Mathematical Modelling and Analysis
Volume 21 Number 1, January 2016, 35–46
http://dx.doi.org/10.3846/13926292.2016.1128990

Publisher: Taylor&Francis and VGTU
http://www.tandfonline.com/TMMA
ISSN: 1392-6292
eISSN: 1648-3510

Universality Theorems for Some Composite Functions

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Received June 15, 2015; revised November 17, 2015; published online January 15, 2016

Abstract. In [5], it was proved that a collection consisting from Dirichlet L-functions and periodic Hurwitz zeta-functions is universal in the sense that the shifts of those functions approximate simultaneously a given collection of analytic functions. In the paper, we prove theorems on the universality of composite functions of the above collection.

Keywords: Dirichlet L-function, Hurwitz zeta-function, mixed joint universality, periodic Hurwitz zeta-function, universality.

AMS Subject Classification: 11M06; 11M41.

1 Introduction

In [16], Voronin discovered the universality property of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, on the approximation of analytic functions from a wide class by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. At the moment, it is known that the majority of zeta and L-functions are universal in the above sense. Also, some zeta and L-functions are jointly universal: their shifts approximate simultaneously a given collection of analytic functions. A series of works are devoted to mixed joint universality when a collection of analytic functions are approximated simultaneously by shifts of zeta-functions with Euler product and without Euler product. The first result in this direction belongs to H. Mishou who proved [14] the joint universality of the function $\zeta(s)$ and the Hurwitz zeta-function $\zeta(s, \alpha)$

with transcendental parameter α. This result has been generalized in [6] for a periodic zeta and a periodic Hurwitz zeta-functions. In [7], the mixed joint universality has been obtained for a wide collection consisting from periodic zeta and periodic Hurwitz zeta-functions. We remind that the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$, where α, $0<\alpha\leq 1$, is a fixed parameter and $\mathfrak{a}=\{a_m:\ m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\}$ is a periodic sequence of complex numbers, is a generalization of the classical Hurwitz zeta-function $\zeta(s,\alpha)$ when $a_m\equiv 1$, and is defined, for $\sigma>1$, by the series

$$\zeta(s,\alpha;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. In [2], the mixed joint universality for a system of functions

$$\zeta(s),\zeta(s,\alpha_1,\mathfrak{a}_{11}),\dots,\zeta(s,\alpha_1,\mathfrak{a}_{1l_1}),\dots,\zeta(s,\alpha_r,\mathfrak{a}_{r1}),\dots,\zeta(s,\alpha_r,\mathfrak{a}_{rl_r})$$

has been considered. In a series of papers [11,12,15], the Riemann zeta-function has been replaced by zeta-functions of certain cusp forms. In [5], in place of the function $\zeta(s)$ a collection of Dirichlet L-functions $L(s,\chi)$ has been put. We will state the latter result.

Let $D=\{s\in\mathbb{C}:\ \frac{1}{2}<\sigma<1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ and $H(K)$, $K\in\mathcal{K}$, the classes of continuous non-vanishing and continuous on K functions, respectively, which are analytic in the interior of K. Let $\text{meas}A$ be the Lebesgue measure of a measurable set $A\subset\mathbb{R}$. Suppose that $\mathfrak{a}_{jl}=\{a_{mjl}:\ m\in\mathbb{N}_0\}$ is a periodic sequence of complex numbers with minimal period $k_{jl}\in\mathbb{N}$, $j=1,\dots,r$, $l=1,\dots,l_j$. Let k_j be the least common multiple of the periods $k_{j1},\dots,k_{jl_j}$, and

$$A_j=\begin{pmatrix} a_{1j1} & a_{1j2} & \dots & a_{1jl_j}\\ a_{2j1} & a_{2j2} & \dots & a_{2jl_j}\\ \dots & \dots & \dots & \dots\\ a_{k_jj1} & a_{k_jj2} & \dots & a_{k_jjl_j}\end{pmatrix},\quad j=1,\dots,r.$$

Then, in [5], the following theorem has been proved.

Theorem 1. *Suppose that $\chi_1,\ \dots,\chi_d$ are pairwise non-equivalent Dirichlet characters, the numbers $\alpha_1,\dots,\alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$, and that* $\text{rank}(A_j)=l_j$, *$j=1,\dots,r$. For $j=1,\dots,d$, let $K_j\in\mathcal{K}$ and $f_j\in H_0(K_j)$, and, for $j=1,\dots,r$, $l=1,\dots,l_j$, let $K_{jl}\in\mathcal{K}$ and $f_{jl}\in H(K_{jl})$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\bigg\{\tau\in[0,T]:\ \sup_{1\leq j\leq d}\sup_{s\in K_j}|L(s+i\tau,\chi_j)-f_j(s)|<\varepsilon,$$
$$\sup_{1\leq j\leq r}\sup_{1\leq l\leq l_j}\sup_{s\in K_{jl}}|\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl})-f_{jl}(s)|<\varepsilon\bigg\}>0.$$

Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta. In [8] and [10], the Voronin theorem has been generalized for $F(\zeta(s))$ with certain operators $F:\ H(D)\to H(D)$, in [9], the universality of $F(\zeta(s),\zeta(s,\alpha))$ has been studied with operators $F:\ H^2(D)\to H(D)$. The papers [3] and [4] are devoted to the universality of the functions $F(L(s,\chi_1),\dots,L(s,\chi_{r_1}),\zeta(s,\alpha_1),\dots,\zeta(s,\alpha_{r_2}))$ for some operators $F:\ H^{r_1+r_2}(D)\to H(D)$. The aim of the present paper is the universality of composite functions of a collection of L and zeta-functions in Theorem 1, i.e., we consider the universality of the functions

$$F\big(L(s,\chi_1),\dots,L(s,\chi_d),\zeta(s,\alpha_1;\mathfrak{a}_{11}),\dots,\zeta(s,\alpha_1;\mathfrak{a}_{1l_1}),\dots,\\ \zeta(s,\alpha_r;\mathfrak{a}_{r1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{rl_r})\big)$$

for some operators F.

First we deal with approximation of functions from the class $H(K)$, $K\in\mathcal{K}$. Let, for brevity, $v=d+l_1+\cdots+l_r$. We say that the operator $F:\ H^v(D)\to H(D)$ belongs to the class $Lip(\beta_1,\dots,\beta_v)$, $\beta_1>0,\dots,\beta_v>0$, if the following hypotheses are satisfied:

1° For every polynomial $p=p(s)$ and all sets $K_1,\dots,K_d\in\mathcal{K}$, there exists an element $\underline{g}=(g_1,\dots,g_d,g_{11},\dots,g_{1l_1},\dots,g_{r1},\dots,g_{rl_r})\in F^{-1}\{p\}\subset H^v(D)$ such that $g_j\neq 0$ on K_j, $j=1,\dots,d$;

2° For all $K\in\mathcal{K}$, there exist a constant $c>0$ and sets $K_1,\dots,K_v\in\mathcal{K}$ such that, for all $(g_{j1},\dots,g_{jv})\in H^v(D)$, $j=1,2$,

$$\sup_{s\in K}|F(g_{11}(s),\dots,g_{1v}(s))-F(g_{21}(s),\dots,g_{2v}(s))|\\ \le c\sup_{1\le j\le v}\sup_{s\in K_j}|g_{1j}(s)-g_{2j}(s)|^{\beta_j}.$$

Theorem 2. *Suppose that $\chi_1,\ \dots,\chi_d$ are pairwise non-equivalent Dirichlet characters, the numbers $\alpha_1,\dots,\alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$,* rank(A_j)=l_j, j=$1,\dots,r$, *and that $F\in Lip(\beta_1,\dots,\beta_v)$. Let $K\in\mathcal{K}$ and $f(s)\in H(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\Big\{\tau\in[0,T]:\ \sup_{s\in K}\big|F\big(L(s+i\tau,\chi_1),\dots,L(s+i\tau,\chi_d),\\ \zeta(s+i\tau,\alpha_1;\mathfrak{a}_{11}),\dots,\zeta(s+i\tau,\alpha_1;\mathfrak{a}_{1l_1}),\dots,\zeta(s+i\tau,\alpha_r;\mathfrak{a}_{r1}),\dots,\\ \zeta(s+i\tau,\alpha_r;\mathfrak{a}_{rl_r})\big)-f(s)\big|<\varepsilon\Big\}>0.$$

We give an example of the operator $F\in Lip(\beta_1,\dots,\beta_v)$. Let, for $(g_1,\dots,g_d,g_{11},\dots,g_{1l_1},\dots,g_{r1},\dots,g_{rl_r})\in H^v(D)$,

$$F(g_1,\dots,g_d,g_{11},\dots,g_{1l_1},\dots,g_{r1},\dots,g_{rl_r})=c_1g_1^{(n_1)}+\cdots+c_dg_d^{(n_d)}\\ +c_{11}g_{11}^{(n_{11})}+\cdots+c_{1l_1}g_{1l_1}^{(n_{1l_1})}+\cdots+c_{r1}g_{r1}^{(n_{r1})}+\cdots+c_{rl_r}g_{rl_r}^{(n_{rl_r})},$$

where $c_1,\dots,c_d,\ c_{11},\dots,c_{1l_1},\dots,c_{r1},\dots,c_{rl_r}\in\mathbb{C}\setminus\{0\}$, $n_1,\dots,n_d$, $n_{11},\dots,n_{1l_1},\dots,n_{r1},\dots,n_{rl_r}\in\mathbb{N}$, and $f^{(n)}$ denotes the nth derivative of f. It is not difficult to see that, for each polynomial $p = p(s)$ and all sets $K_1, \ldots, K_d \in \mathcal{K}$, there exists an element $\underline{g} \in F^{-1}\{p\}$ such that $g_j(s) \neq 0$ on K_j, $j = 1, \ldots, d$. For example, if

$$p(s) = a_k s^k + a_{k-1} s^{k-1} + \cdots + a_0, \quad a_k \neq 0,$$

we can take $\underline{g} = (1, \ldots, 1, 1, \ldots, 1, 1 \ldots, g_{rl_r})$, where

$$g_{rl_r}(s) = \frac{1}{c_{rl_r}} \left(\frac{a_n s^{k+n_{rl_r}}}{(k+1)\cdots(k+n_{rl_r})} + \cdots + \frac{a_0 s^{n_{rl_r}}}{1 \cdots n_{rl_r}} \right).$$

Thus, hypothesis 1° of the class $Lip(\beta_1, \ldots, \beta_v)$ is satisfied.

Hypothesis 2° of the class $Lip(\beta_1, \ldots, \beta_v)$ follows from the Cauchy integral formula. We write F in a more convenient form

$$F(g_1, \ldots, g_v) = \sum_{j=1}^{v} c_j g_j^{(n_j)}.$$

Let $K \in \mathcal{K}$, and $K \subset G \subset K_1$, where G is an open set and $K_1 \in \mathcal{K}$. Moreover, let L be a simple closed contour lying in $K_1 \setminus G$ and containing inside the K. Then the Cauchy integral formula shows that, for $(g_{j1}, \ldots, g_{jv}) \in H^v(D)$, $j = 1, 2$, and $s \in K$,

$$\left|F(g_{11}(s), \ldots, g_{1v}(s)) - F(g_{21}(s), \ldots, g_{2v}(s))\right| = \left| \sum_{j=1}^{v} c_j \frac{n_j!}{2\pi i} \int_L \frac{g_{1j}(z) - g_{2j}(z)}{(z-s)^{n_j+1}} \mathrm{d}z \right|$$

$$\leq \sum_{j=1}^{v} |c_j| C_j \sup_{s \in L} |g_{1j}(s) - g_{2j}(s)| \leq c \sup_{1 \leq j \leq v} \sup_{s \in K_1} |g_{1j}(s) - g_{2j}(s)|$$

with some constants $C_j > 0$, $j = 1, \ldots, v$, and $c > 0$. Thus we have that $F \in Lip(1, \ldots, 1)$, and in this case, $K_1 = \cdots = K_v = K_1$.

Now we give some other classes of operators F. Let

$$S = \{g \in H(D) : \ g(s) \neq 0 \ \text{ or } \ g(s) \equiv 0\}.$$

Moreover, $v_1 = \sum_{j=1}^{r} l_j$.

Theorem 3. *Suppose that the characters $\chi_1, \ldots, \chi_d$, the numbers $\alpha_1, \ldots, \alpha_r$ and the sequences $\mathfrak{a}_{jl}$, $j = 1, \ldots, r$, $l = 1, \ldots, l_j$, satisfy the hypotheses of Theorem 2, and that $F : H^v(D) \to H(D)$ be a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap (S^d \times H^{v_1}(D))$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the assertion of Theorem 2 is true.*

We note that the hypothesis $(F^{-1}G) \cap (S^d \times H^{v_1}(D)) \neq \varnothing$ for every open set $G \subset H(D)$ is general but sufficiently complicated. Obviously, it is satisfied if every $g \in H(D)$ has a preimage in the set $S^d \times H^{v_1}(D)$. On the other hand, Theorem 3 implies the following modification of Theorem 2.

Theorem 4. *Suppose that the characters $\chi_1, \ldots, \chi_d$, the numbers $\alpha_1, \ldots, \alpha_r$ and the sequences $\mathfrak{a}_{jl}$, $j = 1, \ldots, r$, $l = 1, \ldots, l_j$, satisfy the hypotheses of Theorem 2, and that $F : H^v(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S^d \times H^{v_1}(D))$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the assertion of Theorem 2 is true.*

Clearly, hypothesis 2° of the class $Lip(\beta_1, \ldots, \beta_v)$ implies the continuity of F. However, hypothesis 1° is weaker than the requirement $(F^{-1}\{p\}) \cap (S^d \times H^{v_1}(D)) \neq \varnothing$.

Non-vanishing of the polynomial $p(s)$ in a bounded region can be controlled by its constant term. Therefore, sometimes it is more convenient to consider operators F on the space $H^v(D_V, D) = H^d(D_V) \times H^{v_1}(D)$, where, for $V > 0$, $D_V = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, |t| < V\}$. Analogically, let

$$S_V = \{g \in H(D_V) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Then we have the following result.

Theorem 5. *Suppose that the characters $\chi_1, \ldots, \chi_d$, the numbers $\alpha_1, \ldots, \alpha_r$ and the sequences $\mathfrak{a}_{jl}$, $j = 1, \ldots, r$, $l = 1, \ldots, l_j$, satisfy the hypotheses of Theorem 2, $K \in \mathcal{K}$, $f(s) \in H(K)$ and $V > 0$ is such that $K \subset D_V$. Let $F : H^v(D_V, D) \to H(D_V)$ be a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap (S_V^d \times H^{v_1}(D))$ is not empty. Then the assertion of Theorem 2 is true.*

For example, Theorem 5 can be applied for the operator

$$F(g_1, \ldots, g_v) = c_1 g_1^{(n_1)} + \cdots + c_d g_d^{(n_d)}, \quad n_1, \ldots, n_d \in \mathbb{N}.$$

Now we consider approximation of analytic functions from the image of the set $S^d \times H^{v_1}(S)$ of the operator $F : H^v(D) \to H(D)$.

Theorem 6. *Suppose that the characters $\chi_1, \ldots, \chi_d$, the numbers $\alpha_1, \ldots, \alpha_r$ and the sequences $\mathfrak{a}_{jl}$, $j = 1, \ldots, r$, $l = 1, \ldots, l_j$, satisfy the hypotheses of Theorem 2, and that $F : H^v(D) \to H(D)$ is a continuous operator. Let $K \subset D$ be a compact subset, and $f(s) \in F(S^d \times H^{v_1}(D))$. Then the assertion of Theorem 2 is true.*

It is not easy to describe the set $F(S^d \times H^{v_1}(D))$. The next theorem is an example with sufficiently simple set contained in $F(S^d \times H^{v_1}(D))$.

Suppose that $a_1, \ldots, a_k \in \mathbb{C}$ are pairwise different numbers, and

$$H_k(D) = \left\{g \in H(D) : (g(s) - a_j)^{-1} \in H(D), \ j = 1, \ldots, k\right\}.$$

Theorem 7. *Suppose that the characters $\chi_1, \ldots, \chi_d$, the numbers $\alpha_1, \ldots, \alpha_r$ and the sequences $\mathfrak{a}_{jl}$, $j = 1, \ldots, r$, $l = 1, \ldots, l_j$, satisfy the hypotheses of Theorem 2, and that $F : H^v(D) \to H(D)$ is a continuous operator such that $F(S^d \times H^{v_1}(D)) \supset H_k(D)$. For $k = 1$, let $K \in \mathcal{K}$, $f(s) \in H(K)$ and $f(s) \neq a_1$ on K. For $k \geq 2$, let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_k(D)$. Then the assertion of Theorem 2 is true.*

For example, let $k = 2$ and $a_1 = 1$, $a_2 = -1$. Then Theorem 7 implies the universality of the function

$$\sin\big(L(s, \chi_1) + \cdots + L(s, \chi_d) + \zeta(s, \alpha_1; \mathfrak{a}_{11}) + \cdots + \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}) + \cdots \\ + \zeta(s, \alpha_r; \mathfrak{a}_{r1}) + \cdots + \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})\big).$$

For this, it suffices to consider the equation

$$\frac{e^{i\Sigma(s)} - e^{-i\Sigma(s)}}{2i} = f, \quad f \in H(D),\ a_1 = 1,\ a_2 = -1,$$

where $\Sigma(s)$ is the sum under the sign of sin.

2 Proof of Theorem 2

Theorem 2 is a result of Theorem 1, properties of the class $Lip(\beta_1, \dots, \beta_v)$ and of the Mergelyan theorem on the approximation of analytic functions by polynomials. We state this theorem in the form of the next lemma.

Lemma 1. *Suppose that $K \subset \mathbb{C}$ is a compact subset with connected complement, and $f(s)$ is a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

Proof of the lemma can be found in [13] and [17].

Proof of Theorem 2. Lemma 1 implies the existence of the polynomial $p = p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{2.1}$$

Using hypothesis 1° of the class $Lip(\beta_1, \dots, \beta_v)$, we have that, for all sets $K_1, \dots, K_d \in \mathcal{K}$, there exists an element $(g_1, \dots, g_d, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}) \in F^{-1}\{p\}$ such that $g_j(s) \neq 0$ on K_j, $j = 1, \dots, d$. Suppose that $\tau \in \mathbb{R}$ satisfies the inequalities

$$\sup_{1\le j\le d} \sup_{s\in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < c^{-\frac{1}{\beta}} \left(\frac{\varepsilon}{4}\right)^{\frac{1}{\beta}}, \tag{2.2}$$

$$\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} ||\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < c^{-\frac{1}{\beta}} \left(\frac{\varepsilon}{4}\right)^{\frac{1}{\beta}}, \tag{2.3}$$

where the sets $K_1, \dots, K_d, K_{11}, \dots, K_{1l_1}, \dots, K_{r1}, \dots, K_{rl_r} \in \mathcal{K}$ correspond the set K in hypothesis 2° of the class $Lip(\beta_1, \dots, \beta_v)$, and $\beta = \min_{1\le j\le v} \beta_j$, with notation $K_{1l} = K_{d+l}$, $j = 1, \dots, l_1$, $\dots$, $K_{rl} = K_{d+l_1+\dots+l_{r-1}+l}$, $l = 1, \dots, l_r$. Then, in view of Theorem 1, the set of τ satisfying inequalities (2.2) and (2.3) has a positive lower density. Moreover, hypothesis 2° of the class $Lip(\beta_1, \dots, \beta_v)$ shows that, for such τ,

$$\sup_{s\in K} |F(\underline{L}(s + i\tau, \underline{\chi}, \underline{\alpha}, \underline{\mathfrak{a}})) - p(s) \le \sup_{1\le j\le d} \sup_{s\in K_j} |L(s + i\tau, \chi_j) - f_j(s)|^{\beta_j}$$

$$+ c \sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)|^{\beta_{jl}} \le 2cc^{-\frac{\beta}{\beta}} \left(\frac{\varepsilon}{4}\right)^{\frac{\beta}{\beta}} = \frac{\varepsilon}{2}. \tag{2.4}$$

Here $\underline{\chi} = (\chi_1, \dots, \chi_d)$, $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, $\underline{\mathfrak{a}} = (\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1,l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r})$ and

$$\underline{L}(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\mathfrak{a}}) = \big(L(s,\chi_1), \dots, L(s,\chi_d), \zeta(s,\alpha_1;\mathfrak{a}_{11}), \dots, \zeta(s,\alpha_1;\mathfrak{a}_{1l_1}), \dots, \zeta(s,\alpha_r;\mathfrak{a}_{r1}), \dots, \zeta(s,\alpha_r;\mathfrak{a}_{rl_r})\big),$$

and $\beta_{1l} = \beta_{d+l}$, $l = 1, \dots, l_1, \dots,$ $\beta_{rl} = \beta_{d+l_1+\dots+l_{r-1}+l}$, $l = 1, \dots, l_r$. Thus, by the above remark,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} \left| F\left(\underline{L}(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\mathfrak{a}})\right) - p(s) \right| < \frac{\varepsilon}{2} \right\} > 0.$$

Combining this with inequality (2.1) proves the theorem.

3 Elements of probability theory

For the proof of Theorems 3 – 7, we apply a probabilistic approach based on limit theorems for weakly convergent probability measures in the space of analytic functions. We start with a limit theorem for $\underline{L}(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\mathfrak{a}})$ obtained in [5], Theorem 2.

Denote by $\mathcal{B}(X)$ the Borel σ-field of the space X. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on the complex plane, and

$$\Omega = \prod_{p\in\mathcal{P}} \gamma_p, \qquad \hat{\Omega} = \prod_{m\in\mathbb{N}_0} \gamma_m,$$

where $\mathcal{P}$ is the set of all prime numbers, and $\gamma_p = \gamma$ for all $p \in \mathcal{P}$ and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. In view of the classical Tikhonov theorem, the tori Ω and $\hat{\Omega}$ with the product topology and pointwise multiplication are compact topological Abelian groups. Moreover, let

$$\underline{\Omega} = \Omega \times \hat{\Omega}_1 \times \dots \times \hat{\Omega}_r,$$

where $\hat{\Omega}_j = \hat{\Omega}$ for all $j = 1, \dots, r$. Then again $\underline{\Omega}$ is a compact topological Abelian group. This leads to the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, where $\underline{m}_H$ is the probability Haar measure on $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$. Denote by $\omega(p)$ the projection of the element $\omega \in \Omega$ to the coordinate space γ_p, $p \in \mathcal{P}$, and by $\hat{\omega}_j(m)$ the projection of an element $\hat{\omega}_j \in \hat{\Omega}_j$ to the coordinate space γ_m, $m \in \mathbb{N}_0$, $j = 1, \dots, r$. Let $p^k \| m$ mean that $p^k \mid m$ but $p^{k+1} \nmid m$. Extend the function $\omega(p)$ to the set $\mathbb{N}$ by taking

$$\omega(m) = \prod_{p^k \| m} \omega^k(m), \quad m \in \mathbb{N}.$$

Denote by $\underline{\omega} = (\omega, \hat{\omega}_1, \dots, \hat{\omega}_r)$ the elements of $\underline{\Omega}$, and, on the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, define the $H^v(D)$-valued random element $\underline{L}(s, \underline{\chi}, \underline{\alpha}, \underline{\omega}, \underline{\mathfrak{a}})$ by the formula

$$\underline{L}(s, \underline{\chi}, \underline{\alpha}, \underline{\omega}, \underline{\mathfrak{a}}) = \big(L(s,\omega,\chi_1), \dots, L(s,\omega,\chi_d), \zeta(s,\alpha_1,\omega_1;\mathfrak{a}_{11}), \dots, \zeta(s,\alpha_1,\omega_1;\mathfrak{a}_{1l_1}), \dots, \zeta(s,\alpha_r,\omega_r;\mathfrak{a}_{r1}), \dots, \zeta(s,\alpha_r,\omega_r;\mathfrak{a}_{rl_r})\big),$$

where

$$L(s,\omega,\chi_j)=\sum_{m=1}^{\infty}\frac{\chi_j(m)\omega(m)}{m^s},\quad j=1,\dots,d,$$

and

$$\zeta(s,\alpha_j,\omega_j;\mathfrak{a}_{jl})=\sum_{m=1}^{\infty}\frac{a_{mjl}\omega_j(m)}{(m+\alpha_j)^s},\quad j=1,\dots,r,\ l=1,\dots,l_j.$$

We note that the latter series are uniformly convergent on compact subsets of D for almost all $\underline{\omega}\in\underline{\Omega}$. Moreover, for almost $\underline{\omega}\in\underline{\Omega}$, $L(s,\omega,\chi_j)$ can be written in the form

$$L(s,\omega,\chi_j)=\prod_p\left(1-\frac{\chi_j(p)\omega(p)}{p^s}\right)^{-1}.$$

Denote by $P_{\underline{L}}$ the distribution of the random element $\underline{L}(s,\underline{\chi},\underline{\alpha},\underline{\omega},\underline{\mathfrak{a}})$, i.e., the probability measure

$$P_{\underline{L}}(A)=\underline{m}_H\left(\underline{\omega}\in\underline{\Omega}:\ \underline{L}(s,\underline{\chi},\underline{\alpha},\underline{\omega},\underline{\mathfrak{a}})\in A\right),\quad A\in\mathcal{B}(H^v(D)).$$

Then we have the following limit theorem [5].

Lemma 2. *Suppose that the numbers $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$. Then*

$$P_T(A)\overset{def}{=}\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\ \underline{L}(s+i\tau,\underline{\chi},\underline{\alpha},\underline{\mathfrak{a}})\in A\right\},\quad A\in\mathcal{B}(H^v(D))$$

converges weakly to $P_{\underline{L}}$ as $T\to\infty$.

For the proof a limit theorem for composite function $F(\underline{L}(s,\underline{\chi},\underline{\alpha},\underline{\mathfrak{a}}))$, we will apply an assertion on the preservation of the weak convergence under mappings. Let X_1 and X_2 be two metric spaces, and let $u:\ X_1\to X_2$ be a $(\mathcal{B}(X_1),\mathcal{B}(X_2))$-measurable mapping, i.e.,

$$u^{-1}\mathcal{B}(X_2)\subset\mathcal{B}(X_1).$$

Then every probability measure P on $(X_1,\mathcal{B}(X_1))$ induces the unique probability measure Pu^{-1} on $(X_2,\mathcal{B}(X_2))$ defined by

$$Pu^{-1}(A)=P(u^{-1}A),\quad A\in\mathcal{B}(X_2).$$

It is well known that the continuity of u implies its $(\mathcal{B}(X_1),\mathcal{B}(X_2))$-measurability.

Lemma 3. *Suppose that P_n converges weakly to P as $n\to\infty$, and that the mapping $u:\ X_1\to X_2$ is continuous. Then P_nu^{-1} converges weakly to Pu^{-1} as $n\to\infty$.*

Proof of the lemma is given in [1].

Lemma 4. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$ and that the operator $F: H^v(D) \to H(D)$ is continuous. Then*

$$P_{T,F}(A) \stackrel{def}{=} \frac{1}{T}\text{meas}\left\{\tau \in [0,T]: F(\underline{L}(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\mathfrak{a}})) \in A\right\}, \quad A \in \mathcal{B}(H^v(D)),$$

converges weakly to $P_{\underline{L}}F^{-1}$ as $T \to \infty$.

Proof. The lemma is an immediate consequence of Lemmas 2 and 3. □

For the proof of universality theorems for $F(\underline{L}(s+i\tau, \underline{\chi}, \underline{\alpha}, \underline{\mathfrak{a}}))$, we also need the explicit form of the support of the measure $P_{\underline{L}}F^{-1}$. We apply a result of [5] on the support of the measure $P_{\underline{L}}$.

Lemma 5. *Suppose that $\chi_1, \dots, \chi_d$ are pairwise non-equivalent Dirichlet characters, the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that* $\text{rank}(A_j) = l_j$, $j = 1, \dots, r$. *Then the support of $P_{\underline{L}}$ is the set $S^d \times H^{v_1}(D)$.*

Lemma 6. *Suppose that the hypotheses of Lemma 5 are satisfied, and that the operator $F: H^v(D) \to H(D)$ is continuous. Then the support of $P_{\underline{L}}F^{-1}$ is the closure of the set $F(S^d \times H^{v_1}(D))$.*

Proof. Let g be an arbitrary element of the set $F(S^d \times H^{v_1}(D))$, and G be any open neighbourhood of g. Then $F^{-1}G$ is an open neighbourhood of a certain element of the set $S^d \times H^{v_1}(D)$. Therefore, Lemma 5 and properties of a support imply that $P_{\underline{L}}(F^{-1}G) > 0$, hence, $P_{\underline{L}}F^{-1}(G) > 0$. Moreover, in virtue of Lemma 5 again,

$$P_{\underline{L}}F^{-1}(F(S^d \times H^{v_1}(D))) = P_{\underline{L}}(S^d \times H^{v_1}(D)) = 1.$$

Since the support of $P_{\underline{L}}F^{-1}$ is a closed set, from this the lemma follows. □

We also need one equivalent of the weak convergence of probability measures.

Lemma 7. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(X, \mathcal{B}(X))$. Then, P_n, as $n \to \infty$, converges weakly to P if and only if, for every open set $G \subset X$,*

$$\liminf_{n\to\infty} P_n(G) \geq P(G).$$

The lemma is a part of Theorem 2.1 from [1].

4 Proofs of other universality theorems

Proof of Theorem 3. It is not difficult to see that, under hypotheses of Theorem 3, the support of the measure $P_{\underline{L}}F^{-1}$ is the whole of $H(D)$. Indeed, if $(F^{-1}G) \cap (S^d \times H^{v_1}(D)) \neq \varnothing$ for every open set $G \subset H(D)$, then we have that, for every element $g \in H(D)$ and its open neighbourhood G, there exists an element of the set $F(S^d \times H^{v_1}(D))$ which belongs to the set G. This shows that the set $F(S^d \times H^{v_1}(D))$ is dense in $H(D)$. Since, by Lemma 6, the support of $P_{\underline{L}}F^{-1}$ is the closure of $F(S^d \times H^{v_1}(D))$, from this we obtain that the support of $P_{\underline{L}}F^{-1}$ is the whole of $H(D)$.

In view of Lemma 1, there exists a polynomial $p(s)$ satisfying inequality (2.1). Define the set

$$G = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

Obviously, G is an open neighbourhood of $p(s)$ which is an element of the support of $P_{\underline{L}}F^{-1}$. Therefore, $P_{\underline{L}}F^{-1}(G) > 0$, and Lemmas 4 and 7 imply the inequality

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : F(\underline{L}(s + i\tau, \underline{\chi}, \underline{\alpha}, \underline{\mathfrak{a}})) \in G \right\} > 0,$$

or, by definition of G,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |F(\underline{L}(s + i\tau, \underline{\chi}, \underline{\alpha}, \underline{\mathfrak{a}})) - p(s)| < \frac{\varepsilon}{2} \right\} > 0.$$

Combining this with (2.1) gives the assertion of the theorem.

Proof of Theorem 4. Let $\{K_l : l \in \mathbb{N}\} \subset D$ be the sequence of compact subsets such that $K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$,

$$D = \bigcup_{l=1}^{\infty} K_l,$$

and, for every compact subset $K \subset D$, there exists K_l such that $K \subset K_l$. For $g_1, g_2 \in H(D)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}.$$

Then ρ is a metric on $H(D)$ which induces its topology of uniform convergence on compacta. Moreover, from the definition of ρ we have that the function g_2 approximates g_1 with suitable accuracy if g_2 approximate g_1 uniformly on K_l for sufficiently large l. Obviously, we may choose the sets K_l with connected complements, for example, we can take the closed rectangles. Thus, in $H(D)$, we can limit ourselves by approximation of functions on compact subsets with connected complements.

We will show that the hypotheses of the theorem imply those of Theorem 3. Let $G \subset H(D)$ be an arbitrary non-empty open set. Then, in view of Lemma 1 and the above remark on approximation in $H(D)$, there exists a polynomial $p(s) \in G$. Therefore, the hypothesis $(F^{-1}\{p\}) \cap (S^d \times H^{v_1}(D)) \neq \varnothing$ implies that of Theorem 3 that the set $(F^{-1}G) \cap (S^d \times H^{v_1}(D))$ is non-empty. Thus, Theorem 4 is a corollary of Theorem 3.

Proof of Theorem 5. We apply the arguments used in the proof of Theorem 3 with obvious modifications.

Proof of Theorem 6. Since $f(s) \in F(S^d \times H^{v_1}(D))$, it follows from Lemma 6 that $f(s)$ is an element of the support of the measure $P_{\underline{L}}F^{-1}$. Hence, $P_{\underline{L}}F^{-1}(G) > 0$ for

$$G = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \varepsilon \right\}.$$

Therefore, the theorem is a consequence of Lemmas 4 and 7.

Proof of Theorem 7. First suppose that $k = 1$. By Lemma 1, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{4.1}$$

Since $f(s) \neq a_1$ on K, then also $p(s) \neq a_1$ on K if $\varepsilon > 0$ is rather small. Therefore, on K we can define a continuous branch of the logarithm $\log(p(s) - a_1)$ which will be an analytic function in the interior of K. Again by Lemma 1, we can find a polynomial $q(s)$ such that

$$\sup_{s \in K} \left| p(s) - a_1 - e^{q(s)} \right| < \frac{\varepsilon}{4}. \tag{4.2}$$

Let, for brevity, $g_{a_1}(s) = a_1 + e^{q(s)}$. Then, clearly, $g_{a_1}(s) \in H(D)$, and $g_{a_1}(s) \neq a_1$ on D. Thus, $g_{a_1}(s) \in H_1(D)$. In view of Lemma 6, the support of the measure $P_{\underline{L}}F^{-1}$ contains the closure of the set $H_1(D)$. Therefore, the function $g_{a_1}(s)$ is an element of the support of the measure $P_{\underline{L}}F^{-1}$. Hence, $P_{\underline{L}}F^{-1}(G) > 0$, where

$$G = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - g_{a_1}(s)| < \frac{\varepsilon}{2} \right\}.$$

This together with Lemmas 4 and 7 shows that

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |F(\underline{L}(s + i\tau, \underline{\chi}, \underline{\alpha}, \underline{\mathfrak{a}})) - g_{a_1}(s)| < \frac{\varepsilon}{2} \right\} > 0. \tag{4.3}$$

Inequalities (4.1) and (4.2) imply that

$$\sup_{s \in K} |f(s) - g_{a_1}(s)| < \frac{\varepsilon}{2}.$$

This and (4.3) yield the assertion of the theorem in the case $k = 1$.

The case $k \geq 2$ is contained in Theorem 6.

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