbb{R}} u^3_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}dx.\end{aligned} \tag{2.22}$$

Due to (1.1), (1.14), and the Young inequality,

$$\begin{aligned}8|\alpha|\beta\int_{\mathbb{R}} |u^3_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}||\partial^2_{xx}u_{\varepsilon,\beta}|dx &= \beta\int_{\mathbb{R}}\left|\frac{8\beta u^3_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}}{\varepsilon^{1/2}}\right|\left|\varepsilon^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta}\right|dx \\ &\le \frac{32\beta}{\varepsilon}\left\|u^3_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta\varepsilon}{2}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &\le \frac{32\varepsilon}{3}\left\|u^3_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta\varepsilon}{2}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Then, from (2.22), we obtain that

$$\begin{aligned}\beta\frac{d}{dt} \|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3\beta\varepsilon}{2} \left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ + 2\beta^2\varepsilon \left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le \frac{32\varepsilon}{3}\left\|u^3_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

(2.5), (2.13) and an integration on $(0,t)$ give (2.21). □

Lemma 7. *Assume* (2.6). *For each* $t > 0$,

$$\begin{aligned}\varepsilon^2 \|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon^3\int_0^t \left\|\partial^2_{xx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\ + 2\beta\varepsilon^3 \int_0^t \left\|\partial^3_{xxx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C_0.\end{aligned} \tag{2.23}$$

Proof. Multiplying (2.4) by $-2\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta}$, integrating on $\mathbb{R}$, we have

$$\begin{aligned}\varepsilon^2\frac{d}{dt} \|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon^3 \left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta\varepsilon^3 \left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ = 8\alpha\varepsilon^2\int_{\mathbb{R}} u^3_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}dx.\end{aligned} \tag{2.24}$$

Due to (1.1) and the Young inequality,

$$\begin{aligned}8|\alpha|\varepsilon^2\int_{\mathbb{R}} |u^3_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}||\partial^2_{xx}u_{\varepsilon,\beta}|dx &= 2\int_{\mathbb{R}}\left|4\varepsilon^{\frac{1}{2}}u^3_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\right|\left|\varepsilon^{\frac{3}{2}}\partial^2_{xx}u_{\varepsilon,\beta}\right|dx \\ &\le 16\varepsilon\left\|u^3_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \varepsilon^3\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, from (2.24) we gain

$$\varepsilon^2 \frac{d}{dt} \|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon^3 \left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta\varepsilon^3 \left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq 16\varepsilon \left\|u^3_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

(2.5), (2.13) and an integration on $(0,t)$ gives (2.23). □

We begin by proving the following result.

Lemma 8. *Assume that* (1.12), (2.3) *and* (2.5) *hold. Then for any compactly supported entropy-entropy flux pair* (η, q), *there exist two sequences* $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\{\beta_k\}_{k\in\mathbb{N}}$, *with* $\varepsilon_k, \beta_k \to 0$ *and a limit function* $u \in L^\infty((0,\infty); L^2(\mathbb{R}) \cap L^8(\mathbb{R}))$, *such that*

$$u_{\varepsilon_k,\beta_k} \to u \quad \text{in } L^p_{loc}((0,\infty)\times\mathbb{R}), \text{ for each } 1 \leq p < 8, \tag{2.25}$$
$$u \quad \text{is a distributional solution of .}$$

Moreover,

$$\partial_t \left(\frac{u^2}{2}\right) + \frac{4\alpha}{5}\partial_x \left(u^5\right) \leq 0, \quad \text{in weak sense on } (0,\infty)\times\mathbb{R}. \tag{2.26}$$

Proof. Let us consider a compactly supported entropy–entropy flux pair (η, q). Multiplying (2.4) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned}\partial_t \eta(u_{\varepsilon,\beta}) + \alpha\partial_x q(u_{\varepsilon,\beta}) &= \varepsilon\eta'(u_{\varepsilon,\beta})\partial^2_{xx} u_{\varepsilon,\beta} - \beta\varepsilon\eta'(u_{\varepsilon,\beta})\partial^4_{xxxx} u_{\varepsilon,\beta} - a\beta\eta'(u_{\varepsilon,\beta})\partial^3_{xxx} u_{\varepsilon,\beta}\\ &= I_{1,\varepsilon,\beta} + I_{2,\varepsilon,\beta} + I_{3,\varepsilon,\beta} + I_{4,\varepsilon,\beta} + I_{5,\varepsilon,\beta} + I_{6,\varepsilon,\beta},\end{aligned}$$

where

$$\begin{aligned}
I_{1,\varepsilon,\beta} &= \partial_x(\varepsilon\eta'(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}), \quad I_{2,\varepsilon,\beta} = -\varepsilon\eta''(u_{\varepsilon,\beta})(\partial_x u_{\varepsilon,\beta})^2,\\
I_{3,\varepsilon,\beta} &= -\partial_x(\beta\varepsilon\eta'(u_{\varepsilon,\beta})\partial^3_{xxx} u_{\varepsilon,\beta}),\ I_{4,\varepsilon,\beta} = \beta\varepsilon\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^3_{xxx} u_{\varepsilon,\beta},\\
I_{5,\varepsilon,\beta} &= -\partial_x(a\beta\eta'(u_{\varepsilon,\beta})\partial^2_{xx} u_{\varepsilon,\beta}),\ I_{6,\varepsilon,\beta} = a\beta\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^2_{xx} u_{\varepsilon,\beta}.
\end{aligned} \tag{2.27}$$

Fix $T > 0$. Arguing as in [2, Lemma 3.2], we have that $I_{1,\varepsilon,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$ and $\{I_{2,\varepsilon,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$. We claim that

$$I_{3,\varepsilon,\beta} \to 0 \quad \text{in } H^{-1}((0,T)\times\mathbb{R}),\ T > 0, \text{ as } \varepsilon \to 0.$$

By (1.1) and Lemma 6,

$$\begin{aligned}&\left\|a\beta\varepsilon\eta'(u_{\varepsilon,\beta})\partial^3_{xxx} u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})}\\ &\quad\leq \beta^2\varepsilon^2 \|\eta'\|_{L^\infty(\mathbb{R})} \left\|\partial^3_{xxx} u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})} \leq C_0 \|\eta'\|_{L^\infty(\mathbb{R})} \varepsilon \to 0.\end{aligned}$$

We have that

$$\{I_{4,\varepsilon,\beta}\}_{\varepsilon,\beta>0} \quad \text{is bounded in } L^1((0,T)\times\mathbb{R}),\ T > 0.$$

By Lemmas 2, 6 and the Hölder inequality,

$$\begin{aligned}
&\left\|\beta\varepsilon\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{xxx}^3 u_{\varepsilon,\beta}\right\|_{L^1((0,T)\times\mathbb{R})}\\
&\le \beta\varepsilon\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\int_0^T\!\!\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial_{xxx}^3 u_{\varepsilon,\beta}|dtdx\\
&\le \beta\varepsilon\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\le C_0\left\|\eta''\right\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

We have that

$$I_{5,\,\varepsilon,\,\beta}\to 0\quad \text{in } H^{-1}((0,T)\times\mathbb{R}),\, T>0, \text{ as } \varepsilon\to 0.$$

By (1.1), (1.12) and Lemma 6,

$$\begin{aligned}
\left\|a\beta\eta'(u_{\varepsilon,\beta})\partial_{xx}^2 u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})} &\le \beta^2\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})}\\
&= \frac{\beta^2\varepsilon}{\varepsilon}\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\le \frac{\beta}{\varepsilon}C_0\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\le C_0\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\varepsilon\to 0.
\end{aligned}$$

We show that

$$\{I_{6,\,\varepsilon,\,\beta}\}_{\varepsilon,\beta>0}\quad \text{is bounded in } L^1((0,T)\times\mathbb{R}),\, T>0.$$

By (1.1), (1.12), Lemmas 2, 6 and the Hölder inequality,

$$\begin{aligned}
&\left\|a\beta\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}\right\|_{L^1((0,T)\times\mathbb{R})}\\
&\le \beta\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\int_0^T\!\!\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dtdx\\
&\le \frac{\beta\varepsilon}{\varepsilon}\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\\
&\le \frac{\beta^{\frac{1}{2}}}{\varepsilon}C_0\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\le C_0\left\|\eta''\right\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

Therefore, (2.25) follows from Lemmas 2, 4, 1 and the L^p compensated compactness of [21].

We begin by proving that u is a distributional solution of (2.2). Let $\phi\in C^2(\mathbb{R}^2)$ be a test function with compact support. We have to prove that

$$\int_0^\infty\!\!\int_{\mathbb{R}}\left(u\partial_t\phi+\alpha u^4\partial_x\phi\right)dx+\int_{\mathbb{R}}u_0(x)\phi(0,x)dx=0. \tag{2.28}$$

We define $u_k:=u_{\varepsilon_k,\,\beta_k}$, then we have

$$\begin{aligned}
&\int_0^\infty\!\!\int_{\mathbb{R}}\left(u_k\partial_t\phi+\alpha u_k^4\partial_x\phi\right)dtdx+\int_{\mathbb{R}}u_{0,\,k}(x)\phi(0,x)dx\\
&=-\varepsilon_k\int_0^\infty\!\!\int_{\mathbb{R}}u_k\partial_{xx}^2\phi dtdx+\beta_k\varepsilon_k\int_0^\infty\!\!\int_{\mathbb{R}}u_k\partial_{xxxx}^4\phi dtdx-\beta_k\int_0^\infty\!\!\int_{\mathbb{R}}u_k\partial_{xxx}^3\phi dtdx.
\end{aligned}$$

Therefore, (2.28) follows from (2.3) and (2.25).

We conclude by proving (2.26). Multiplying (2.4) by $u_{\varepsilon,\beta}$, we have

$$\begin{aligned}\partial_t\left(\frac{u_{\varepsilon,\beta}^2}{2}\right)+\frac{4\alpha}{5}\partial_x(u_{\varepsilon,\beta}^5)=\varepsilon u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}\\ -\beta\varepsilon u_{\varepsilon,\beta}\partial_{xxxx}^4 u_{\varepsilon,\beta}-a\beta u_{\varepsilon,\beta}\partial_{xxx}^3 u_{\varepsilon,\beta}.\end{aligned}\tag{2.29}$$

Let $\phi\in C_c^\infty((0,\infty)\times\mathbb{R})$ be a non-negative test function. Fix $T>0$. Multiplying (2.29) by ϕ, we get

$$\begin{aligned}&\int_0^\infty\!\!\int_{\mathbb{R}}\Big[\partial_t\Big(\frac{u_{\varepsilon,\beta}^2}{2}\Big)+\frac{4\alpha}{5}\partial_x(u_{\varepsilon,\beta}^5)\Big]\phi dtdx=\varepsilon\int_0^\infty\!\!\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_{xx}^2u_{\varepsilon,\beta}\phi dtdx\\&\quad-\beta\varepsilon\int_0^\infty\!\!\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_{xxxx}^4u_{\varepsilon,\beta}\phi dtdx-a\beta\int_0^\infty\!\!\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_{xxx}^3u_{\varepsilon,\beta}\phi dtdx\\&=-\varepsilon\int_0^\infty\!\!\int_{\mathbb{R}}(\partial_xu_{\varepsilon,\beta})^2\phi dtdx-\varepsilon\int_0^\infty\!\!\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_xu_{\varepsilon,\beta}\partial_x\phi dtdx\\&\quad-\beta\varepsilon\int_0^\infty\!\!\int_{\mathbb{R}}(\partial_{xx}^2u_{\varepsilon,\beta})^2\phi dtdx-\beta\varepsilon\int_0^\infty\!\!\int_{\mathbb{R}}\partial_xu_{\varepsilon,\beta}\partial_{xx}^2u_{\varepsilon,\beta}\partial_x\phi dtdx\\&\quad+\beta\varepsilon\int_0^\infty\!\!\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_{xxx}^3u_{\varepsilon,\beta}\partial_x\phi dtdx+a\beta\int_0^\infty\!\!\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_{xx}^2u_{\varepsilon,\beta}\partial_x\phi dtdx\\&\quad+a\beta\int_0^\infty\!\!\int_{\mathbb{R}}\partial_xu_{\varepsilon,\beta}\partial_{xx}^2u_{\varepsilon,\beta}\phi dtdx.\end{aligned}\tag{2.30}$$

Observe that

$$a\beta\int_0^\infty\!\!\int_{\mathbb{R}}\partial_xu_{\varepsilon,\beta}\partial_{xx}^2u_{\varepsilon,\beta}\phi dtdx=-\frac{a\beta}{2}\int_0^\infty\!\!\int_{\mathbb{R}}(\partial_xu_{\varepsilon,\beta})^2\partial_x\phi dtdx.\tag{2.31}$$

It follows from (1.1), (2.30), (2.31) that

$$\begin{aligned}&\int_0^\infty\!\!\int_{\mathbb{R}}\Big[\partial_t\Big(\frac{u_{\varepsilon,\beta}^2}{2}\Big)+\frac{4\alpha}{5}\partial_x(u_{\varepsilon,\beta}^5)\Big]\phi dtdx\\&\le\varepsilon\int_0^\infty\!\!\int_{\mathbb{R}}|u_{\varepsilon,\beta}||\partial_xu_{\varepsilon,\beta}|\partial_x\phi|dtdx+\beta\varepsilon\int_0^\infty\!\!\int_{\mathbb{R}}|\partial_xu_{\varepsilon,\beta}||\partial_{xx}^2u_{\varepsilon,\beta}||\partial_x\phi|dtdx\\&\quad+\beta\varepsilon\int_0^\infty\!\!\int_{\mathbb{R}}|u_{\varepsilon,\beta}||\partial_{xxx}^3u_{\varepsilon,\beta}||\partial_x\phi|dtdx+\beta\int_0^\infty\!\!\int_{\mathbb{R}}|u_{\varepsilon,\beta}||\partial_{xx}^2u_{\varepsilon,\beta}||\partial_x\phi|dtdx\\&\quad+\beta\int_0^\infty\!\!\int_{\mathbb{R}}(\partial_xu_{\varepsilon,\beta})^2|\partial_x\phi|dtdx\\&\le\varepsilon\|\partial_x\phi\|_{L^\infty((0,\infty)\times\mathbb{R})}\|u_{\varepsilon,\beta}\|_{L^2((0,T)\times\mathbb{R})}\|\partial_xu_{\varepsilon,\beta}\|_{L^2((0,T)\times\mathbb{R})}\\&\quad+\beta\varepsilon\|\partial_x\phi\|_{L^\infty((0,\infty)\times\mathbb{R})}\|\partial_xu_{\varepsilon,\beta}\|_{L^2((0,T)\times\mathbb{R})}\left\|\partial_{xx}^2u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\\&\quad+\beta\varepsilon\|\partial_x\phi\|_{L^\infty((0,\infty)\times\mathbb{R})}\|u_{\varepsilon,\beta}\|_{L^2((0,T)\times\mathbb{R})}\left\|\partial_{xxx}^3u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\\&\quad+\beta\|\partial_x\phi\|_{L^\infty((0,\infty)\times\mathbb{R})}\|u_{\varepsilon,\beta}\|_{L^2((0,T)\times\mathbb{R})}\left\|\partial_{xx}^2u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\\&\quad+\beta\|\partial_x\phi\|_{L^\infty((0,\infty)\times\mathbb{R})}\|\partial_xu_{\varepsilon,\beta}\|_{L^2((0,T)\times\mathbb{R})}^2.\end{aligned}\tag{2.32}$$

We prove that

$$\varepsilon \left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \to 0. \tag{2.33}$$

Due to Lemma 2,

$$\begin{aligned}\varepsilon \left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \\ \leq C_0\sqrt{T}\left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \varepsilon^{\frac{1}{2}} \to 0.\end{aligned}$$

We have that

$$\beta\varepsilon \left\|\phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \left\|\partial_{xxx}^3 u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \to 0. \tag{2.34}$$

Due to Lemmas 2 and 6,

$$\begin{aligned}\beta\varepsilon \left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \left\|\partial_{xx}^2 u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \\ \leq C_0 \left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \beta^{\frac{1}{2}} \to 0.\end{aligned}$$

We get

$$\beta\varepsilon \left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \left\|\partial_{xxx}^3 u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \to 0. \tag{2.35}$$

Again by Lemmas 2.7 and 6,

$$\begin{aligned}\beta\varepsilon \left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \left\|\partial_{xxx}^3 u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \\ \leq C_0 \left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \varepsilon^{\frac{1}{2}} \to 0.\end{aligned}$$

We show that

$$\beta \left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \left\|\partial_{xx}^2 u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \to 0. \tag{2.36}$$

By (1.12) and Lemmas 2, 6,

$$\begin{aligned}&\beta \left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \left\|\partial_{xx}^2 u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})} \\ &\leq C_0\sqrt{T}\left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \frac{\beta^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}} \leq C_0\sqrt{T}\left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \varepsilon^{\frac{1}{2}} \to 0.\end{aligned}$$

We have that

$$\beta \left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|\partial_x u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})} \to 0. \tag{2.37}$$

From (1.14) and Lemma 2

$$\begin{aligned}\beta \left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|\partial_x u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})} \leq C_0\varepsilon^2 \left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \\ \times \left\|\partial_x u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})} \leq C_0 \left\|\partial_x \phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \varepsilon \to 0.\end{aligned}$$

It follows from (2.25), (2.32), (2.33), (2.34), (2.35), (2.36) and (2.37) that

$$\int_0^\infty\!\!\int_{\mathbb{R}} \left[\partial_t\left(\frac{u^2}{2}\right) + \frac{4\alpha}{5}\partial_x\left(u^5\right)\right]\phi \leq 0,$$

that is (2.26). □

Following [17], we prove the following result.

Lemma 9. *Assume* (1.13)*,* (2.3)*,* (2.5) *and* (2.6) *hold. Then for any compactly supported entropy-entropy flux pair* (η, q)*, there exist two sequences* $\{\varepsilon_k\}_{k\in\mathbb{N}}$*,* $\{\beta_k\}_{k\in\mathbb{N}}$*, with* $\varepsilon_k, \beta_k \to 0$ *and a limit function*

$$u \in L^\infty((0,\infty); L^2(\mathbb{R}) \cap L^8(\mathbb{R})),$$

such that (2.25) *holds and* u *is the unique entropy solution of* (2.2)*.*

Proof. Let us consider a compactly supported entropy-entropy flux pair (η, q). Multiplying (2.4) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned}\partial_t\eta(u_{\varepsilon,\beta}) &+ \alpha\partial_x q(u_{\varepsilon,\beta})\\ &=\varepsilon\eta'(u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\eta'(u_{\varepsilon,\beta})\partial^4_{xxxx}u_{\varepsilon,\beta} - a\beta\eta'(u_{\varepsilon,\beta})\partial^3_{xxx}u_{\varepsilon,\beta}\\ &=I_{1,\,\varepsilon,\,\beta} + I_{2,\,\varepsilon,\,\beta} + I_{3,\,\varepsilon,\,\beta} + I_{4,\,\varepsilon,\,\beta} + I_{5,\,\varepsilon,\,\beta} + I_{6,\,\varepsilon,\,\beta},\end{aligned}$$

where $I_{1,\,\varepsilon,\,\beta}$, $I_{2,\,\varepsilon,\,\beta}$, $I_{3,\,\varepsilon,\,\beta}$, $I_{4,\,\varepsilon,\,\beta}$, $I_{5,\,\varepsilon,\,\beta}$, $I_{6,\,\varepsilon,\,\beta}$ are defined in (2.27).

Fix $T > 0$. Arguing as Lemma 8, we have that $I_{1,\,\varepsilon,\,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$ and $\{I_{2,\,\varepsilon,\,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$, $I_{3,\,\varepsilon,\,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$ and $I_{5,\,\varepsilon,\,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$. We claim

$$I_{4,\,\varepsilon,\,\beta} \to 0 \quad \text{in } L^1((0,T)\times\mathbb{R}),\ T>0, \text{ as } \varepsilon \to 0.$$

By (1.13), Lemmas 2, 7 and the Hölder inequality,

$$\begin{aligned}&\left\|\beta\varepsilon\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}\right\|_{L^1((0,T)\times\mathbb{R})}\\ &\quad\le \frac{\beta\varepsilon^{\frac12}}{\varepsilon^{\frac12}}\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\int_0^T\!\!\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial^3_{xxx}u_{\varepsilon,\beta}|dtdx\\ &\quad\le \frac{\beta\varepsilon^{\frac12}}{\varepsilon^{\frac12}}\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\left\|\partial^3_{xxx}u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\\ &\quad\le C_0\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\frac{\beta^{\frac12}}{\varepsilon^{\frac12}} \le C_0\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\varepsilon^{\frac12}\to 0.\end{aligned}$$

We have that

$$I_{6,\,\varepsilon,\,\beta} \to 0 \quad \text{in } L^1((0,T)\times\mathbb{R}),\ T>0, \text{ as } \varepsilon \to 0.$$

From (1.1), (1.13), Lemmas 2, 7 and the Hölder inequality,

$$\begin{aligned}&\left\|a\beta\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}\right\|_{L^1((0,T)\times\mathbb{R})}\\ &\quad\le \beta\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\int_0^T\!\!\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial^2_{xx}u_{\varepsilon,\beta}|dtdx\\ &\quad\le \frac{\beta\varepsilon}{\varepsilon}\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\left\|\partial^2_{xx}u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\\ &\quad\le \frac{\beta^{\frac12}}{\varepsilon}C_0\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\to 0.\end{aligned}$$

Therefore, (2.25) follows from Lemmas 2, 4, 1 and the L^p compensated compactness of [21].

We conclude by proving that u is the unique entropy solution. Let us consider a compactly supported entropy-entropy flux pair (η, q) and $\phi \in C^2_c((0,\infty)\times\mathbb{R})$ a non-negative function. Fix $T>0$. We have to prove,

$$\int_0^\infty\int_{\mathbb{R}} (\partial_t \eta(u) + \partial_x q(u))\phi dtdx \leq 0. \tag{2.38}$$

Due to (1.1), we have

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}} \left(\partial_t \eta(u_k) + \partial_x q(u_k)\right)\phi dtdx \\
&= \varepsilon_k \int_0^\infty\int_{\mathbb{R}} \partial_x \left(\eta'(u_k)\partial_x u_k\right)\phi dtdx - \varepsilon_k \int_0^\infty\int_{\mathbb{R}} \eta''(u_k)\left(\partial_x u_k\right)^2 \phi dtdx \\
&- \beta_k\varepsilon_k \int_0^\infty\int_{\mathbb{R}} \partial_x \left(\eta'(u_k)\partial^3_{xxx} u_k\right)\phi dtdx + \beta_k\varepsilon_k \int_0^\infty\int_{\mathbb{R}} \eta''(u_k)\partial_x u_k \partial^3_{xxx} u_k \phi dtdx \\
&- a\beta_k \int_0^\infty\int_{\mathbb{R}} \partial_x \left(\eta'(u_k)\partial^2_{xx} u_k\right)\phi dtdx + a\beta_k \int_0^\infty\int_{\mathbb{R}} \eta''(u_k)\partial_x u_k \partial^2_{xx} u_k dtdx \\
&\leq -\varepsilon_k \int_0^\infty\int_{\mathbb{R}} \eta'(u_k)\partial_x u_k \partial_x \phi dtdx + \beta_k\varepsilon_k \int_0^\infty\int_{\mathbb{R}} \eta'(u_k)\partial^3_{xxx} u_k \partial_x \phi dtdx \\
&+ \beta_k\varepsilon_k \int_0^\infty\int_{\mathbb{R}} \eta''(u_k)\partial_x u_k \partial^3_{xxx} u_k \phi dtdx - a\beta_k \int_0^\infty\int_{\mathbb{R}} \eta'(u_k)\partial^2_{xx} u_k \partial_x \phi dtdx \\
&+ a\beta_k \int_0^\infty\int_{\mathbb{R}} \eta''(u_k)\partial_x u_k \partial^2_{xx} u_k dtdx \\
&\leq \varepsilon_k \left\|\eta'\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|\partial_x u_k\right\|_{L^2((0,T)\times\mathbb{R})} \left\|\partial_x \phi\right\|_{L^2((0,T)\times\mathbb{R})} \\
&+ \beta_k\varepsilon_k \left\|\eta'\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|\partial^3_{xxx} u_k\right\|_{L^2((0,T)\times\mathbb{R})} \left\|\partial_x \phi\right\|_{L^2((0,T)\times\mathbb{R})} \\
&+ \beta_k\varepsilon_k \left\|\eta''\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|\phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|\partial_x u_k \partial^3_{xxx} u_k\right\|_{L^1((0,T)\times\mathbb{R})} \\
&+ \beta_k \left\|\eta'\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|\partial^2_{xx} u_k\right\|_{L^2((0,T)\times\mathbb{R})} \left\|\partial_x \phi\right\|_{L^2((0,T)\times\mathbb{R})} \\
&+ \beta_k \left\|\eta''\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|\phi\right\|_{L^\infty((0,\infty)\times\mathbb{R})} \left\|\partial_x u_k \partial^2_{xx} u_k\right\|_{L^1((0,T)\times\mathbb{R})}.
\end{aligned}$$

(2.38) follows from (1.13), (2.25) and Lemmas 2 and 7. □

Proof. [Proof of Theorem 1.] Theorem 1 follows from Lemmas 8 and 9. □

3 The Korteweg-de Vries equation with flux u^6.

In this section, we Theorem 1 when $n=6$. In this case, (1.1) reads

$$\partial_t u + \alpha\partial_x(u^6) + a\beta\partial^3_{xxx} u = 0, \tag{3.1}$$

while (1.9) reads

$$\partial_t u + \alpha\partial_x(u^6) = 0. \tag{3.2}$$

We augment (3.1) with the initial condition

$$u(0,x) = u_0(x),$$

on which we assume (2.3), or

$$u_0 \in L^2(\mathbb{R}) \cap L^{12}(\mathbb{R}). \tag{3.3}$$

We study the dispersion-diffusion limit for (3.1). Therefore, we fix two small numbers ε, β and consider the following fourth order approximation

$$\begin{cases} \partial_t u_{\varepsilon,\beta} + \alpha\partial_x(u^6_{\varepsilon,\beta}) + a\beta\partial^3_{xxx}u_{\varepsilon,\beta} = \varepsilon\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^4_{xxxx}u_{\varepsilon,\beta}, & t>0,\ x\in\mathbb{R},\\ u_{\varepsilon,\beta}(0,x) = u_{\varepsilon,\beta,0}(x), & x\in\mathbb{R}, \end{cases} \tag{3.4}$$

where $u_{\varepsilon,\beta,0}$ is a C^∞ approximation of u_0, on which we assume (2.5), or

$$\begin{aligned} &u_{\varepsilon,\,\beta,\,0} \to u_0 \quad \text{in } L^p_{loc}(\mathbb{R}),\ 1\le p<12, \text{ as } \varepsilon,\ \beta\to 0,\\ &\|u_{\varepsilon,\beta,0}\|^{12}_{L^{12}(\mathbb{R})} + \|u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \varepsilon^2\|\partial_x u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \le C_0, \quad \varepsilon,\beta>0, \end{aligned} \tag{3.5}$$

and C_0 is a constant independent on ε and β.

Remark 1. Since

$$6\int_{\mathbb{R}} u^6_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx = 6\int_{\mathbb{R}} u^8_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx = 6\int_{\mathbb{R}} u^{12}_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx = 0,$$

for (3.1), L^2-, L^4-, L^8- norms are conserved.

Lemma 10. *Assume* (2.5)*. For each* $t>0$*,* (2.21) *holds.*

Proof. We multiply (3.4) by $-2\beta\partial^2_{xx}u_{\varepsilon,\beta}$ and argue as in Lemma 6. □

Lemma 11. *Assume* (3.5)*. For each* $t>0$*,*

$$\begin{aligned} \frac{1}{12}\|u_{\varepsilon,\beta}(t,\cdot)\|^{12}_{L^{12}(\mathbb{R})} &+ \frac{55\varepsilon}{6}\int_0^t \left\|u^5_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &+ \frac{11\beta\varepsilon}{2}\int_0^t \left\|u^5_{\varepsilon,\beta}(s,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C_0. \end{aligned} \tag{3.6}$$

The proof of the previous lemma is based on the regularity of the functions $u_{\varepsilon,\beta}$ and the following result.

Lemma 12 [Connected Inequality]. *We have that*

$$\begin{aligned} \int_{\mathbb{R}} u^8_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^4dx \le &\frac{4c_1^3}{9}\int_{\mathbb{R}}u^6_{\varepsilon,\beta}dx\int_{\mathbb{R}}u^4_{\varepsilon,\beta}dx\int_{\mathbb{R}}u^2_{\varepsilon,\beta}dx\int_{\mathbb{R}}(\partial^2_{xx}u_{\varepsilon,\beta})^2dx\\ &+\frac{4c_1^2}{9}\int_{\mathbb{R}}u^6_{\varepsilon,\beta}dx\int_{\mathbb{R}}u^4_{\varepsilon,\beta}dx\int_{\mathbb{R}}u^2_{\varepsilon,\beta}(\partial^2_{xx}u_{\varepsilon,\beta})^2dx\\ &+\frac{8c_1^2}{9}\int_{\mathbb{R}}u^6_{\varepsilon,\beta}dx\int_{\mathbb{R}}u^2_{\varepsilon,\beta}dx\int_{\mathbb{R}}(\partial^2_{xx}u_{\varepsilon,\beta})^2dx\\ &+\frac{2c_1}{9}\int_{\mathbb{R}}u^6_{\varepsilon,\beta}dx\int_{\mathbb{R}}u^4_{\varepsilon,\beta}(\partial^2_{xx}u_{\varepsilon,\beta})^2dx. \end{aligned} \tag{3.7}$$

Proof. Consider the auxiliary function

$$v=\frac{1}{3}u^3_{\varepsilon,\beta},\quad \partial_x v=u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta},\quad \partial^2_{xx}v=2u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2+u^2_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}. \quad (3.8)$$

It follows from (2.12) that

$$\int_{\mathbb{R}}(\partial_x v)^4dx\le c_1\int_{\mathbb{R}}v^2dx\int_{\mathbb{R}}(\partial^2_{xx}v)^2dx. \quad (3.9)$$

Therefore, from (3.8) and (3.9),

$$\int_{\mathbb{R}}u^8_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^4dx\le\frac{c_1}{9}\int_{\mathbb{R}}u^6_{\varepsilon,\beta}dx\int_{\mathbb{R}}\left[2u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2+u^2_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}\right]^2dx. \quad (3.10)$$

Due the Young inequality,

$$\left[2u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2+u^2_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}\right]^2\le 8u^2_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^4+2u^4_{\varepsilon,\beta}(\partial^2_{xx}u_{\varepsilon,\beta})^2.$$

Again by the Young inequality,

$$\begin{aligned}&\int_{\mathbb{R}}\left[2u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2+u^2_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}\right]^2dx\\ &\quad\le 8\int_{\mathbb{R}}u^2_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^4dx+2\int_{\mathbb{R}}u^4_{\varepsilon,\beta}(\partial^2_{xx}u_{\varepsilon,\beta})^2dx\\ &\quad\le 8\int_{\mathbb{R}}u^4_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^4dx+8\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx+2\int_{\mathbb{R}}u^4_{\varepsilon,\beta}(\partial^2_{xx}u_{\varepsilon,\beta})^2dx\end{aligned} \quad (3.11)$$

(3.10) and (3.11) give

$$\begin{aligned}\int_{\mathbb{R}}u^8_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^4dx&\le\frac{8c_1}{9}\int_{\mathbb{R}}u^6_{\varepsilon,\beta}dx\int_{\mathbb{R}}u^4_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^4dx\\ &+\frac{8c_1}{9}\int_{\mathbb{R}}u^6_{\varepsilon,\beta}dx\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx+\frac{2c_1}{9}\int_{\mathbb{R}}u^6_{\varepsilon,\beta}dx\int_{\mathbb{R}}u^4_{\varepsilon,\beta}(\partial^2_{xx}u_{\varepsilon,\beta})^2dx.\end{aligned} \quad (3.12)$$

(2.12), (2.14) and (3.12) give (3.7). □

Proof. [Proof of Lemma 11] Multiplying (2.4) by $u^{11}_{\varepsilon,\beta}$, integrating on $\mathbb{R}$, we have that

$$\begin{aligned}&\frac{1}{12}\frac{d}{dt}\left\|u_{\varepsilon,\beta}(t,\cdot\right\|^{12}_{L^{12}(\mathbb{R})}+11\varepsilon\left\|u^5_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\qquad+11\beta\varepsilon\left\|u^5_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad=11a\beta\int_{\mathbb{R}}u^{10}_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}dx+330\beta\varepsilon\int_{\mathbb{R}}u^8_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^4dx.\end{aligned} \quad (3.13)$$

Thanks to (1.1), (1.14) and the Young inequality,

$$\begin{aligned}11|a|\beta\int_{\mathbb{R}}u^{10}_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}dx&=11\beta\int_{\mathbb{R}}\left|\frac{u^5_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}}{\varepsilon^{\frac{1}{2}}}\right|\left|\varepsilon^{\frac{1}{2}}u^5_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}\right|dx\\ &\le\frac{11\beta}{2\varepsilon}\left\|u^5_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{11\beta\varepsilon}{2}\left\|u^5_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\le\frac{11\varepsilon}{6}\left\|u^5_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{11\beta\varepsilon}{2}\left\|u^5_{\varepsilon,\beta}(t,\cdot)\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Observe that, from (2.9), (2.13) and the Young inequality,

$$\int_{\mathbb{R}} u_{\varepsilon,\beta}^6 dx \leq \frac{1}{2}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{1}{2}\|u_{\varepsilon,\beta}(t,\cdot)\|^8_{L^8(\mathbb{R})} \leq C_0. \tag{3.14}$$

Hence, due to (2.7), (2.9), (12) and (3.14),

$$\begin{aligned}\int_{\mathbb{R}} u_{\varepsilon,\beta}^8 (\partial_x u_{\varepsilon,\beta})^4 dx \leq & C_0\beta\varepsilon \left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &+ C_0\beta\varepsilon \left\|u_{\varepsilon,\beta}(t,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &+ C_0\beta\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\beta}^4 (\partial_{xx}^2 u_{\varepsilon,\beta})^2 dx.\end{aligned}$$

Therefore, the Young inequality, an integration on $(0,t)$, (2.7), (2.9), (2.13), and (3.13) give (3.6). □

Lemma 13. *Assume* (3.5)*. For each* $t > 0$*,* (2.23) *holds*

Proof. From (1.1), the Young inequality and (3.6), arguing as in Lemma 7, we have (2.23). □

We begin by proving the following result.

Lemma 14. *Assume that* (1.12)*,* (2.3) *and* (2.5) *hold. Then for any compactly supported entropy-entropy flux pair* (η, q)*, there exist two sequences* $\{\varepsilon_k\}_{k\in\mathbb{N}}$*,* $\{\beta_k\}_{k\in\mathbb{N}}$*, with* ε_k*,* $\beta_k \to 0$ *and a limit function* $u \in L^\infty((0,\infty); L^2(\mathbb{R}) \cap L^8(\mathbb{R}))$*, such that* (2.25) *holds and* u *is a distributional solution of* (3.2)*. Moreover,*

$$\partial_t\left(\frac{u^2}{2}\right) + \frac{6\alpha}{7}\partial_x\left(u^7\right) \leq 0, \quad \textit{in weak sense on } (0,\infty)\times\mathbb{R}. \tag{3.15}$$

Proof. We consider a compactly supported entropy-entropy flux pair (η, q), multiply (3.4) by $\eta'(u_{\varepsilon,\beta})$, argue as in Lemma 8 and have (2.25), u is a distributional solution and (3.15). □

Lemma 15. *Assume* (1.13)*,* (3.3) *and* (3.5) *hold. Then for any compactly supported entropy-entropy flux pair* (η, q)*, there exist two sequences* $\{\varepsilon_k\}_{k\in\mathbb{N}}$*,* $\{\beta_k\}_{k\in\mathbb{N}}$*, with* ε_k*,* $\beta_k \to 0$ *and a limit function*

$$u \in L^\infty((0,\infty); L^2(\mathbb{R}) \cap L^{12}(\mathbb{R})),$$

such that

$$u_{\varepsilon_k,\beta_k} \to u \quad \textit{in } L^p_{loc}((0,\infty)\times\mathbb{R}), \textit{ for each } 1 \leq p < 12, \tag{3.16}$$

$$u \quad \textit{is the unique entropy solution of } (3.2). \tag{3.17}$$

Proof. We consider a compactly supported entropy-entropy flux pair (η, q), multiplying (3.4) by $\eta'(u_{\varepsilon,\beta})$, argue as in Lemma 9 and get (3.16) and (3.17). □

Proof. [Proof of Theorem 1.] Theorem 1 follows from Lemmas 14 and 15. □

4 The Korteweg-de Vries equation with flux u^5.

In this section, we Theorem 1 when $n = 5$. In this case, (1.1) reads

$$\partial_t u + \alpha\partial_x(u^5) + a\beta\partial^3_{xxx}u = 0, \tag{4.1}$$

while (1.9) reads

$$\partial_t u + \alpha\partial_x(u^5) = 0. \tag{4.2}$$

We augment (4.1) with the initial condition

$$u(0, x) = u_0(x),$$

on which we assume (2.3), or (3.3).

We study the dispersion-diffusion limit for (4.1). Therefore, we fix two small numbers ε, β and consider the following fourth order approximation

$$\begin{cases} \partial_t u_{\varepsilon,\beta}+\alpha\partial_x(u^5_{\varepsilon,\beta})+a\beta\partial^3_{xxx}u_{\varepsilon,\beta}=\varepsilon\partial^2_{xx}u_{\varepsilon,\beta}-\beta\varepsilon\partial^4_{xxxx}u_{\varepsilon,\beta}, & t > 0,\ x \in \mathbb{R}, \\ u_{\varepsilon,\beta}(0, x) = u_{\varepsilon,\beta,0}(x), & x \in \mathbb{R}, \end{cases} \tag{4.3}$$

where $u_{\varepsilon,\beta,0}$ is a C^∞ approximation of u_0, on which we assume (2.5), or (3.5).

Remark 2. Since

$$5\int_{\mathbb{R}} u^5_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx = 5\int_{\mathbb{R}} u^6_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx = 5\int_{\mathbb{R}} u^{11}_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx = 0,$$

for (4.1), the L^2, L^4, L^8 norms are conserved. Moreover, assuming (3.5), since

$$5\int_{\mathbb{R}} u^{15}_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx = 0,$$

for (4.1), the L^{12} norm is also conserved.

Lemma 16. *Assume* (2.5)*. For each* $t > 0$*,* (2.21) *holds.*

Proof. We multiply (4.3) by $-2\beta\partial^2_{xx}u_{\varepsilon,\beta}$ and argue as in Lemma 6. □

Lemma 17. *Assume* (3.5)*. For each* $t > 0$*,* (2.23) *holds.*

Proof. From (1.1), the Young inequality, (2.13), (3.6), arguing as in Lemma 7, we have (2.23). □

We begin by proving the following result.

Lemma 18. *Assume that* (1.12)*,* (2.3) *and* (2.5) *hold. Then for any compactly supported entropy-entropy flux pair* (η, q)*, there exist two sequences* $\{\varepsilon_k\}_{k\in\mathbb{N}}$*,* $\{\beta_k\}_{k\in\mathbb{N}}$*, with* ε_k*,* $\beta_k \to 0$ *and a limit function* $u\in L^\infty((0,\infty); L^2(\mathbb{R})\cap L^8(\mathbb{R}))$*, such that* (2.25) *holds and* u *is a distributional solution of* (4.2)*. Moreover,*

$$\partial_t\left(\frac{u^2}{2}\right) + \frac{5\alpha}{6}\partial_x\left(u^6\right) \le 0, \quad \textit{in weak sense on } (0,\infty)\times\mathbb{R}. \tag{4.4}$$

Proof. Let us consider a compactly supported entropy-entropy flux pair (η, q). Multiplying (3.4) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned}&\partial_t\eta(u_{\varepsilon,\beta}) + \alpha\partial_x q(u_{\varepsilon,\beta})\\ &\qquad=\varepsilon\eta'(u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\eta'(u_{\varepsilon,\beta})\partial^4_{xxxx}u_{\varepsilon,\beta} - a\beta\eta'(u_{\varepsilon,\beta})\partial^3_{xxx}u_{\varepsilon,\beta}\\ &\qquad=I_{1,\,\varepsilon,\,\beta} + I_{2,\,\varepsilon,\,\beta} + I_{3,\,\varepsilon,\,\beta} + I_{4,\,\varepsilon,\,\beta} + I_{5,\,\varepsilon,\,\beta} + I_{6,\,\varepsilon,\,\beta},\end{aligned}$$

where $I_{1,\,\varepsilon,\,\beta}$, $I_{2,\,\varepsilon,\,\beta}$, $I_{3,\,\varepsilon,\,\beta}$, $I_{4,\,\varepsilon,\,\beta}$, $I_{5,\,\varepsilon,\,\beta}$, $I_{6,\,\varepsilon,\,\beta}$ are defined in (2.27). Arguing as in Lemma 8, we get the proof. □

Lemma 19. *Assume* (1.13)*,* (3.3) *and* (3.5) *hold. Then for any compactly supported entropy-entropy flux pair* $(\eta,\, q)$*, there exist two sequences* $\{\varepsilon_k\}_{k\in\mathbb{N}}$*,* $\{\beta_k\}_{k\in\mathbb{N}}$*, with* $\varepsilon_k,\, \beta_k \to 0$ *and a limit function* $u \in L^\infty((0,\infty); L^2(\mathbb{R})\cap L^{12}(\mathbb{R}))$*, such that* (3.16) *holds and* u *is the unique entropy solution of* (4.2)*.*

Proof. Let us consider a compactly supported entropy-entropy flux pair (η, q). Multiplying (3.4) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned}&\partial_t\eta(u_{\varepsilon,\beta}) + \alpha\partial_x q(u_{\varepsilon,\beta})\\ &\qquad=\varepsilon\eta'(u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\eta'(u_{\varepsilon,\beta})\partial^4_{xxxx}u_{\varepsilon,\beta} - a\beta\eta'(u_{\varepsilon,\beta})\partial^3_{xxx}u_{\varepsilon,\beta}\\ &\qquad=I_{1,\,\varepsilon,\,\beta} + I_{2,\,\varepsilon,\,\beta} + I_{3,\,\varepsilon,\,\beta} + I_{4,\,\varepsilon,\,\beta} + I_{5,\,\varepsilon,\,\beta} + I_{6,\,\varepsilon,\,\beta},\end{aligned}$$

where $I_{1,\,\varepsilon,\,\beta}$, $I_{2,\,\varepsilon,\,\beta}$, $I_{3,\,\varepsilon,\,\beta}$, $I_{4,\,\varepsilon,\,\beta}$, $I_{5,\,\varepsilon,\,\beta}$, $I_{6,\,\varepsilon,\,\beta}$ are defined in (2.27). Arguing as in Lemma 9, we get the proof. □

Proof. of Theorem 1. Theorem 1 follows from Lemmas 18 and 19. □

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