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On Fractional Volterra Integrodifferential Equations with Fractional Integrable Impulses

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Abstract. We consider a class of nonlinear fractional Volterra integrodifferential equation with fractional integrable impulses and investigate the existence and uniqueness results in the Bielecki's normed Banach spaces. Further, Bielecki-Ulam type stabilities have been demonstrated on a compact interval. A concrete example is provided to illustrate the outcomes we acquired.

Keywords: fractional Volterra integrodifferential equation, integrable impulses, Banach contraction principle, existence of solutions, Bielecki norm, Bielecki-Ulam type stability.

AMS Subject Classification: 34A37; 45M10; 34G20; 34A08.

1 Introduction

Famous "Ulam stability problem" of functional differential equation raised by Ulam [10] have been extended to different kinds of equations. Wang et al. [13] are the first mathematicians who investigated the Ulam stability and data dependence for fractional differential equations. Thereafter, several interesting works on different Ulam type stabilities of fractional differential and integral equations have been reported (see for instance [3], [14], [11], [18], [19]).

An overview pertaining to impulsive differential equations with instantaneous impulses and its applicability in the practical dynamical systems have provided in the monograph [1, 2, 9]. The impulsive differential equations with time variable impulses dealt in an interesting papers [4,5,6]. Wang et al. [12,17]

studied existence and uniqueness of solutions and established generalized β-Ulam-Hyers-Rassias stability to differential equations with non instantaneous impulses in a $P\beta$-normed Banach space. Zada et al. [15, 20, 21, 22, 23, 24, 25] investigated Ulam-types stabilities for various classes of differential equations with and without impulse effect.

Recently, Wang and Zang [16] introduced a new class of impulsive differential equations of the form

$$\begin{cases} x'(\tau) = f\left(\tau, x(\tau)\right), \tau \in (\sigma_i, \tau_{i+1}],\ i = 0, 1, \ldots, m, \\ x(\tau) = I^{\beta}_{\tau_i,\tau} h_i(\tau, x(\tau)),\ \tau \in (\tau_i, \sigma_i],\ i = 1, 2, \ldots, m,\ \beta \in (0,1) \end{cases}$$

and examined the existence and uniqueness of solutions in Bielecki's normed Banach spaces. Further, demonstrated that the corresponding equations are Bielecki-Ulam-Hyer stable. It is seen that such sort of formulations are adequate to depict the memory procedures of the drugs in the circulation system and the subsequent absorption for the body.

Motivated by the work of Wang and Zang [16], we consider the following class of nonlinear fractional Volterra integrodifferential equation with fractional order integrable impulses of the form

$$\begin{cases} {}^{c}_{\sigma_i}\mathcal{D}^{\alpha}_{\tau} x(\tau) = f\left(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma))d\sigma\right), \tau \in (\sigma_i, \tau_{i+1}], \\ \qquad\qquad\qquad\qquad i = 0, 1, \ldots, m, \quad \alpha \in (0,1), \\ x(\tau) = \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, x(\tau)),\ \tau \in (\tau_i, \sigma_i],\ i = 1, 2, \ldots, m,\ \beta \in (0,1) \end{cases} \tag{1.1}$$

and research the existence and uniqueness of solutions and examine the outcomes relating to Bielecki-Ulam type stabilities viz. Bielecki-Ulam-Hyers and Bielecki-Ulam-Hyers-Rassias stabilities on a compact interval. Here τ_i and σ_i are pre-fixed numbers satisfying $0 = \tau_0 = \sigma_0 < \tau_1 \leq \sigma_1 \leq \tau_2 < \ldots < \tau_m \leq \sigma_m < \tau_{m+1} = T$, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ and for each $i = 1, 2 \ldots m$, $h_i : (\tau_i, \sigma_i] \times \mathbb{R} \to \mathbb{R}$ is continuous function, ${}^{c}_{\sigma_i}\mathcal{D}^{\alpha}_{\tau}$ is the Caputo fractional derivative of order α with lower terminal at σ_i and $\mathcal{I}^{\beta}_{\tau_i,\tau}$ is the Riemann-Liouville fractional integral of order β with lower terminal τ_i.

We comment that within this scope, the class of equations considered in the present paper is more broad and the outcomes acquired are the generalization of the fundamental results obtained by Wang and Zang [16]. We support our main results with the examples.

In Section 2, we introduce some preliminaries and auxiliary lemmas related to fractional calculus. In Section 3, we prove existence and uniqueness results for (1.1) by using Banach contraction principle via Bielecki norm. In Section 4, adopting the idea of Wang and Zang [16] we examine different Bielecki-Ulam's type stability for the problem (1.1). Finally, an example has been provided to illustrate the results we obtained.

2 Preliminaries

Let $J = [0, T]$. Let $C(J, \mathbb{R}) = \{x : J \to \mathbb{R}\,;\, x \,\text{is continuous function}\}$ be the Banach space endowed with a Bielecki norm

$$\|x\|_B = \sup_{\tau \in J} \frac{|x(\tau)|}{e^{\theta\tau}},$$

where $\theta > 0$ is a fixed real number. Let

$$PC(J,\mathbb{R}) = \Big\{x : J \to \mathbb{R} \,:\, x \in C((\tau_i, \tau_{i+1}], \mathbb{R}),\ \text{both } x(\tau_i^+) \text{ and } x(\tau_i^-) \text{ exists} \\ \text{and } x(\tau_i) = x(\tau_i^-),\ i = 0, 1, \ldots, m\Big\}.$$

If $PC(J, \mathbb{R})$ is endowed with the norm

$$\|x\|_{PB} = \sup_{\tau \in J} \frac{|x(\tau)|}{e^{\theta\tau}},\ \theta > 0,$$

then $\Big(PC(J, \mathbb{R}),\ \|\cdot\|_{PB}\Big)$ is a Banach space. The space

$$PC^1(J, \mathbb{R}) = \Big\{x \in PC(J, \mathbb{R}) \,:\, x' \in PC(J, \mathbb{R})\Big\}$$

with the norm $\|x\|_{PB'} = \max\{\|x\|_{PB},\ \|x'\|_{PB}\}$ then $\Big(PC^1(J, \mathbb{R}),\ \|\cdot\|_{PB'}\Big)$ is a Banach space.

Next, we use definitions and the results listed bellow from fractional calculus. For more details, we refer the readers to the monograph [7].

Definition 1. Let $g \in C[a, T]$ with $T > a \geq 0$ and $\beta \geq 0$, then the Riemann-Liouville fractional integral $\mathcal{I}^{\beta}_{a,\tau}$ of order β of a function g is defined as

$$\mathcal{I}^{\beta}_{a,\tau}\, g(\tau) = \frac{1}{\Gamma(\beta)} \int_a^{\tau} (\tau - \sigma)^{\beta - 1} g(\sigma) d\sigma,\ \ \tau > a \geq 0,$$

provided the integral exists.

Definition 2. Let $0 < \alpha \leq 1$ then the Caputo fractional derivative ${}^{c}_{a}\mathcal{D}^{\alpha}_{\tau}$ of order α with lower terminal a of a function $g \in C^1[a, T]$ is defined as

$${}^{c}_{a}\mathcal{D}^{\alpha}_{\tau} g(\tau) = \frac{1}{\Gamma(1-\alpha)} \int_a^{\tau} (\tau - \sigma)^{-\alpha} g'(\sigma) d\sigma,\ \ \tau > a \geq 0.$$

Lemma 1. *Let $m - 1 < \alpha \leq m \in \mathbb{N}$ and $g \in C^m[a, T]$. Then*

$$\mathcal{I}^{\alpha}_{a,\tau}[{}^{c}_{a}\mathcal{D}^{\alpha}_{\tau} g(\tau)] = g(\tau) - \sum_{k=0}^{m-1} \frac{g^{(k)}(a)}{\Gamma(k+1)} \tau^k,\ \ \tau > a \geq 0.$$

The following lemma plays an important role to obtain our results.

Lemma 2. *[8] Let α,β,γ and p be constants such that $\alpha>0,\ p(\gamma-1)+1>0$ and $p(\beta-1)+1>0$. Then*

$$\int_0^\tau (\tau^\alpha-\sigma^\alpha)^{p(\beta-1)}\sigma^{p(\gamma-1)}d\sigma=\frac{\tau^\theta}{\alpha}\mathbb{B}\left(\frac{p(\gamma-1)+1}{\alpha},p(\beta-1)+1\right),\ \ \forall\tau\in\mathbb{R}_+,$$

where

$$\mathbb{B}(\xi,\sigma)=\int_0^1\sigma^{\xi-1}(1-\sigma)^{\sigma-1}d\sigma,\ \ Re(\xi)>0,\ Re(\sigma)>0$$

is a Beta function and $\theta=p[\alpha(\beta-1)+\gamma-1]+1$.

3 Existence and uniqueness results

Lemma 3. *A function $x\in PC^1(J,\mathbb{R})$ is a classical solution of the problem*

$$\begin{cases} {}^c_{\sigma_i}\mathcal{D}^\alpha_\tau x(\tau)=f\left(\tau,x(\tau),\int_{\sigma_i}^\tau h(\sigma,x(\sigma))d\sigma\right),\ \tau\in(\sigma_i,\tau_{i+1}],\\ \qquad\qquad\qquad\qquad\qquad i=0,1,\ldots,m,,\quad \alpha\in(0,1),\\ x(\tau)=\mathcal{I}^\beta_{\tau_i,\tau}h_i(\tau,x(\tau)),\ \tau\in(\tau_i,\sigma_i],\ i=1,2,\ldots,m,\ \beta\in(0,1),\\ x(0)=x_0,\end{cases}\tag{3.1}$$

if x satisfies the fractional Volterra integral equations

$$x(\tau)=\begin{cases} x_0,\ \tau=0,\\ \mathcal{I}^\beta_{\tau_i,\tau}h_i(\tau,x(\tau)),\ \tau\in(\tau_i,\sigma_i],\ i=1,\ldots,m,\\ x_0+\mathcal{I}^\alpha_{0,\tau}f\left(\tau,x(\tau),\int_0^\tau h(\sigma,x(\sigma))d\sigma\right),\ \tau\in(0,\tau_1],\\ \mathcal{I}^\beta_{\tau_i,\sigma_i}h_i(\sigma_i,x(\sigma_i))+\mathcal{I}^\alpha_{\sigma_i,\tau}f\left(\tau,x(\tau),\int_{\sigma_i}^\tau h(\sigma,x(\sigma))d\sigma\right),\\ \qquad\qquad\qquad\qquad\qquad \tau\in(\sigma_i,\tau_{i+1}],i=1,\ldots,m.\end{cases}$$

Proof. For $i=0$, on operating Riemann-Liouville fractional integral operator $\mathcal{I}^\alpha_{\sigma_i,\tau}$ on both sides of fractional differential equation (3.1), we get

$$\mathcal{I}^\alpha_{0,\tau}[{}^c_0\mathcal{D}^\alpha_\tau x(\tau)]=\mathcal{I}^\alpha_{0,\tau}f\left(\tau,x(\tau),\int_0^\tau h(\sigma,x(\sigma))d\sigma\right),\ \tau\in(0,\tau_1].$$

As $0<\alpha<1$, in view of Lemma 1, we obtain

$$x(\tau)-x(0)=\mathcal{I}^\alpha_{0,\tau}f\left(\tau,x(\tau),\int_0^\tau h(\sigma,x(\sigma))d\sigma\right),\ \tau\in(0,\tau_1].$$

Therefore

$$x(\tau)=x_0+\mathcal{I}^\alpha_{0,\tau}f\left(\tau,x(\tau),\int_0^\tau h(\sigma,x(\sigma))d\sigma\right),\ \ \tau\in(0,\tau_1].$$

Similarly, for each i ($i=1,2,\ldots,m$), operating $\mathcal{I}^\alpha_{\sigma_i,\tau}$ on both sides of (3.1), we get

$$x(\tau)=x(\sigma_i)+\mathcal{I}^\alpha_{\sigma_i,\tau}f\left(\tau,x(\tau),\int_{\sigma_i}^\tau h(\sigma,x(\sigma))d\sigma\right),\ \ \tau\in(\sigma_i,\tau_{i+1}].$$

But from equation (3.1), we have

$$x(\sigma_i) = \left[\mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, x(\tau))\right]_{\tau=\sigma_i} = \mathcal{I}^{\beta}_{\tau_i,\sigma_i} h_i(\sigma_i, x(\sigma_i)), \quad i = 1, 2, \ldots, m.$$

Therefore

$$x(\tau) = \mathcal{I}^{\beta}_{\tau_i,\sigma_i} h_i(\sigma_i, x(\sigma_i)) + \mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma))d\sigma\right), \quad \tau \in (\sigma_i, \tau_{i+1}].$$

□

We list the following hypotheses in order to establish our main results.

(H1) The function $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ satisfies the Lipschitz condition

$$|f(\tau, x, y) - f(\tau, \bar{x}, \bar{y})| \le M_f|x - \bar{x}| + N_f|y - \bar{y}|, \ \tau \in J; \ x, \bar{x}, y, \bar{y} \in \mathbb{R},$$

where $M_f > 0, \ N_f > 0$.

(H2) The function $h \in C(J \times \mathbb{R}, \mathbb{R})$ satisfies the Lipschitz condition

$$|h(\tau, x) - h(\tau, \bar{x})| \le K_h|x - \bar{x}|, \ \tau \in J; \ x, \bar{x} \in \mathbb{R},$$

where $K_h > 0$.

(H3) For each $i = 1, 2, \ldots, m$; $h_i \in C((\sigma_i, \tau_i] \times \mathbb{R}, \mathbb{R})$ and satisfies Lipschitz condition

$$|h_i(\tau, x) - h_i(\tau, \bar{x})| \le L_{h_i}|x - \bar{x}|, \text{for each } \tau \in (\sigma_i, \tau_i]; \ x, \bar{x} \in \mathbb{R},$$

where $L_{h_i} > 0$.

Theorem 1. *Assume that hypotheses* (H1)–(H3) *hold. Then the problem* (3.1) *has a unique solution, provided that* $\alpha, \beta \in \left(\frac{1}{2}, 1\right)$.

Proof. Define an operator $T : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ by

$$(Tx)(\tau) = \begin{cases} x_0, \text{if } \tau = 0, \\ \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, x(\tau)), \ \text{if } \tau \in (\tau_i, \sigma_i], \ i = 1, \ldots, m, \\ x_0 + \mathcal{I}^{\alpha}_{0,\tau} f\left(\tau, x(\tau), \int_0^{\tau} h(\sigma, x(\sigma))d\sigma\right), \ \text{if } \tau \in (0, \tau_1], \\ \mathcal{I}^{\beta}_{\tau_i,\sigma_i} h_i(\sigma_i, x(\sigma_i)) + \mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma))d\sigma\right), \\ \qquad\qquad\qquad \text{if } \tau \in (\sigma_i, \tau_{i+1}], \ i = 1, \ldots, m. \end{cases}$$

We shall show that the operator T is contraction with respect to Bielecki norm. Let $x, y \in PC(J, \mathbb{R})$ and $\tau \in (\tau_i, \sigma_i], \ i = 1, 2, \ldots, m,$ then using hypothesis

(H3), we have

$$\begin{aligned}
&\left|\mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, x(\tau)) - \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, y(\tau))\right| \\
&= \left|\frac{1}{\Gamma(\beta)}\int_{\tau_i}^{\tau}(\tau-\sigma)^{\beta-1}h_i(\sigma, x(\sigma))d\sigma - \frac{1}{\Gamma(\beta)}\int_{\tau_i}^{\tau}(\tau-\sigma)^{\beta-1}h_i(\sigma, y(\sigma))d\sigma\right| \\
&\leq \frac{1}{\Gamma(\beta)}\int_{\tau_i}^{\tau}(\tau-\sigma)^{\beta-1}\left|h_i(\sigma, x(\sigma)) - h_i(\sigma, y(\sigma))\right| d\sigma \\
&\leq \frac{L_{h_i}}{\Gamma(\beta)}\int_{\tau_i}^{\tau}(\tau-\sigma)^{\beta-1}\left|x(\sigma) - y(\sigma)\right| d\sigma \\
&\leq \frac{L_{h_i}}{\Gamma(\beta)}\int_{\tau_i}^{\tau}(\tau-\sigma)^{\beta-1}e^{\theta\sigma}\left(\sup_{\sigma\in(\tau_i,\sigma_i]} e^{-\theta\sigma}|x(\sigma) - y(\sigma)|\right) d\sigma \\
&\leq \frac{L_{h_i}}{\Gamma(\beta)}\|x-y\|_{PB}\int_{\tau_i}^{\tau}(\tau-\sigma)^{\beta-1}e^{\theta\sigma}d\sigma.
\end{aligned}$$

By Hölders inequality, for $\beta \in (\frac{1}{2}, 1)$ and $\tau \in (\tau_i, \sigma_i]$, we have

$$\begin{aligned}
&\int_{\tau_i}^{\tau}(\tau-\sigma)^{\beta-1}e^{\theta\sigma}d\sigma \leq \left(\int_{\tau_i}^{\tau}(\tau-\sigma)^{2(\beta-1)}d\sigma\right)^{\frac{1}{2}}\left(\int_{\tau_i}^{\tau}e^{2\theta\sigma}d\sigma\right)^{\frac{1}{2}} \\
&= \left(\frac{(\tau-\tau_i)^{2\beta-1}}{2\beta-1}\right)^{\frac{1}{2}}\left(\frac{e^{2\theta\tau}-e^{2\theta\tau_i}}{2\theta}\right)^{\frac{1}{2}} \leq \frac{(\sigma_i-\tau_i)^{\beta-\frac{1}{2}}}{(2\beta-1)^{\frac{1}{2}}}\left(\frac{e^{\theta\tau}-e^{\theta\tau_i}}{(2\theta)^{\frac{1}{2}}}\right) \\
&\leq \frac{(\sigma_i-\tau_i)^{\beta-\frac{1}{2}}}{(2\beta-1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}}e^{\theta\tau}.
\end{aligned} \tag{3.2}$$

Thus for any $x, y \in PC(J, \mathbb{R})$ and $\tau \in (\tau_i, \sigma_i],\ i = 1, 2, \ldots, m$, we have

$$\left|\mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, x(\tau)) - \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, y(\tau))\right| \leq \frac{L_{h_i}}{\Gamma(\beta)}\frac{(\sigma_i-\tau_i)^{\beta-\frac{1}{2}}}{(2\beta-1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}}e^{\theta\tau}\|x-y\|_{PB}. \tag{3.3}$$

Now, for any $x, y \in PC(J, \mathbb{R})$ and $\tau \in (\sigma_i, \tau_{i+1}], i = 1, 2, \ldots, m$ by using hypotheses (H1)–(H2), we obtain

$$\begin{aligned}
&\left|\mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma))d\sigma\right) - \mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))d\sigma\right)\right| \\
&\quad \leq \mathcal{I}^{\alpha}_{\sigma_i,\tau}\left|f\left(\tau, x(\tau), \mathcal{I}^{1}_{\sigma_i,\tau}h(\tau, x(\tau))\right) - f\left(\tau, y(\tau), \mathcal{I}^{1}_{\sigma_i,\tau}h(\tau, y(\tau))\right)\right| \\
&\quad \leq \mathcal{I}^{\alpha}_{\sigma_i,\tau}\left(M_f|x(\tau)-y(\tau)| + N_f|\mathcal{I}^{1}_{\sigma_i,\tau}h(\tau, x(\tau)) - \mathcal{I}^{1}_{\sigma_i,\tau}h(\tau, y(\tau))|\right) \\
&\quad \leq \mathcal{I}^{\alpha}_{\sigma_i,\tau}\left(M_f|x(\tau)-y(\tau)| + N_f\mathcal{I}^{1}_{\sigma_i,\tau}|h(\tau, x(\tau)) - h(\tau, y(\tau))|\right) \\
&\quad \leq \mathcal{I}^{\alpha}_{\sigma_i,\tau}\left(M_f|x(\tau)-y(\tau)| + N_f\mathcal{I}^{1}_{\sigma_i,\tau}K_h|x(\tau) - y(\tau))|\right) \\
&\quad = M_f\mathcal{I}^{\alpha}_{\sigma_i,\tau}|x(\tau)-y(\tau)| + N_fK_h\mathcal{I}^{\alpha}_{\sigma_i,\tau}\left(\mathcal{I}^{1}_{\sigma_i,\tau}|x(\tau)-y(\tau)|\right) \\
&\quad = M_f\mathcal{I}^{\alpha}_{\sigma_i,\tau}|x(\tau)-y(\tau)| + N_fK_h\mathcal{I}^{\alpha+1}_{\sigma_i,\tau}|x(\tau)-y(\tau)| \\
&= \frac{M_f}{\Gamma(\alpha)}\int_{\sigma_i}^{\tau}(\tau-\sigma)^{\alpha-1}|x(\sigma)-y(\sigma)|d\sigma + \frac{N_fK_h}{\Gamma(\alpha+1)}\int_{\sigma_i}^{\tau}(\tau-\sigma)^{\alpha}|x(\sigma)-y(\sigma)|d\sigma
\end{aligned}$$

$$= \frac{M_f}{\Gamma(\alpha)} \int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha-1} e^{\theta\sigma} \left(\sup_{\sigma\in(\sigma_i,\tau_{i+1}]} e^{-\theta\sigma} |x(\sigma)-y(\sigma)| \right) d\sigma + \frac{N_f K_h}{\Gamma(\alpha+1)}$$
$$\times \int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha} |x(\sigma)-y(\sigma)| e^{\theta\sigma} \left(\sup_{\sigma\in(\sigma_i,\tau_{i+1}]} e^{-\theta\sigma} |x(\sigma)-y(\sigma)| \right) d\sigma$$
$$\leq \frac{M_f}{\Gamma(\alpha)} \times \|x-y\|_{PB} \int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha-1} e^{\theta\sigma} d\sigma + \frac{N_f K_h}{\Gamma(\alpha+1)} \|x-y\|_{PB} \int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha} e^{\theta\sigma} d\sigma. \tag{3.4}$$

Since $\alpha \in (\frac{1}{2}, 1)$, from the inequality (3.2), for any $\tau \in (\sigma_i, \tau_{i+1}]$,$i$=$1, 2, \ldots, m$, we have

$$\int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha-1} e^{\theta\sigma} d\sigma \leq \frac{(\tau-\sigma_i)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}} (2\theta)^{\frac{1}{2}}} e^{\theta\tau} \leq \frac{(\tau_{i+1}-\sigma_i)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}} (2\theta)^{\frac{1}{2}}} e^{\theta\tau}. \tag{3.5}$$

By replacing α by $(\alpha+1)$ in the above inequality, we get

$$\int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha} e^{\theta\sigma} d\sigma \leq \frac{(\tau_{i+1}-\sigma_i)^{\alpha+\frac{1}{2}}}{(2\alpha+1)^{\frac{1}{2}} (2\theta)^{\frac{1}{2}}} e^{\theta\tau}, \quad \tau \in (\sigma_i, \tau_{i+1}],\ i = 1, 2, \ldots, m. \tag{3.6}$$

Using the inequalities (3.5) and (3.6), the inequality (3.4) takes the form

$$\left| \mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma)) d\sigma\right) - \mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma)) d\sigma\right) \right|$$
$$\leq \left[\frac{M_f}{\Gamma(\alpha)} \frac{(\tau_{i+1}-\sigma_i)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}} (2\theta)^{\frac{1}{2}}} + \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{(\tau_{i+1}-\sigma_i)^{\alpha+\frac{1}{2}}}{(2\alpha+1)^{\frac{1}{2}} (2\theta)^{\frac{1}{2}}} \right] e^{\theta\tau} \|x-y\|_{PB}. \tag{3.7}$$

From the definition of an operator T and hypothesis (H1)–(H3), for any $x, y \in PC(J, \mathbb{R})$, we obtain that
Case 1: For $\tau \in (\tau_i, \sigma_i],\ i = 1, 2, \ldots, m$, from the inequality (3.3), we have

$$\begin{aligned} |Tx(\tau) - Ty(\tau)| e^{-\theta\tau} &\leq \left| \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, x(\tau)) - \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, y(\tau)) \right| e^{-\theta\tau} \\ &\leq \frac{L_{h_i}}{\Gamma(\beta)} \frac{(\sigma_i-\tau_i)^{\beta-\frac{1}{2}}}{(2\beta-1)^{\frac{1}{2}} (2\theta)^{\frac{1}{2}}} \|x-y\|_{PB}. \end{aligned}$$

Case 2: For $\tau \in (0, \tau_1]$, on similar line of (3.7), we have

$$\begin{aligned} |Tx(\tau) - Ty(\tau)| e^{-\theta\tau} &= \Big| \mathcal{I}^{\alpha}_{0,\tau} f\left(\tau, x(\tau), \int_0^{\tau} h(\sigma, x(\sigma)) d\sigma\right) \\ &\quad - \mathcal{I}^{\alpha}_{0,\tau} f\left(\tau, y(\tau), \int_0^{\tau} h(\sigma, y(\sigma)) d\sigma\right) \Big| e^{-\theta\tau} \\ &\leq \left[\frac{M_f}{\Gamma(\alpha)} \frac{\tau_1^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}} (2\theta)^{\frac{1}{2}}} + \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{\tau_1^{\alpha+\frac{1}{2}}}{(2\alpha+1)^{\frac{1}{2}} (2\theta)^{\frac{1}{2}}} \right] \|x-y\|_{PB}. \end{aligned}$$

Case 3: For $\tau \in (\sigma_i, \tau_{i+1}]$, $i = 1, 2, \ldots, m$, using the inequalities (3.3) and (3.7), we have

$$\begin{aligned}
|Tx(\tau) - Ty(\tau)|e^{-\theta\tau} &= \left|\mathcal{I}^{\beta}_{\tau_i,\sigma_i} h_i(\sigma_i, x(\sigma_i)) - \mathcal{I}^{\beta}_{\tau_i,\sigma_i} h_i(\sigma_i, y(\sigma_i))\right| e^{-\theta\tau} \\
&+ \left|\mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma))d\sigma\right) - \mathcal{I}^{\alpha}_{\sigma_i,t} f\left(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))d\sigma\right)\right| e^{-\theta\tau} \\
&\leq \frac{L_{h_i}}{\Gamma(\beta)} \frac{(\sigma_i - \tau_i)^{\beta-\frac{1}{2}}}{(2\beta-1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}} e^{\theta(\sigma_i - \tau)} \|x - y\|_{PB} \\
&+ \left[\frac{M_f}{\Gamma(\alpha)} \frac{(\tau_{i+1} - \sigma_i)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}} + \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{(\tau_{i+1} - \sigma_i)^{\alpha+\frac{1}{2}}}{(2\alpha+1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}}\right] \|x - y\|_{PB} \\
&\leq \left(\frac{L_{h_i}}{\Gamma(\beta)} \frac{(\sigma_i - \tau_i)^{\beta-\frac{1}{2}}}{(2\beta-1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}} + \frac{M_f}{\Gamma(\alpha)} \frac{(\tau_{i+1} - \sigma_i)^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}}\right. \\
&\left.+ \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{(\tau_{i+1} - \sigma_i)^{\alpha+\frac{1}{2}}}{(2\alpha+1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}}\right) \|x - y\|_{PB}.
\end{aligned}$$

From Cases 1–3, we can conclude that

$$\|Tx - Ty\|_{PB} \leq \mathcal{L}\|x - y\|_{PB}, \text{ for any } x, y \in PC(J, \mathbb{R}),$$

where

$$\begin{aligned}
\mathcal{L} = \max\Bigg\{ &\frac{L_{h_i}}{\Gamma(\beta)} \frac{(\sigma_i - \tau_i)^{\beta-\frac{1}{2}}}{\sqrt{(2\beta-1)}\sqrt{2\theta}} + \frac{M_f}{\Gamma(\alpha)} \frac{(\tau_{i+1} - \sigma_i)^{\alpha-\frac{1}{2}}}{\sqrt{(2\alpha-1)}\sqrt{2\theta}} \\
&+ \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{(\tau_{i+1} - \sigma_i)^{\alpha+\frac{1}{2}}}{\sqrt{2\alpha+1}\sqrt{2\theta}}; i = 0, 1, \ldots, m \Bigg\}.
\end{aligned}$$

Choose sufficiently large value of θ so that $\mathcal{L} < 1$. Thus, T becomes a contraction mapping and has a unique fixed point due to Banach contraction principle. This fixed point of T act as unique solution of the problem (3.1). □

Next, our aim is to extend the restriction of $\alpha, \beta \in (\frac{1}{2}, 1)$ to $\alpha, \beta \in (0, 1)$. For this purpose we use the following variants of the hypotheses (H1) and (H3):

($\tilde{\text{H}}$1) The function $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ satisfies the Lipschitz condition

$$|f(\tau, x, y) - f(\tau, \bar{x}, \bar{y})| \leq M_f \tau^{\gamma_f}|x - \bar{x}| + N_f|y - \bar{y}|, \tau \in J;\ x, \bar{x}, y, \bar{y} \in \mathbb{R},$$

where $M_f, N_f > 0$ and $\gamma_f > -\alpha$.

($\tilde{\text{H}}$3) For each $i = 1, 2, \ldots, m$; $h_i \in C((\sigma_i, \tau_i] \times \mathbb{R}, \mathbb{R})$ and satisfies Lipschitz condition

$$|h_i(\tau, x) - h_i(\tau, \bar{x})| \leq \tau^{\gamma} L_{h_i}|x - \bar{x}|, \tau \in [\tau_i, \sigma_i];\ x, \bar{x} \in \mathbb{R},$$

where $L_{h_i} > 0$ and $\gamma > -\beta$.

Theorem 2. *Assume that hypotheses* ($\tilde{\text{H}}1$), (H2) *and* ($\tilde{\text{H}}3$) *hold. Then the problem* (3.1) *has a unique solution.*

Proof. Consider the operator T defined in the Theorem 1. By ($\tilde{\text{H}}1$), $\gamma_f > -\alpha = (1-\alpha)-1$ where $\alpha \in (0,1)$. Choose $\sigma > 1$ such that $\sigma\gamma_f > \sigma(1-\alpha)-1$ and $\sigma(\alpha-1)+1>0$. Define $\sigma^* = \frac{\sigma}{\sigma-1}$. Then $\sigma + \sigma^* = 1$. By ($\tilde{\text{H}}3$), for any $\tau \in (\sigma_i, \tau_{i+1}]$, we have

$$\begin{aligned}
&\Big|\mathcal{I}^{\alpha}_{\sigma_i,\tau} f\Big(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma))d\sigma\Big) - \mathcal{I}^{\alpha}_{\sigma_i,\tau} f\Big(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))d\sigma\Big)\Big| \\
&\le \mathcal{I}^{\alpha}_{\sigma_i,\tau}\left|f\left(\tau, x(\tau), \mathcal{I}^{1}_{\sigma_i,\tau} h(\tau, x(\tau))\right) - f\left(\tau, y(\tau), \mathcal{I}^{1}_{\sigma_i,\tau} h(\tau, y(\tau))\right)\right| \\
&\le \mathcal{I}^{\alpha}_{\sigma_i,\tau}\left(M_f \tau^{\gamma_f}|x(t) - y(\tau)| + N_f|\mathcal{I}^{1}_{\sigma_i,\tau} h(\tau, x(\tau)) - \mathcal{I}^{1}_{\sigma_i,\tau} h(\tau, y(\tau))|\right) \\
&\le \mathcal{I}^{\alpha}_{\sigma_i,\tau}\left(M_f \tau^{\gamma_f}|x(\tau) - y(\tau)| + N_f\mathcal{I}^{1}_{\sigma_i,\tau}|h(\tau, x(\tau)) - h(\tau, y(\tau))|\right) \\
&\le \mathcal{I}^{\alpha}_{\sigma_i,\tau}\left(M_f \tau^{\gamma_f}|x(\tau) - y(\tau)| + N_f\mathcal{I}^{1}_{\sigma_i,\tau} K_h|x(\tau) - y(\tau)|\right) \\
&= M_f \mathcal{I}^{\alpha}_{\sigma_i,\tau} \tau^{\gamma_f}|x(\tau) - y(\tau)| + N_f K_h \mathcal{I}^{\alpha}_{\sigma_i,\tau}\left(\mathcal{I}^{1}_{\sigma_i,\tau}|x(\tau) - y(\tau)|\right) \\
&= M_f \mathcal{I}^{\alpha}_{\sigma_i,\tau}\left(\tau^{\gamma_f}|x(\tau) - y(\tau)|\right) + N_f K_h \mathcal{I}^{\alpha+1}_{\sigma_i,\tau}\left(|x(\tau) - y(\tau)|\right) = \frac{M_f}{\Gamma(\alpha)} \\
&\times \int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha-1}\sigma^{\gamma_f}|x(\sigma) - y(\sigma)|d\sigma + \frac{N_f K_h}{\Gamma(\alpha+1)} \int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha}|x(\sigma) - y(\sigma)|d\sigma \\
&\le \frac{M_f}{\Gamma(\alpha)} \int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha-1}\sigma^{\gamma_f} e^{\theta\sigma}\big(\sup_{\sigma\in(\sigma_i,\tau_{i+1}]} e^{-\theta\sigma}|x(\sigma) - y(\sigma)|\big)d\sigma \\
&\quad + \frac{N_f K_h}{\Gamma(\alpha+1)} \int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha}|x(\sigma) - y(\sigma)|e^{\theta\sigma}\big(\sup_{\sigma\in(\sigma_i,\tau_{i+1}]} e^{-\theta\sigma}|x(\sigma) - y(\sigma)|\big)d\sigma \\
&\le \frac{M_f}{\Gamma(\alpha)}\|x-y\|_{PB} \int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha-1}\sigma^{\gamma_f} e^{\theta\sigma} d\sigma \\
&\quad + \frac{N_f K_h}{\Gamma(\alpha+1)}\|x-y\|_{PB} \int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha} e^{\theta\sigma} d\sigma.
\end{aligned} \tag{3.8}$$

Using Hölders inequality, we get

$$\int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha-1}\sigma^{\gamma_f} e^{\theta\sigma} d\sigma \le \left(\int_{\sigma_i}^{\tau} (\tau-\sigma)^{\sigma(\alpha-1)}\sigma^{\sigma\gamma_f} d\sigma\right)^{\frac{1}{\sigma}} \left(\int_{\sigma_i}^{\tau} e^{\theta\sigma^*\sigma} d\sigma\right)^{\frac{1}{\sigma^*}}.$$

But

$$\left(\int_{\sigma_i}^{\tau} e^{\theta\sigma^*\sigma} d\sigma\right)^{\frac{1}{\sigma^*}} = \left(\frac{e^{\theta\sigma^*\tau} - e^{\theta\sigma^*\sigma_i}}{\theta\sigma^*}\right)^{\frac{1}{\sigma^*}} \le \left(\frac{e^{\theta\sigma^*\tau}}{\theta\sigma^*}\right)^{\frac{1}{\sigma^*}} = \frac{e^{\theta\tau}}{(\theta\sigma^*)^{\frac{1}{\sigma^*}}}.$$

By Lemma 2,

$$\begin{aligned}
\left(\int_{\sigma_i}^{\tau} (\tau-\sigma)^{\sigma(\alpha-1)}\sigma^{\sigma\gamma_f} d\sigma\right)^{1/\sigma} &\le \left(\int_{0}^{\tau} (\tau-\sigma)^{\sigma(\alpha-1)}\sigma^{\sigma\gamma_f} d\sigma\right)^{1/\sigma} \\
&= \left(\tau^{\sigma(\alpha-1)+\sigma\gamma_f+1}\mathbb{B}(\sigma\gamma_f+1, \sigma(\alpha-1)+1)\right)^{1/\sigma} \le \omega_1,
\end{aligned}$$

where $\omega_1 = \left(T^{\sigma(\alpha-1)+\sigma\gamma_f+1}\mathbb{B}(\sigma\gamma_f+1, \sigma(\alpha-1)+1)\right)^{1/\sigma}$. Therefore,

$$\int_{\sigma_i}^{\tau} (\tau-\sigma)^{\alpha-1}\sigma^{\gamma_f} e^{\theta\sigma} d\sigma \leq \omega_1 \frac{e^{\theta\tau}}{(\theta\sigma^*)^{1/\sigma^*}}. \tag{3.9}$$

Using equations (3.6) and (3.9), from (3.8) we get

$$\begin{aligned}
&\left|\mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))d\sigma\right) - \mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))d\sigma\right)\right| e^{-\theta\tau} \\
&\leq \left(\omega_1 \frac{M_f}{\Gamma(\alpha)(\theta\sigma^*)^{\frac{1}{\sigma^*}}} + \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{(\tau_{i+1}-\sigma_i)^{\alpha+\frac{1}{2}}}{(2\alpha+1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}}\right) \|x-y\|_{PB}.
\end{aligned} \tag{3.10}$$

On the similar line by ($\tilde{\mathrm{H}}3$), as discussed above, we can choose $\sigma_1 > 1$ such that $\sigma_1\gamma > \sigma_1(1-\beta) - 1$ and $\sigma_1(\beta-1)+1 > 0$. Then for $\sigma_1^* = \frac{\sigma_1}{\sigma_1-1}$, we have $\sigma_1 + \sigma_1^* = 1$. By ($\tilde{\mathrm{H}}3$) and the Hölders inequality, for any $\tau \in (\tau_i, \sigma_i]$,

$$\begin{aligned}
&|\mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, x(\tau)) - \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, y(\tau))| \leq \frac{L_{h_i}}{\Gamma(\beta)} \int_{\tau_i}^{\tau} (\tau-\sigma)^{\beta-1}\sigma^{\gamma} \\
&\quad \times |x(\sigma) - y(\sigma)|\, d\sigma \leq \frac{L_{h_i}}{\Gamma(\beta)} \|x-y\|_{PB} \int_{\tau_i}^{\tau} (\tau-\sigma)^{\beta-1}\sigma^{\gamma} e^{\theta\sigma} d\sigma \\
&\leq \frac{L_{h_i}}{\Gamma(\beta)} \|x-y\|_{PB} \left(\int_{\tau_i}^{\tau} (\tau-\sigma)^{\sigma_1(\beta-1)}\sigma^{\sigma_1\gamma} d\sigma\right)^{\frac{1}{\sigma_1}} \left(\int_{\tau_i}^{\tau} e^{\theta\sigma_1^*\sigma} d\sigma\right)^{\frac{1}{\sigma_1^*}} \\
&\leq \frac{L_{h_i}}{\Gamma(\beta)} \|x-y\|_{PB} \left(\int_{\tau_i}^{\tau} (\tau-\sigma)^{\sigma_1(\beta-1)}\sigma^{\sigma_1\gamma} d\sigma\right)^{\frac{1}{\sigma_1}} \left(\int_{\tau_i}^{\tau} e^{\theta\sigma_1^*\sigma} d\sigma\right)^{\frac{1}{\sigma_1^*}} \\
&\leq \frac{L_{h_i}}{\Gamma(\beta)} \|x-y\|_{PB} \left(\int_{0}^{\tau} (\tau-\sigma)^{\sigma_1(\beta-1)}\sigma^{\sigma_1\gamma} d\sigma\right)^{\frac{1}{\sigma_1}} \frac{e^{\theta\tau}}{(\theta\sigma_1^*)^{\frac{1}{\sigma_1^*}}} \\
&\leq \frac{L_{h_i}\,\omega_2}{\Gamma(\beta)\,(\theta\sigma_1^*)^{\frac{1}{\sigma_1^*}}} e^{\theta\tau} \|x-y\|_{PB},
\end{aligned}$$

where $\omega_2 = \left(T^{\sigma_1(\beta-1)+\sigma_1\gamma+1}\mathbb{B}(\sigma_1\gamma+1, \sigma_1(\beta-1)+1)\right)^{1/\sigma_1}$. Thus

$$\left|\mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, x(\tau)) - \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, y(\tau))\right| e^{-\theta\tau} \leq \frac{L_{h_i}\omega_2}{\Gamma(\beta)(\theta\sigma_1^*)^{\frac{1}{\sigma_1^*}}} \|x-y\|_{PB},\ \tau \in (\tau_i, \sigma_i]. \tag{3.11}$$

By definition of an operator T and using equations (3.10) and (3.11), for any $x, y \in PC(J, \mathbb{R})$, we have:
Case (i): For $\tau \in (\sigma_i, \tau_{i+1}],\ i = 1, 2, \ldots, m$,

$$\begin{aligned}
&|Tx(\tau) - Ty(\tau)|e^{-\theta\tau} \leq \mathcal{I}^{\beta}_{\tau_i,\sigma_i} |h_i(\sigma_i, x(\sigma_i)) - h_i(\sigma_i, y(\sigma_i))| e^{-\theta\tau} \\
&\quad + \mathcal{I}^{\alpha}_{\sigma_i,\tau} \left| f\left(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma))d\sigma\right) - f\left(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))d\sigma\right)\right| e^{-\theta\tau} \\
&\leq \left(\frac{L_{h_i}\omega_2}{\Gamma(\beta)(\theta\sigma_1^*)^{\frac{1}{\sigma_1^*}}} + \omega_1 \frac{M_f}{\Gamma(\alpha)(\theta\sigma^*)^{\frac{1}{\sigma^*}}} + \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{(\tau_{i+1}-\sigma_i)^{\alpha+\frac{1}{2}}}{(2\alpha+1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}}\right) \|x-y\|_{PB}.
\end{aligned}$$

Case (ii): For $\tau \in (0, \tau_1]$,

$$\begin{aligned}
&|Tx(\tau) - Ty(\tau)|e^{-\theta\tau} \\
&\leq \mathcal{I}_{0,\tau}^{\alpha} \left| f\left(\tau, x(\tau), \int_0^{\tau} h(\sigma, x(\sigma))d\sigma\right) - f\left(\tau, y(\tau), \int_0^{\tau} h(\sigma, y(\sigma))d\sigma\right) \right| e^{-\theta\tau} \\
&\leq \left(\omega_1 \frac{M_f}{\Gamma(\alpha)(\theta\sigma^*)^{\frac{1}{\sigma^*}}} + \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{\tau_1^{\alpha+\frac{1}{2}}}{(2\alpha+1)^{\frac{1}{2}} (2\theta)^{\frac{1}{2}}} \right) \|x - y\|_{PB}.
\end{aligned}$$

Case (iii): For $\tau \in (\sigma_i, \tau_i]$, $i = 1, 2, \ldots, m$,

$$\begin{aligned}
|Tx(\tau) - Ty(\tau)|e^{-\theta\tau} &\leq \left| \mathcal{I}_{\tau_i,\tau}^{\beta} h_i(\tau, x(\tau)) - \mathcal{I}_{\tau_i,\tau}^{\beta} h_i(\tau, y(\tau)) \right| e^{-\theta\tau} \\
&\leq \frac{L_{h_i}\omega_2}{\Gamma(\beta)(\theta\sigma_1^*)^{1/\sigma_1^*}} \|x - y\|_{PB}.
\end{aligned}$$

By Cases (i)–(iii), we have

$$\|Tx - Ty\|_{PB} \leq \mathcal{L}_1 \|x - y\|_{PB}, \text{ for any } x, y \in PC(J, \mathbb{R}),$$

where

$$\begin{aligned}
\mathcal{L}_1 = \max \Bigg\{ &\frac{L_{h_i}\omega_2}{\Gamma(\beta)(\theta\sigma_1^*)^{1/\sigma_1^*}} + \omega_1 \frac{M_f}{\Gamma(\alpha)(\theta\sigma^*)^{1/\sigma^*}} \\
&+ \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{(\tau_{i+1} - \sigma_i)^{\alpha+1/2}}{(2\alpha+1)^{1/2} (2\theta)^{1/2}} : \quad i = 0, 1, \ldots, m \Bigg\}.
\end{aligned}$$

By choosing sufficient large value of θ, we get $\mathcal{L}_1 < 1$ and in this case T is a contraction and hence T has a unique fixed point, which is the unique solution of (3.1). □

4 Bielecki-Ulam-Hyers stability

We adopt the idea of Wang and Zang [16] to investigate the concepts of Bielecki-Ulams type stability for the class of nonlinear fractional order Volterra integrodifferential equation (1.1).

For any $\theta > 0$, $\epsilon > 0$, $\psi \geq 0$, $\varphi \in PC(J, \mathbb{R}_+)$ is nondecreasing and $\alpha, \beta \in (0, 1)$, consider the following inequalities:

$$\begin{cases}
\left| {}_{\sigma_i}^{c}\mathcal{D}_{\tau}^{\alpha} y(\tau) - f(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))d\sigma) \right| \leq \epsilon, \ \tau \in (\sigma_i, \tau_{i+1}], \ i{=}0, 1, \ldots, m, \\
|y(\tau) - \mathcal{I}_{\tau_i,\tau}^{\beta} h_i(\tau, y(\tau))| \leq \epsilon, \ \tau \in (\tau_i, \sigma_i], \ i = 1, 2, \ldots, m,
\end{cases} \tag{4.1}$$

$$\begin{cases}
\left| {}_{\sigma_i}^{c}\mathcal{D}_{\tau}^{\alpha} y(\tau) - f(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))d\sigma) \right| \leq \varphi(\tau), \ \tau \in (\sigma_i, \tau_{i+1}], \\
\qquad\qquad\qquad\qquad\qquad\qquad i = 0, 1, \ldots, m, \\
|y(\tau) - \mathcal{I}_{\tau_i,\tau}^{\beta} h_i(\tau, y(\tau))| \leq \psi, \ \tau \in (\tau_i, \sigma_i], \ i = 1, 2, \ldots, m
\end{cases} \tag{4.2}$$

and

$$\begin{cases} \left|{}^{c}_{\sigma_i}\mathcal{D}^{\alpha}_{\tau}y(\tau) - f(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))d\sigma)\right| \leq \epsilon\varphi(\tau),\ \tau \in (\sigma_i, \tau_{i+1}], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 0, 1, \ldots, m, \\ |y(\tau) - \mathcal{I}^{\beta}_{\tau_i,\tau}h_i(\tau, y(\tau))| \leq \epsilon\psi,\ \tau \in (\tau_i, \sigma_i],\ i = 1, 2, \ldots, m. \end{cases} \tag{4.3}$$

Definition 3. The equation (1.1) is Bielecki-Ulam-Hyers stable if there exists a real number $C_{f,\theta,h,\alpha,\beta,h_i} > 0$ such that for each $\epsilon > 0$ for each solution $y \in PC^1(J, \mathbb{R})$ of inequality (4.1) there exists a solution $x \in PC^1(J, \mathbb{R})$ of equation (1.1) with

$$|y(\tau) - x(\tau)|e^{-\theta\tau} \leq C_{f,\theta,h,\alpha,\beta,h_i}\epsilon,\ \ \tau \in J.$$

Definition 4. The equation (1.1) is generalized Bielecki-Ulam-Hyers stable if there exists $\theta_{f,\theta,h,\alpha,\beta,h_i} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_{f,\theta,h,\alpha,\beta,h_i}(0) = 0$ such that for each $\epsilon > 0$ for each solution $y \in PC^1(J, \mathbb{R})$ of inequality (4.1) there exists a solution $x \in PC^1(J, \mathbb{R})$ of equation (1.1) with

$$|y(\tau) - x(\tau)|e^{-\theta\tau} \leq \theta_{f,\theta,h,\alpha,\beta,h_i}(\epsilon),\ \ \tau \in J.$$

Definition 5. The equation (1.1) is Bielecki-Ulam-Hyers-Rassias stable with respect to (ϕ, ψ) if there exists $C_{f,\theta,h,\alpha,\beta,h_i,\varphi} > 0$ such that for each $\epsilon > 0$ for each solution $y \in PC^1(J, \mathbb{R})$ of inequality (4.3) there exists a solution $x \in PC^1(J, \mathbb{R})$ of equation (1.1) with

$$|y(\tau) - x(\tau)|e^{-\theta\tau} \leq C_{f,\theta,h,\alpha,\beta,h_i,\varphi}\epsilon(\psi + \varphi(\tau)),\ \ \tau \in J.$$

Definition 6. The equation (1.1) is generalized Bielecki-Ulam-Hyers-Rassias stable with respect to (φ, ψ) if there exists $C_{f,\theta,h,\alpha,\beta,h_i,\varphi} > 0$ such that for each solution $y \in PC^1(J, \mathbb{R})$ of inequality (4.2) there exists a solution $x \in PC^1(J, \mathbb{R})$ of equation (1.1) with

$$|y(\tau) - x(\tau)|e^{-\theta\tau} \leq C_{f,\theta,h,\alpha,\beta,h_i,\varphi}(\psi + \varphi(\tau)),\ \ \tau \in J.$$

Lemma 4. *If $y \in PC^1(J, \mathbb{R})$ is a solution of inequality (4.3) then y satisfies the following integral inequalities*

$$\begin{cases} |y(\tau) - \mathcal{I}^{\beta}_{\tau_i,\tau}h_i(\tau, y(\tau))| \leq \epsilon\psi,\ \tau \in (\tau_i, \sigma_i],\ i = 1, 2, \ldots, m, \\ \left|y(\tau) - y(0) - \mathcal{I}^{\alpha}_{0,\tau}f\left(\tau, y(\tau), \int_0^{\tau} h(\sigma, y(\sigma))\,d\sigma\right|\right. \leq \epsilon\mathcal{I}^{\alpha}_{0,\tau}\varphi(\tau),\ \textit{if}\ \tau \in (0, \tau_1], \\ \left|y(\tau) - \mathcal{I}^{\beta}_{\tau_i,\sigma_i}h_i(\sigma_i, y(\sigma_i)) - \mathcal{I}^{\alpha}_{\sigma_i,\tau}f\left(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma)\right)d\sigma\right| \\ \leq \epsilon\left(\psi + \mathcal{I}^{\alpha}_{\sigma_i,\tau}\varphi(\tau)\right), \tau \in (\sigma_i, \tau_{i+1}],\ i = 1, 2, \ldots, m. \end{cases} \tag{4.4}$$

Proof. If $y \in PC^1(J, \mathbb{R})$ is a solution of the inequality (4.3) then there exists $H \in PC(J, \mathbb{R})$ and constants H_i, $i = 1, 2, \ldots, m$ (which depend on y) such that

(i) $|H(\tau)| \leq \epsilon\,\varphi(\tau),\ \tau \in J$ and $|H_i| \leq \epsilon\,\psi$ for $i = 1, 2, \ldots, m.$

$$\text{(ii)} \begin{cases} {}^{c}_{\sigma_i}\mathcal{D}^{\alpha}_{\tau} y(\tau) = f\left(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))d\sigma\right) + H(\tau),\ \tau \in (\sigma_i, \tau_{i+1}], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 0, 1, \ldots, m, \\ y(\tau) = \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, y(\tau)) + H_i,\ \tau \in (\tau_i, \sigma_i],\ i = 1, 2, \ldots, m. \end{cases}$$

In view of Lemma 3, above equation is equivalent to the integral equations

$$y(\tau) = \begin{cases} \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, y(\tau)) + H_i,\ \tau \in (\tau_i, \sigma_i],\ i = 1, \ldots, m, \\ y(0) + \mathcal{I}^{\alpha}_{0,\tau}\left[f\left(\tau, y(\tau), \int_0^{\tau} h(\sigma, y(\sigma))d\sigma\right) + H(\tau)\right], \tau \in [0, \tau_1], \\ \mathcal{I}^{\beta}_{\tau_i,\sigma_i} h_i(\sigma_i, y(\sigma_i)) + H_i + \mathcal{I}^{\alpha}_{\sigma_i,\tau}\Big[f\left(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))d\sigma\right) \\ \qquad +H(\tau)\Big], \quad \tau \in (\sigma_i, \tau_{i+1}], i = 1, \ldots, m. \end{cases}$$

For any $\tau \in (\sigma_i, \tau_{i+1}],\ i = 1, 2, \ldots, m$,

$$\begin{aligned} &\left| y(\tau) - \mathcal{I}^{\beta}_{\tau_i,\sigma_i} h_i(\sigma_i, y(\sigma_i)) - \mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))\right) d\sigma \right| \\ &= |H_i + \mathcal{I}^{\alpha}_{\sigma_i,\tau} H(\tau)| \le |H_i| + \mathcal{I}^{\alpha}_{\sigma_i,\tau}|H(\tau)| \le \epsilon\left(\psi + \mathcal{I}^{\alpha}_{\sigma_i,\tau}\varphi(\tau)\right). \end{aligned}$$

For $\tau \in (0, \tau_1]$,

$$\left| y(\tau) - y(0) - \mathcal{I}^{\alpha}_{0,\tau} f\left(\tau, y(\tau), \int_0^{\tau} h(\sigma, y(\sigma))\right) d\sigma \right| = \mathcal{I}^{\alpha}_{0,\tau}|H(\tau)| \le \epsilon \mathcal{I}^{\alpha}_{0,\tau}\varphi(\tau).$$

For $\tau \in (\tau_i, \sigma_i],\ i = 1, 2, \ldots, m$,

$$\left| y(\tau) - \mathcal{I}^{\beta}_{\tau_i,\sigma_i} h_i(\tau, y(\tau)) \right| \le |H_i| \le \epsilon\psi,\ \tau \in (\tau_i, \sigma_i],\ i = 1, 2, \ldots, m.$$

The last three inequalities are the required equivalent integral inequalities in (4.4). □

Following additional hypothesis is needed to prove the Bielecki-Ulam-Hyers-Rassias stability of equation (1.1).

(H4) Let $\varphi \in C(J, \mathbb{R}_+)$ is nondecreasing and there exists $c_\varphi > 0$ such that

$$\mathcal{I}^{\alpha}_{0,\tau}\varphi \le c_\varphi \varphi(\tau),\ \tau \in J.$$

Theorem 3. *Assume that hypotheses* ($\tilde{\text{H}}1$), (H2)*,* ($\tilde{\text{H}}3$) *and* (H4) *hold. Then, the equation* (1.1) *is Bielecki-Ulam-Hyers-Rassias stable with respect to* (φ, ψ)*, where* $\alpha,\ \beta \in (0, 1)$*.*

Proof. Let $y \in PC^1(J, \mathbb{R})$ be a solution of inequality (4.3). Then by Lemma 4, y satisfies the integral inequalities

$$\begin{cases} |y(\tau) - \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, y(\tau))| \le \epsilon\psi,\ \tau \in (\tau_i, \sigma_i],\ i = 1, 2, \ldots, m, \\ \left| y(\tau) - y(0) - \mathcal{I}^{\alpha}_{0,\tau} f\left(\tau, y(\tau), \int_0^{\tau} h(\sigma, y(\sigma)\right) d\sigma \right| \le \epsilon c_\varphi \varphi(\tau),\ \tau \in (0, \tau_1], \\ \left| y(\tau) - \mathcal{I}^{\beta}_{\tau_i,\sigma_i} h_i(\sigma_i, y(\sigma_i)) - \mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma)\right) d\sigma \right| \\ \qquad \le \epsilon\left(\psi + c_\varphi \varphi(\tau)\right), \quad \tau \in (\sigma_i, \tau_{i+1}],\ i = 1, 2, \ldots, m. \end{cases}$$

Denote by x the classical solution of fractional Volterra integrodifferential equations

$$\begin{cases} {}^{c}_{\sigma_i}\mathcal{D}^{\alpha}_{\tau}x(\tau) = f\left(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma))d\sigma\right), \ \tau \in (\sigma_i, \tau_{i+1}], \ i = 0, 1, \ldots, m, \\ x(\tau) = \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, x(\tau)), \ \tau \in (\tau_i, \sigma_i], \ i = 1, 2, \ldots, m, \\ x(0) = y(0). \end{cases}$$

Then, in view of Lemma 3, x satisfies the fractional Volterra integral equations

$$x(\tau) = \begin{cases} y(0), \ \tau = 0, \\ \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, x(\tau)), \ \tau \in (\tau_i, \sigma_i], \ i = 1, \ldots, m, \\ y(0) + \mathcal{I}^{\alpha}_{0,\tau} f\left(\tau, x(\tau), \int_0^{\tau} h(\sigma, x(\sigma))d\sigma\right), \ \tau \in (0, \tau_1], \\ \mathcal{I}^{\beta}_{\tau_i,\sigma_i} h_i(\sigma_i, x(\sigma_i)) + \mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma))d\sigma\right), \\ \qquad\qquad \tau \in (\sigma_i, \tau_{i+1}], i = 1, \ldots, m. \end{cases}$$

Proceeding as in the proof of Theorem 2, for any $\tau \in (\sigma_i, \tau_{i+1}], \ i = 1, 2, \ldots, m$, we obtain

$$\begin{aligned} &|y(\tau) - x(\tau)|e^{-\theta\tau} \\ &= \left| y(\tau) - \left(\mathcal{I}^{\beta}_{\tau_i,\sigma_i} h_i(\sigma_i, x(\sigma_i)) + \mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma))d\sigma\right)\right)\right| e^{-\theta\tau} \\ &\le \left| y(\tau) - \mathcal{I}^{\beta}_{\tau_i,\sigma_i} h_i(\sigma_i, y(\sigma_i)) - \mathcal{I}^{\alpha}_{\sigma_i,t} f\left(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))d\sigma\right)\right| e^{-\theta\tau} \\ &\quad + \left| \mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, y(\tau), \int_{\sigma_i}^{\tau} h(\sigma, y(\sigma))d\sigma\right) - \mathcal{I}^{\alpha}_{\sigma_i,\tau} f\left(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma))d\sigma\right)\right| e^{-\theta\tau} \\ &= \epsilon\left(\psi + c_{\varphi}\varphi(\tau)\right) e^{-\theta\tau} + \frac{L_{h_i}\omega_2}{\Gamma(\beta)(\theta\sigma_1^*)^{\frac{1}{\sigma_1^*}}} \left(\sup_{\tau\in(\tau_i,\sigma_i]} e^{-\theta\tau}|y(\tau) - x(\tau)| \right) \\ &\quad + \left[\frac{\omega_1 M_f}{\Gamma(\alpha)(\theta\sigma^*)^{\frac{1}{\sigma^*}}} + \frac{N_f K_h(\tau_{i+1} - \sigma_i)^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+1)\,(2\alpha+1)^{\frac{1}{2}}\,(2\theta)^{\frac{1}{2}}} \right] \left(\sup_{\tau\in(\sigma_i,\tau_{i+1}]} e^{-\theta\tau}|y(\tau) - x(\tau)| \right). \end{aligned}$$

This gives

$$\begin{aligned} \sup_{\tau\in(\sigma_i,\tau_{i+1}]} e^{-\theta\tau}|x(\tau) - y(\tau)| &\le \epsilon(1 + c_{\varphi})\left(\psi + \varphi(\tau)\right) e^{-\theta\sigma_i} + \Bigg(\frac{L_{h_i}\omega_2}{\Gamma(\beta)(\theta\sigma_1^*)^{\frac{1}{\sigma_1^*}}} \\ &\quad + \frac{\omega_1 M_f}{\Gamma(\alpha)(\theta\sigma^*)^{\frac{1}{\sigma^*}}} + \frac{N_f K_h(\tau_{i+1} - \sigma_i)^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+1)\,(2\alpha+1)^{\frac{1}{2}}\,(2\theta)^{\frac{1}{2}}} \Bigg) \sup_{\tau\in(\sigma_i,\tau_{i+1}]} e^{-\theta\tau}|x(\tau) - y(\tau)|. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sup_{\tau\in(\sigma_i,\tau_{i+1}]} e^{-\theta\tau}|x(\tau) - y(\tau)| \\ &\le \frac{\epsilon(1 + c_{\varphi})\left(\psi + \varphi(\tau)\right) e^{-\theta\sigma_i}}{1 - \left(\frac{L_{h_i}\omega_2}{\Gamma(\beta)(\theta\sigma_1^*)^{\frac{1}{\sigma_1^*}}} + \frac{\omega_1 M_f}{\Gamma(\alpha)(\theta\sigma^*)^{\frac{1}{\sigma^*}}} + \frac{N_f K_h(\tau_{i+1} - \sigma_i)^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+1)\,(2\alpha+1)^{\frac{1}{2}}\,(2\theta)^{\frac{1}{2}}} \right)}. \qquad (4.5) \end{aligned}$$

Now, for any $\tau \in (\tau_i, \sigma_i], i = 1, 2, \ldots, m$,

$$\begin{aligned}
&|y(\tau) - x(\tau)|e^{-\theta\tau} \leq |y(\tau) - \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, x(\tau))|e^{-\theta\tau} \\
&\quad \leq |y(\tau) - \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, y(\tau))|e^{-\theta\tau} + |\mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, y(\tau)) - \mathcal{I}^{\beta}_{\tau_i,\tau} h_i(\tau, x(\tau))|e^{-\theta\tau} \\
&\quad \leq \epsilon\psi e^{-\theta\tau_i} + \frac{L_{h_i}\omega_2}{\Gamma(\beta)(\theta\sigma_1^*)^{\frac{1}{\sigma_1^*}}} \left(\sup_{\tau\in(\tau_i,\sigma_i]} e^{-\theta\tau}|x(\tau) - y(\tau)| \right).
\end{aligned}$$

Therefore,

$$\sup_{\tau\in(\tau_i,\sigma_i]} e^{-\theta\tau}|x(\tau) - y(\tau)| \leq \epsilon\psi e^{-\theta\tau_i} + \frac{L_{h_i}\omega_2}{\Gamma(\beta)(\theta\sigma_1^*)^{\frac{1}{\sigma_1^*}}} \left(\sup_{\tau\in(\tau_i,\sigma_i]} e^{-\theta\tau}|x(\tau) - y(\tau)| \right).$$

Thus, we have

$$\sup_{\tau\in(\tau_i,\sigma_i]} e^{-\theta\tau}|x(\tau) - y(\tau)| \leq \frac{\epsilon\psi}{\left(1 - L_{h_i}\omega_2/\left(\Gamma(\beta)(\theta\sigma_1^*)^{\frac{1}{\sigma_1^*}}\right)\right)} e^{-\theta\tau_i}. \tag{4.6}$$

Next, for any $\tau \in (0, \tau_1]$,

$$\begin{aligned}
&|y(\tau) - x(\tau)|e^{-\theta\tau} \leq \left| y(\tau) - y(0) - \mathcal{I}^{\alpha}_{0,\tau} f\left(\tau, x(\tau), \int_0^\tau h(\sigma, x(\sigma))d\sigma\right) \right| e^{-\theta\tau} \\
&\leq \left| y(\tau) - y(0) - \mathcal{I}^{\alpha}_{0,\tau} f\left(\tau, y(\tau), \int_0^\tau h(\sigma, y(\sigma))d\sigma\right) \right| e^{-\theta\tau} \\
&\quad + \mathcal{I}^{\alpha}_{0,\tau} \left| f(\tau, y(\tau), \int_0^\tau h(\sigma, y(\sigma))d\sigma) - f(\tau, x(\tau), \int_0^\tau h(\sigma, x(\sigma))d\sigma) \right| e^{-\theta\tau} \\
&\leq \epsilon c_\varphi \varphi(\tau) \\
&\quad + \left(\frac{\omega_1 M_f}{\Gamma(\alpha)(\theta\sigma^*)^{\frac{1}{\sigma^*}}} + \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{\tau_1^{\alpha+\frac{1}{2}}}{(2\alpha+1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}} \right) \sup_{\tau\in(0,\tau_1]} e^{-\theta\tau}|x(\tau) - y(\tau)|.
\end{aligned}$$

Just computed as above, we get

$$\sup_{\tau\in(0,\tau_1]} |y(\tau) - x(\tau)|e^{-\theta\tau} \leq \frac{\epsilon c_\varphi \varphi(\tau)}{1 - \left(\frac{\omega_1 M_f}{\Gamma(\alpha)(\theta\sigma^*)^{\frac{1}{\sigma^*}}} + \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{\tau_1^{\alpha+\frac{1}{2}}}{(2\alpha+1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}} \right)}. \tag{4.7}$$

Since

$$J = (0, T] = \left[\bigcup_{i=0}^{m} (\sigma_i, \tau_{i+1}] \right] \left[\bigcup_{i=1}^{m} (\tau_i, \sigma_i] \right],$$

from the inequalities (4.5),(4.6) and (4.7), we obtain

$$\begin{aligned}
&\sup_{\tau\in(0,T]} |y(\tau) - x(\tau)|e^{-\theta\tau} \leq \sup_{\tau\in(0,\tau_1]} |y(\tau) - x(\tau)|e^{-\theta\tau} \\
&\quad + \sum_{i=1}^{m} \sup_{\tau\in(\sigma_i,\tau_{i+1}]} |y(\tau) - x(\tau)|e^{-\theta\tau} + \sum_{i=1}^{m} \sup_{\tau\in(\tau_i,\sigma_i]} |y(\tau) - x(\tau)|e^{-\theta\tau} \\
&\quad \leq \epsilon c_{f,\theta,h,\alpha,\beta,h_i,\varphi}(\psi + \varphi(\tau)),
\end{aligned} \tag{4.8}$$

where

$$c_{f,\theta,h,\alpha,\beta,h_i,\varphi}=\frac{c_\varphi}{1-\left(\frac{\omega_1 M_f}{\Gamma(\alpha)(\theta\sigma^*)^{\frac{1}{\sigma^*}}}+\frac{N_f K_h}{\Gamma(\alpha+1)}\frac{\tau_1^{\alpha+\frac{1}{2}}}{(2\alpha+1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}}\right)}+\sum_{i=1}^{m}\frac{\psi}{\left(1-\frac{L_{h_i}\omega_2}{\Gamma(\beta)(\theta\sigma_1^*)^{\frac{1}{\sigma_1^*}}}\right)}$$

$$+\sum_{i=1}^{m}\frac{1+c_\varphi}{1-\left(\frac{L_{h_i}\omega_2}{\Gamma(\beta)(\theta\sigma_1^*)^{\frac{1}{\sigma_1^*}}}+\frac{\omega_1 M_f}{\Gamma(\alpha)(\theta\sigma^*)^{\frac{1}{\sigma^*}}}+\frac{N_f K_h(\tau_{i+1}-\sigma_i)^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+1)(2\alpha+1)^{\frac{1}{2}}(2\theta)^{\frac{1}{2}}}\right)}.$$

Finally, from inequality (4.8), we have

$$|y(\tau)-x(\tau)|e^{-\theta\tau}\leq \epsilon c_{f,\theta,h,\alpha,\beta,h_i,\varphi}(\psi+\varphi(\tau)),\ \tau\in J.$$

This shows that equation (1.1) Bielecki-Ulam-Hyers-Rassias stable with respect to (φ,ψ). □

Corollary 1. Assume that hypotheses ($\tilde{\text{H}}1$), (H2), ($\tilde{\text{H}}3$) and (H4) hold. Then, the equation (1.1) is generalized Bielecki-Ulam-Hyers-Rassias stable with respect to (φ,ψ), where $\alpha,\beta\in(0,1)$.

Proof. Set $\epsilon=1$ in the proof of Theorem 2, we obtain

$$|y(\tau)-x(\tau)|e^{-\theta\tau}\leq c_{f,\theta,h,\alpha,\beta,i,\varphi}(\psi+\varphi(\tau)),\ \tau\in J.$$

This proves the equation (1.1) is generalized Bielecki-Ulam-Hyers-Rassias stable with respect to (φ,ψ). □

Corollary 2. Assume that hypotheses ($\tilde{\text{H}}1$), (H2), ($\tilde{\text{H}}3$) and (H4) hold. Then, the equation (1.1) is Bielecki-Ulam-Hyers stable, where $\alpha,\beta\in(0,1)$.

Proof. Set $\psi=1$ and $\varphi(\tau)=1,\ \tau\in J$ in the proof of Theorem 2. Then $\varphi\in C(J,\mathbb{R}_+)$ and $\mathcal{I}^{\alpha}_{0,\tau}\varphi\leq c_\varphi\varphi(\tau),\ \tau\in J$, where $c_\varphi=\frac{T^\alpha}{\Gamma(1+\alpha)}$. Thus the hypothesis (H4) is satisfied. Further, we have

$$|y(\tau)-x(\tau)|e^{-\theta\tau}\leq \epsilon c_{f,\theta,h,\alpha,\beta,h_i},\ \tau\in J. \tag{4.9}$$

Therefore the equation (1.1) is Bielecki-Ulam-Hyers stable. □

Corollary 3. Assume that hypotheses ($\tilde{\text{H}}1$), (H2), ($\tilde{\text{H}}3$) and (H4) hold. Then, the equation (1.1) is generalized Bielecki-Ulam-Hyers stable, where $\alpha,\beta\in(0,1)$.

Proof. Define $\theta_{f,\theta,h,\alpha,\beta,h_i}:\mathbb{R}_+\to\mathbb{R}_+$ by $\theta_{f,\theta,h,\alpha,\beta,h_i}(\epsilon)=\epsilon c_{f,\theta,h,\alpha,\beta,h_i}$. Then $\theta_{f,\theta,h,\alpha,\beta,h_i}\in C(\mathbb{R}_+,\mathbb{R}_+)$ and $\theta_{f,\theta,h,\alpha,\beta,h_i}(0)=0$. Further, from the equation (4.9), we have

$$|y(\tau)-x(\tau)|e^{-\theta t}\leq\theta(\epsilon),\ \tau\in J,$$

which shows that (1.1) is generalized Bielecki-Ulam-Hyers stable. □

Remark 1. Under the hypotheses of Theorem 1, one can obtain the Bielecki - Ulam-Hyers stability and Bielecki-Ulam-Hyers-Rassias stability of equation for $\alpha,\beta\in(\frac{1}{2},1)$.

5 Example

Example 1. Consider Caputo fractional differential equation with fractional integrable impulse

$$ {}^{c}_{\sigma_i}\mathcal{D}^{\frac{2}{3}}_{\tau} x(\tau) = f\Big(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma))d\sigma\Big), \;\; \tau \in (\sigma_i, \tau_{i+1}], i = 0, 1, \tag{5.1} $$

$$ x(\tau) = \mathcal{I}^{\frac{2}{3}}_{\tau_1,\tau} h_1(\tau, x(\tau)), \;\; \tau \in (\tau_1, \sigma_1], \tag{5.2} $$

$$ x(0) = 0, \tag{5.3} $$

where $0 = \sigma_0 = \tau_0 < \tau_1 = 1 < \sigma_1 = 2 < \tau_2 = 3$ and the functions $f : [0, 3] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $h : [0, 3] \times \mathbb{R} \to \mathbb{R}$ and $h_1 : (1, 2] \times \mathbb{R} \to \mathbb{R}$ are defined as follows:

$$ f\left(\tau, x(\tau), \int_{\sigma_i}^{\tau} h(\sigma, x(\sigma))d\sigma\right) = g(\tau) + \frac{e^{-\tau^2}}{4} (\sin(x(\tau)) + \cos x(\tau)) + \int_{\sigma_i}^{\tau} \frac{\sigma}{e^{\sigma^2}} \sin(x(\sigma))d\sigma, $$

$$ h(\tau, x(\tau)) = \frac{\tau}{e^{\tau^2}} \sin(x(\tau)), \quad h_1(\tau, x(\tau)) = \frac{1}{\Gamma(\frac{4}{3})} \frac{\tau(\tau-1)^{\frac{1}{3}}}{(\tau-4)} \left(\frac{|x(\tau)| - 3}{|x(\tau)| + 1}\right), $$

where

$$ g(\tau) = \begin{cases} \frac{9}{2\Gamma(1/3)}\tau^{4/3} - \frac{1}{4}, \quad \tau \in [0, 1], \\ 0, \; \tau \in (1, 2], \\ \frac{9}{2\Gamma(1/3)}(\tau - 2)^{4/3} - \frac{1}{4e^4} (\cos 4 + \sin 4), \;\; \tau \in (2, 3]. \end{cases} $$

For any $x, y, \bar{x}, \bar{y} \in PC(J, \mathbb{R})$ and $\tau \in [0, 3]$,

$$ \begin{aligned} &|f(\tau, x, y) - f(\tau, \bar{x}, \bar{y})| \\ &= \left|\Big(g(\tau) + \frac{e^{-\tau^2}}{4} (\sin x + \cos x) + y\Big) - \Big(g(\tau) + \frac{e^{-\tau^2}}{4} (\sin \bar{x} + \cos \bar{x}) + \bar{y}\Big)\right| \\ &\le \frac{e^{-\tau^2}}{4} (|\sin x - \sin \bar{x}| + |\cos x - \cos \bar{x}|) + |y - \bar{y}| = \frac{e^{-\tau^2}}{4} \\ &\times \left(\left|2\cos\left(\frac{x+\bar{x}}{2}\right)\sin\left(\frac{x-\bar{x}}{2}\right)\right| + \left|2\sin\left(\frac{x+\bar{x}}{2}\right)\sin(\frac{x-\bar{x}}{2})\right|\right) + |y - \bar{y}| \\ &\le \frac{e^{-\tau^2}}{2}|x - \bar{x}| + |y - \bar{y}| \le \frac{1}{2}|x - \bar{x}| + |y - \bar{y}|. \end{aligned} $$

Further, for $\tau \in [1, 2]$,

$$ \begin{aligned} &|h_1(\tau, x) - h_1(\tau, \bar{x})| = \left|\frac{1}{\Gamma(\frac{4}{3})} \frac{\tau(\tau-1)^{\frac{1}{3}}}{(\tau-4)} \Big(\frac{|x| - 3}{|x| + 1}\Big) - \frac{1}{\Gamma(\frac{4}{3})} \frac{\tau(\tau-1)^{\frac{1}{3}}}{(\tau-4)} \Big(\frac{|y| - 3}{|y| + 1}\Big)\right| \\ &\le \frac{1}{\Gamma(\frac{4}{3})} \left|\frac{|x| - 3}{|x| + 1} - \frac{|y| - 3}{|y| + 1}\right| \le \frac{4}{\Gamma(\frac{4}{3})} \frac{||x| - |y||}{(|x| + 1)(|y| + 1)} \le \frac{4}{\Gamma(\frac{4}{3})}|x - y| \end{aligned} $$

and for $\tau \in [0,3]$, we have

$$|h(\tau,x(\tau)) - h(\tau,\bar{x})| = \left|\frac{\tau}{e^{\tau^2}}\sin x - \frac{\tau}{e^{\tau^2}}\sin\bar{x}\right| = \frac{\tau}{e^{\tau^2}}|\sin x - \sin\bar{x}|$$
$$= \frac{\tau}{e^{\tau^2}}|\sin x - \sin\bar{x}| = \frac{\tau}{e^{\tau^2}}\left|2\cos(\frac{x+\bar{x}}{2})\sin(\frac{x-\bar{x}}{2})\right| \leq 3|x-\bar{x}|.$$

Thus f, h and h_1 satisfies Lipschitz condition with Lipschitz constants

$$M_f = \frac{1}{2},\ N_f = 1,\ K_h = 3 \text{ and } L_{h_1} = \frac{4}{\Gamma(4/3)}.$$

Thus by Theorem 1, the problem (5.1)–(5.3) has a unique solution on $[0,3]$. Let $\varphi(\tau) = 2.8361\mathbb{E}_{\frac{2}{3}}(\tau^{\frac{2}{3}}),\ \tau \in J$ and $\psi = 0$. Then $\varphi \in C(J,\mathbb{R}_+)$ is nondecreasing and satisfy the condition

$$\mathcal{I}^{\frac{2}{3}}_{0,\tau}\varphi(\tau) = 2.8361\mathcal{I}^{\frac{2}{3}}_{\sigma_i,\tau}(\mathbb{E}_{\frac{2}{3}}(\tau^{\frac{2}{3}})) \leq 2.8361\mathbb{E}_{\frac{2}{3}}(\tau^{\frac{2}{3}}) = c_\varphi\varphi(\tau), \tau \in J,$$

where $c_\varphi = 1$. Note that all the hypotheses of the Corollary 1 hold. Therefore problem (5.1)–(5.2) is generalized Bielecki-Ulam-Hyers-Rassias stable with respect to (φ,ψ).

Next, we shall discuss the generalized Bielecki-Ulam-Hyers-Rassias stablity of the equation (5.1)–(5.2) by showing that there exists an exact solution $x(\tau)$ of the problem (5.1)–(5.3) corresponding to (φ,ψ) and the given solution $y(\tau)$ of the inequalities

$$\left|{}^c_{\sigma_i}\mathcal{D}^{\frac{2}{3}}_{\tau}y(\tau) - f(\tau,y(\tau),\int_{\sigma_i}^{\tau}h(\sigma,y(\sigma))d\sigma)\right| \leq \varphi(\tau),\ \tau \in (\sigma_i,\tau_{i+1}], i = 0,1, \tag{5.4}$$
$$\left|y(\tau) - \mathcal{I}^{\frac{2}{3}}_{\tau_1,\tau}h_1(\tau,y(\tau))\right| \leq 0,\ \ \tau \in (\tau_1,\sigma_1], \tag{5.5}$$

where $0 = \sigma_0 = \tau_0 < \tau_1 = 1 < \sigma_1 = 2 < \tau_2 = 3$. Let

$$y(\tau) = \begin{cases} \tau,\ \tau \in [0,1] \cup (2,3], \\ (\tau-1),\ \tau \in (1,2]. \end{cases}$$

Then for $\tau \in (0,1]$,

$$\left|{}^c_0\mathcal{D}^{\frac{2}{3}}_{\tau}y(t) - g(\tau) - \frac{e^{-\tau^2}}{4}(\sin y(\tau) + \cos y(\tau)) - \int_0^{\tau}\frac{\sigma}{e^{\sigma^2}}\sin y(\sigma)d\sigma\right|$$
$$= \left|{}^c_0\mathcal{D}^{\frac{2}{3}}_{\tau}(\tau) - \frac{9}{2\Gamma(1/3)}\tau^{\frac{4}{3}} + \frac{1}{4} - \frac{e^{-\tau^2}}{4}(\sin\tau + \cos\tau) - \int_0^{\tau}\frac{\sigma}{e^{\sigma^2}}\sin\sigma d\sigma\right|$$
$$\leq \frac{1}{\Gamma(\frac{4}{3})}\tau^{\frac{1}{3}} + \frac{9}{2\Gamma(\frac{1}{3})}\tau^{\frac{4}{3}} + \frac{1}{4} + \frac{e^{-\tau^2}}{4}(|\sin\tau| + |\cos\tau|) + \int_0^{\tau}\frac{\sigma}{e^{\sigma^2}}|\sin\sigma|d\sigma$$
$$\leq 4.5495$$

and for $\tau \in (2,3]$,

$$\begin{aligned}
&\left| {}_{2}^{c}\mathcal{D}_{\tau}^{\frac{2}{3}} y(\tau) - g(\tau) - \frac{e^{-\tau^2}}{4}\left(\sin(y(\tau)) + \cos y(\tau)\right) - \int_2^{\tau} \frac{\sigma}{e^{\sigma^2}} \sin(y(\sigma)) d\sigma \right| \\
&= \left| {}_{2}^{c}\mathcal{D}_{\tau}^{\frac{2}{3}} [(\tau-2)+2] - \frac{9}{2\Gamma(1/3)}(\tau-2)^{\frac{4}{3}} + \frac{1}{4e^4}\left(\cos 4 + \sin 4\right) \right. \\
&\quad \left. - \frac{e^{-\tau^2}}{4}\left(\sin\tau + \cos\tau\right) - \int_2^{\tau} \frac{\sigma}{e^{\sigma^2}} \sin\sigma d\sigma \right| \leq \frac{1}{\Gamma(4/3)}(\tau-2)^{\frac{1}{3}} + 0 \\
&\quad + \frac{9}{2\Gamma(1/3)}(\tau-2)^{\frac{4}{3}} + \frac{1}{4e^4}\left(|\cos 4| + |\sin 4|\right) + \frac{1}{4e^4}\left(|\sin\tau| + |\cos\tau|\right) \\
&\quad + \int_2^{\tau} \frac{\sigma}{e^{\sigma^2}} |\sin\sigma| d\sigma \leq 2.8361.
\end{aligned}$$

Therefore, for $\tau \in (\sigma_i, \tau_i],\ i = 0,1$, we have

$$\begin{aligned}
&\left| {}_{\sigma_i}^{c}\mathcal{D}_{\tau}^{\frac{2}{3}} y(\tau) - g(\tau) - \frac{e^{-\tau^2}}{4}\left(\sin y(\tau) + \cos y(\tau)\right) - \int_{\sigma_i}^{\tau} \frac{\sigma}{e^{\sigma^2}} \sin y(\sigma) d\sigma \right| \\
&\qquad \leq \min\{4.5495, 2.8361\} = 2.8361 \leq 2.8361\mathbb{E}_{\frac{2}{3}}(\tau^{\frac{2}{3}}) = \varphi(\tau).
\end{aligned}$$

Also, for $\tau \in (1,2]$, we get

$$|y(\tau) - \mathcal{I}_{1,\tau}^{\frac{2}{3}}(h_1(\tau, y(\tau)))| = |(\tau-1) - (\tau-1)| = 0.$$

Hence, $y(\tau)$ is a solution of an inequality (5.4)–(5.5).

Next, one can easily verify that

$$x(\tau) = \begin{cases} \tau^2, & \tau \in (0,1] \cup (2,3], \\ (\tau-1), & \tau \in (1,2] \end{cases}$$

is the unique solution of the problem (5.1)–(5.3).

As discussed in the proof of the Theorem 1, we have

$$\begin{aligned}
\mathcal{L} &= \max\left\{ \frac{L_{h_i}}{\Gamma(\beta)} \frac{(\sigma_i - \tau_i)^{\beta-\frac{1}{2}}}{\sqrt{(2\beta-1)}\sqrt{2\theta}} + \frac{M_f}{\Gamma(\alpha)} \frac{(\tau_{i+1} - \sigma_i)^{\alpha-\frac{1}{2}}}{\sqrt{(2\alpha-1)}\sqrt{2\theta}} \right. \\
&\qquad \left. + \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{(\tau_{i+1} - \sigma_i)^{\alpha+\frac{1}{2}}}{\sqrt{2\alpha+1}\sqrt{2\theta}}; i = 0,1 \right\} \\
&= \max\left\{ \frac{M_f}{\Gamma(\alpha)} \frac{1}{\sqrt{(2\alpha-1)}\sqrt{2\theta}} + \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{1}{\sqrt{\sqrt{2\alpha+1}}\sqrt{2\theta}}, \right. \\
&\qquad \frac{L_{h_1}}{\Gamma(\beta)} \frac{1}{\sqrt{(2\beta-1)}\sqrt{2\theta}} + \frac{M_f}{\Gamma(\alpha)} \frac{1}{\sqrt{(2\alpha-1)}\sqrt{2\theta}} \\
&\qquad \left. + \frac{N_f K_h}{\Gamma(\alpha+1)} \frac{1}{\sqrt{2\alpha+1}\sqrt{2\theta}} \right\} = \frac{7.5188\sqrt{3}}{\Gamma(\frac{2}{3})\sqrt{2\theta}}.
\end{aligned}$$

Choose $\theta > \left(\frac{7.5188\sqrt{3}}{\sqrt{2}\Gamma(\frac{2}{3})}\right)^2 = 46.2473$, so that $\mathcal{L} < 1$. For this choice of θ, we have: for $\tau \in (0,1]$,

$$|y(\tau) - x(\tau)|e^{-\theta\tau} = |\tau - \tau^2|e^{-\theta\tau} \leq (\tau + \tau^2) \leq 2,$$

for $\tau \in (1,2]$,

$$|y(\tau) - x(\tau)|e^{-\theta\tau} = 0$$

and for $\tau \in (2,3]$,

$$|y(\tau) - x(\tau)|e^{-\theta\tau} = |\tau - \tau^2|e^{-\theta\tau} \leq (\tau + \tau^2)e^{-2\theta} = 11e^{-2\theta}.$$

Thus,

$$|y(\tau) - x(\tau)|e^{-\theta\tau} \leq C_{f,\theta,h,\alpha,\beta,h_i,\varphi}(\psi + \varphi(\tau)), \quad \tau \in J = [0,3],$$

where $C_{f,\theta,h,\alpha,\beta,h_i,\ \varphi} = 1, \psi = 0$ and $\varphi(\tau) = 2.8361\mathbb{E}_{\frac{2}{3}}(\tau^{\frac{2}{3}})$.

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