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A QUASISTATIC UNILATERAL CONTACT PROBLEM WITH FRICTION FOR NONLINEAR ELASTIC MATERIALS

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Abstract. The aim of this paper is to prove the existence of a solution to the quasistatic unilateral contact problem with a modified version of Coulomb's law of dry friction for nonlinear elastic materials. We derive a variational incremental problem which admits a solution if the friction coefficient is sufficiently small and then by passing to the limit with respect to time we obtain the existence of a solution.
Key words: nonlinear elasticity, quasistatic frictional process, incremental problems, variational inequality

1. Introduction

Contact mechanics is the branch of solid mechanics which typically involves two bodies instead of one and focuses its objective on their common interface rather than their interiors. Contact problems involving deformable bodies are quite frequent in the industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve a complicated surface phenomena, and are modeled with highly nonlinear initial boundary value problems. Taking into account various frictional contact conditions associated with behavior laws becoming more and more complex leads to the introduction of new and non standard models, expressed by the aid of evolution variational inequalities. An early attempt to study frictional contact problems within the framework of variational inequalities was made in [7]. The mathematical, mechanical and numerical state of the art can be found in [11]. In this paper we consider a problem of frictional contact between a body and a rigid foundation in nonlinear elasticity. We assume that the forces applied to the body vary slowly in time so that the acceleration in the system is negligible. In this case we can study a quasistatic approach to the process. We model the friction with a modified version of Coulomb's law which has been derived in [14, 15]. In linear elasticity the quasistatic problem of unilateral contact using a normal compliance law has been studied in [1] by considering incremental problems and in [9] by a different method, based on a time regularization. The quasistatic contact problem with local or non-local friction has been solved respectively in [12] and in [4] by using a time discretization method. In [2] the quasistatic contact problem with Coulomb friction was solved by an established shifting technique used to obtain increased regularity at the contact surface and by the aid of auxiliary problems involving regularized friction terms and a so-called normal compliance penalization technique. Signorini 's problem with friction for nonlinear elastic materials has been solved in [5] by using the fixed point's method. In viscoelasticity, the contact problem with a normal compliance law has been solved in [13] by the same method. The book [8] introduces a mathematical theory of contact problems involving deformable bodies. In carrying out the variational analysis, the authors systematically use results on elliptic and evolutionary variational inequalities, convex analysis, nonlinear equations with monotone operators, and fixed points of operators.

The novelty of the present paper is to extend the results in [4] in the case when the elasticity operator is nonlinear, strongly monotone and Lipschitz continuous. As in [4], we propose a variational formulation written in the form of two inequalities: an inequality which describes the contact under a differential form with the velocity field as test function and an inequality which represents the unilateral condition. By means of Euler's implicit scheme as in [4, 12], the unilateral contact problem leads us to solve a well-posed variational inequality at each time step. Finally by using lower semicontinuity and compactness arguments we prove that the limit of the discrete solution is a solution to the continuous problem.

2. Problem Statement and Variational Formulation

The physical setting is as follows. Let $\Omega \subset \mathbf{R}^d$, $(d = 2, 3)$, be the reference domain occupied by the nonlinear elastic body. Ω is supposed to be open, bounded, with a sufficiently regular boundary Γ. Γ is partitioned into three parts $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where $\Gamma_1, \Gamma_2, \Gamma_3$ are disjoint open sets and $meas\Gamma_1 > 0$. The body is acted upon by a volume force of density φ_1 on Ω and a surface traction of density φ_2 on Γ_2. On Γ_3 the body is in unilateral contact with friction with a rigid foundation. Under these conditions the classical formulation of the mechanical problem is the following.

Problem P_1. *Find a displacement field* $u : \Omega \times [0, T] \to \mathbf{R}^d$ *such that*

$$div\sigma + \varphi_1 = 0 \quad \text{in } \Omega \times (0,T), \tag{2.1}$$

$$\sigma = \mathcal{F}\varepsilon(u) \quad \text{in } \Omega \times (0,T), \tag{2.2}$$

$$u = 0 \quad \text{on} \Gamma_1 \times (0,T), \tag{2.3}$$

$$\sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.4}$$

$$\sigma_\nu \leq 0,\ u_\nu \leq 0,\ \sigma_\nu u_\nu = 0 \quad \text{on } \Gamma_3 \times (0,T), \tag{2.5}$$

$$\begin{cases} |\sigma_\tau| \leq \mu p\left(|R\sigma_\nu(u)|\right), \\ |\sigma_\tau| < \mu p\left(|R\sigma_\nu(u)|\right) \Longrightarrow \dot{u}_\tau = 0, & \text{on } \Gamma_3 \times (0,T), \\ |\sigma_\tau| = \mu p\left(|R\sigma_\nu(u)|\right) \Longrightarrow \exists\lambda \geq 0 \text{ s.t. } \sigma_\tau = -\lambda \dot{u}_\tau \end{cases} \quad (2.6)$$

$$u(0) = u_0 \quad \text{in } \Omega. \quad (2.7)$$

Here equality (2.1) represents the equilibrium equation. Equality (2.2) represents the elastic constitutive law of the material in which $\mathcal{F}$ is a given function and $\varepsilon(u)$ denotes the small strain tensor; (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal on Γ and $\sigma\nu$ represents the Cauchy stress vector. Conditions (2.5) represent the unilateral contact boundary conditions. Conditions (2.6) represent the nonlocal friction law in which σ_τ denotes the tangential stress, $\dot{u}_\tau$ denotes the tangential velocity on the boundary and μ is the coefficient of friction. R is a continuous regularization operator representing the averaging of the normal stress over a small neighborhood of the contact point. The choice $p(r) = r_+(1-\delta r)_+$, where $r_+ = \max(r,0)$ and δ is a small positive coefficient related to the wear and hardness of the surface, was employed in [14, 15]. In (2.6) and below, a dot above a variable represents its derivative with respect to time. Finally (2.7) represents the initial condition. We denote by S_d the space of second order symmetric tensors on $\mathbf{R}^d$ $(d = 2,3)$. We recall that the inner products and the corresponding norms are given by

$$u.v = u_i v_i, \quad |v| = (u.v)^{\frac{1}{2}}, \quad \forall u, v \in \mathbf{R}^d,$$
$$\sigma.\tau = \sigma_{ij}\tau_{ij}, \quad |\tau| = (\tau.\tau)^{\frac{1}{2}}, \quad \forall \sigma, \tau \in S_d.$$

Here and below, $i, j = 1, \ldots, d$, and the summation convention over repeated indices is adopted. Moreover, in the sequel, the index that follows a comma indicates a partial derivative, e.g., $u_{i,j} = \partial u_i / \partial x_j$. To proceed with the variational formulation, we need the following functional spaces:

$$H = \left(L^2(\Omega)\right)^d, \quad H_1 = \left(H^1(\Omega)\right)^d, \quad Q = \left\{\tau = (\tau_{ij}); \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\},$$
$$H(div;\Omega) = \{\sigma \in Q; div\sigma \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products given by

$$\langle u, v\rangle_H = \int_\Omega u_i v_i \, dx, \quad \langle \sigma, \tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} \, dx.$$

The small strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = \{1, \ldots, d\},$$

here $div\sigma = (\sigma_{ij,j})$ is the divergence of σ. Let $\mathbf{H}^{\frac{1}{2}}(\Gamma) = \left(H^{\frac{1}{2}}(\Gamma)\right)^d$ and let $\gamma : H_1 \to \mathbf{H}^{\frac{1}{2}}(\Gamma)$ be the trace map. For every element $v \in H_1$, we also use the notation $\mathbf{v}$ for the trace γv of v on Γ and we denote by v_ν and v_τ the normal and tangential components of $\mathbf{v}$ on Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

Let $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ be the dual of $\mathbf{H}^{\frac{1}{2}}(\Gamma)$, for every $\sigma \in H(div;\Omega)$, $\sigma\nu$ can be defined as the element in $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ which satisfies the Green's formula:

$$\langle \sigma, \varepsilon(v)\rangle_Q + \langle div\sigma, v\rangle_H = \langle \sigma\nu, v\rangle_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)\times \mathbf{H}^{\frac{1}{2}}(\Gamma)} \quad \forall v \in H_1. \tag{2.8}$$

Denote by σ_ν and σ_τ the normal and tangential traces of σ, respectively. If σ is regular $\left(\text{say } C^1\right)$, then

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu,$$
$$\langle \sigma\nu, v\rangle_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)\times \mathbf{H}^{\frac{1}{2}}(\Gamma)} = \int_\Gamma \sigma\nu.v\, da \quad \forall v \in H_1,$$

where da is the surface measure element. Let V be the closed subspace of H_1 defined by $V = \{v \in H_1; v = 0 \text{ on } \Gamma_1\}$ and K be the set of admissible displacements given by $K = \{v \in V; v_\nu \leq 0 \text{ on } \Gamma_3\}$. Since $meas\Gamma_1 > 0$, the following Korn's inequality holds (see [7]):

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V,$$

where the constant c_Ω depends only on Ω and Γ_1. We equip V with the inner product

$$\langle u, v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_Q$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space. Moreover by the Sobolev trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.9}$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1,\infty}(0, T; V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|.\|_X)$ and $T > 0$ we use the notation $C([0,T]; X)$ for the space of continuous functions from $[0,T]$ to X; recall that $C([0,T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

The force and the traction densities are assumed to satisfy

$$\varphi_1 \in W^{1,\infty}(0,T;H), \quad \varphi_2 \in W^{1,\infty}\left(0,T;\left(L^2(\Gamma_2)\right)^d\right). \tag{2.10}$$

We denote by $\phi(t)$ the element of V defined by

$$\langle \phi(t), v\rangle_V = \int_\Omega \varphi_1(t).v\,dx + \int_{\Gamma_2} \varphi_2(t).v\,da \quad \forall v \in V,\ t \in [0,T].$$

Using (2.10) yields $\phi \in W^{1,\infty}(0,T;V)$. Let

$$H(\Gamma_3) = \left\{ w\,|_{\Gamma_3} : w \in H^{\frac{1}{2}}(\Gamma),\ w = 0 \text{ on } \Gamma_1 \right\}$$

equipped with the norm of $H^{\frac{1}{2}}(\Gamma)$. $\langle .,.\rangle$ shall denote the duality pairing on $H(\Gamma_3) \times H'(\Gamma_3)$ where $H'(\Gamma_3)$ denotes the dual of $H(\Gamma_3)$.

Before we start with the variational formulation of problem P_1 let us state in which sense the duality pairing $\langle .,.\rangle$ is taken. For $\sigma \in H(div;\Omega)$, if $\sigma\nu \in (L^2(\Gamma_2))^d$ in the sense of distributions, i.e. $\exists h \in (L^2(\Gamma_2))^d$ such that

$$\langle \sigma\nu, \varphi\rangle_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)\times\mathbf{H}^{\frac{1}{2}}(\Gamma)} = \int_{\Gamma_2} h.\varphi\,da \quad \forall \varphi \in (C_0^\infty(\Gamma_2))^d,$$

we define the normal stress σ_ν on Γ_3 as follows:

$$\begin{cases} \forall w \in H(\Gamma_3): \\ \langle \sigma_\nu, w\rangle = \langle \sigma, \varepsilon(v)\rangle_Q + \langle div\sigma, v\rangle_H - \int_{\Gamma_2} h.v\,da \\ \forall v \in V; \quad v_\nu = w,\ v_\tau = 0 \quad \text{on } \Gamma_3. \end{cases} \tag{2.11}$$

We assume that $R : H'(\Gamma_3) \to L^2(\Gamma_3)$ is a linear continuous mapping. It is obvious to check that, when δ is a given positive constant and sufficiently small that there exists a constant $L_p > 0$ such that

$$|p(r_1) - p(r_2)| \le L_p |r_1 - r_2|, \quad \forall r_1, r_2 \in \mathbf{R}. \tag{2.12}$$

We assume that $\mathcal{F} : \Omega \times S_d \to S_d$ satisfies the following conditions:

$$\begin{cases} (a) \text{ There exists } L_1 > 0 \text{ such that} \\ \quad |\mathcal{F}(x,\varepsilon_1) - \mathcal{F}(x,\varepsilon_2)| \le L_1 |\varepsilon_1 - \varepsilon_2| \\ \quad \text{for all } \varepsilon_1, \varepsilon_2 \in S_d, a.e.\ x \in \Omega. \\ (b) \text{ There exists } L_2 > 0 \text{ such that} \\ \quad (\mathcal{F}(x,\varepsilon_1) - \mathcal{F}(x,\varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \ge L_2 |\varepsilon_1 - \varepsilon_2|^2, \\ \quad \text{for all } \varepsilon_1, \varepsilon_2 \in S_d,\ a.e.\ x \in \Omega. \\ (c) \text{ The mapping } x \to \mathcal{F}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \quad \text{for any } \varepsilon \in S_d. \\ (d)\ \mathcal{F}(x,0) = 0 \text{ for } a.e.\ x \in \Omega. \end{cases} \tag{2.13}$$

First we note that the condition (2.13) is satisfied in the case of the linear elastic constitutive law $\sigma = \mathcal{F}\varepsilon(u)$ in which $\mathcal{F}\xi = a_{ijkh}\xi_{kh}$, provided that $a_{ijkh} \in L^\infty(\Omega)$ and there exists $\alpha > 0$ such that

$$a_{ijkh}(x)\,\xi_k\xi_h \ge \alpha|\xi|^2 \quad \forall \xi \in S_d,\ a.e.\ x \in \Omega.$$

Examples of nonlinear constitutive law which satisfy (2.13) can be find in [10] and [16].

Remark 1. $\mathcal{F}(x,\tau(x)) \in Q$, $\forall \tau \in Q$ and thus it is possible to consider $\mathcal{F}$ as an operator defined from Q into Q.

Next, we assume that the friction coefficient satisfies

$$\mu \in L^{\infty}(\Gamma_3) \quad \text{and} \quad \mu \geq 0 \ \ a.e. \text{ on } \Gamma_3, \tag{2.14}$$

and we define the functional $j : V_0 \times V \to \mathbf{R}$ by

$$j(u,v) = \int_{\Gamma_3} \mu p(|R\sigma_\nu(u)|)|v_\tau|\,da \quad \forall (u,v) \in V_0 \times V,$$

where V_0 is the subset of H_1 defined as follows

$$V_0 = \{v \in H_1;\ \ div\sigma(v) \in H\}.$$

If u is a solution of problem P_2 stated below then $\sigma(u(t)) \in H(div;\Omega)$ *a.e.* $t \in [0,T]$ and therefore

$$j(u(t),v) = \int_{\Gamma_3} \mu p(|R\sigma_\nu(u(t))|)|v_\tau|\,da \quad \forall v \in V.$$

Finally we assume that the initial data u_0 satisfies $u_0 \in K \cap V_0$,

$$\langle \mathcal{F}\varepsilon(u_0), \varepsilon(v-u_0)\rangle_Q + j(u_0, v-u_0) \geq \langle \phi(0), v-u_0\rangle_V \quad \forall v \in K. \tag{2.15}$$

Using Green's formula (2.8) it is straighforward to see that if u is a sufficiently regular function which satisfy (2.1)–(2.6) then,

$$\begin{cases} u(t) \in K \text{ for all } t \in [0,T], \text{and for almost all } t \in (0,T), \\ \langle \mathcal{F}\varepsilon(u(t)), \varepsilon(v-\dot{u}(t))\rangle_Q + j(u(t),v) - j(u(t),\dot{u}(t)) \geq \\ \qquad \langle \phi(t), v-\dot{u}(t)\rangle_V + \langle \sigma_\nu(u(t)), v_\nu - \dot{u}_\nu(t)\rangle \quad \forall v \in V, \\ \langle \sigma_\nu(u(t)), z - u_\nu(t)\rangle \geq 0 \quad \forall z \in K. \end{cases}$$

Therefore, using (2.7) and the previous inequalities yields to the following variational formulation of problem P_1.

Problem P_2. *Find a displacement field* $u \in W^{1,\infty}(0,T;V)$ *such that* $u(0) = u_0$ *in* Ω, $u(t) \in K \cap V_0$, *for all* $t \in [0,T]$, *and for almost all* $t \in (0,T)$,

$$\langle \mathcal{F}\varepsilon(u(t)), \quad \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_Q + j(u(t),v) - j(u(t),\dot{u}(t)) \geq$$
$$\langle \phi(t), v-\dot{u}(t)\rangle_V + \langle \sigma_\nu(u(t)), v_\nu - \dot{u}_\nu(t)\rangle \quad \forall v \in V, \tag{2.16}$$

$$\langle \sigma_\nu(u(t)), z_\nu - u_\nu(t)\rangle \geq 0 \quad \forall z \in K. \tag{2.17}$$

Our main result of this section which will be established in the next section is the following theorem.

Theorem 1. *Let* $T > 0$ *and assume that* (2.10), (2.12), (2.13), (2.14) *and* (2.15) *hold. Then there exists at least one solution* u *of problem* P_2 *for a sufficiently small friction coefficient* μ.

3. Incremental Formulation

The proof of Theorem 1 is based on a time discretization method followed by a fixed point arguments, similar to those used in [4] and is carried out in several steps. For this, let $n \in \mathbf{N}^*$, $\Delta t = \frac{T}{n}$, $t_i = i\Delta t$, $0 \leq i \leq n$. We denote by u^i the approximation of u at the time t_i and by the symbol Δu^i the backward difference $u^{i+1} - u^i$. For a continuous function $w(t)$ we use the notation $w^i = w(t_i)$. Then we obtain a sequence of incremental problems P_n^i defined for $u^0 = u_0$ by

Problem P_n^i. *Find* $u^{i+1} \in K \cap V_0$ *such that*

$$\begin{cases} \left\langle \mathcal{F}\varepsilon(u^{i+1}), \varepsilon(w) - \varepsilon\left(u^{i+1}\right)\right\rangle_Q + j\left(u^{i+1}, w - u^i\right) - j\left(u^{i+1}, u^{i+1} - u^i\right) \geq \\ \qquad \left\langle \phi^{i+1}, w - u^{i+1}\right\rangle_V + \left\langle \sigma_\nu\left(u^{i+1}\right), w_\nu - u_\nu^{i+1}\right\rangle \quad \forall w \in V, \\ \left\langle \sigma_\nu\left(u^{i+1}\right), z_\nu - u_\nu^{i+1}\right\rangle \geq 0 \qquad \forall z \in K. \end{cases}$$

We have the following result.

Proposition 1. *Problem* P_n^i *is equivalent to the following*
Problem Q_n^i. *Find* $u^{i+1} \in K \cap V_0$ *such that*

$$\begin{cases} \left\langle \mathcal{F}\varepsilon\left(u^{i+1}\right), \varepsilon(w) - \varepsilon\left(u^{i+1}\right)\right\rangle_Q + j\left(u^{i+1}, w - u^i\right) \\ -j\left(u^{i+1}, u^{i+1} - u^i\right) \geq \left\langle \phi^{i+1}, w - u^{i+1}\right\rangle_V \ \forall\, w \in K. \end{cases} \tag{3.1}$$

Proof. We refer the reader to [4] for the linear case. The proof is easily extended to the nonlinear case as we replace only the bilinear form $a : V \times V \to \mathbf{R}$ continuous and $V-$elliptic by a nonlinear operator $A : V \to V$ strongly monotone and Lipschitz continuous. ■

Lemma 1. *There exists* $\mu_0 > 0$ *such that for* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, *problem* Q_n^i *has a unique solution.*

To show lemma 1 we introduce an intermediate problem. We define the convex set $C_+^* = \left\{g \in L^2(\Gamma_3); g \geq 0 \text{ a.e. on } \Gamma_3\right\}$ and

$$\varphi(w) = \int_{\Gamma_3} \mu g \left|w_\tau\right| da.$$

We introduce the intermediate problem Q_{ng}^i for $g \in C_+^*$ by replacing in (3.1) $R\sigma_\nu\left(u^{i+1}\right)$ by g as follows.

Problem Q_{ng}^i. *Find* $u_g \in K$ *such that*

$$\left\langle \mathcal{F}\varepsilon(u_g), \varepsilon(w) - \varepsilon(u_g)\right\rangle_Q + \varphi\left(w - u^i\right) - \varphi\left(u_g - u^i\right) \geq \left\langle \phi^{i+1}, w - u_g\right\rangle_V \quad \forall\, w \in K. \tag{3.2}$$

Lemma 2. *For any* $g \in C_+^*$ *problem* Q_{ng}^i *has a unique solution* u_g. *Furthermore, there exist constants* $c_i > 0$, $i = 1, 2$, *such that*

$$\left\|u_g\right\|_V \leq c_1 \left\|\mu\right\|_{L^\infty(\Gamma_3)} \left\|g\right\|_{L^2(\Gamma_3)} + c_2 \left\|\phi^{i+1}\right\|_V. \tag{3.3}$$

Proof. Using Riesz's representation theorem we define the nonlinear operator $A : V \to V$ by

$$\langle Av, w \rangle_V = \langle \mathcal{F}\varepsilon(v), \varepsilon(w) \rangle_Q .$$

Then hypotheses (2.13) (b) and (2.13) (d) on $\mathcal{F}$ imply that the operator A is strongly monotone and Lipschitz continuous; on the other hand the functional φ is proper, convex and lower semicontinuous. Some results from the theory of elliptic variational inequalities (see [3]) imply that the inequality (3.2) has a unique solution u_g. Setting $w = 0$ in the inequality (3.2) and using both the hypothesis (2.13) (d) on $\mathcal{F}$ and the inequality

$$\left| \left| (u_g - u^i)_\tau \right| - \left| u^i_\tau \right| \right| \le |u_{g\tau}| ,$$

we see that there exist constants $c_i > 0,\ i = 1, 2$, such that

$$\|u_g\|^2_V \le c_1 \|\mu\|_{L^\infty(\Gamma_3)} \|g\|_{L^2(\Gamma_3)} \|u_g\|_V + c_2 \left\|\phi^{i+1}\right\|_V \|u_g\|_V .$$

Simplifying by the norm $\|u_g\|_V$ we have the inequality (3.3) . ■

Lemma 3. *Let $\Psi : C^*_+ \to C^*_+$ be the mapping defined by*

$$g \to \Psi(g) = p\left(|R\sigma_\nu(u_g)|\right).$$

There exists $\mu_0 > 0$ such that for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, Ψ has a fixed point g^* and u_{g^*} is a solution of problem Q^i_n.

Proof. As for $g \in L^2(\Gamma_3)$, $\sigma_\nu(u_g)$ is defined on Γ_3 and belongs to $H'(\Gamma_3)$, using (2.12) we have

$$\begin{aligned} |\Psi(g_1) - \Psi(g_2)| &= |p(|R\sigma_\nu(u_{g_1})|) - p(|R\sigma_\nu(u_{g_2})|)| \\ &\le L_p \left||R\sigma_\nu(u_{g_1})| - |R\sigma_\nu(u_{g_2})|\right| \le L_p \left|R(\sigma_\nu(u_{g_1}) - \sigma_\nu(u_{g_2}))\right| . \end{aligned}$$

Thus we deduce that

$$\|\Psi(g_1) - \Psi(g_2)\|_{L^2(\Gamma_3)} \le L_p \|R(\sigma_\nu(u_{g_1}) - \sigma_\nu(u_{g_2}))\|_{L^2(\Gamma_3)} .$$

Then using the continuity of the mapping R, there exists a constant $c > 0$ such that

$$\|R(\sigma_\nu(u_{g_1}) - \sigma_\nu(u_{g_2}))\|_{L^2(\Gamma_3)} \le c \|\sigma_\nu(u_{g_1}) - \sigma_\nu(u_{g_2})\|_{H'(\Gamma_3)} ,$$

and using the relation (2.11) and (2.13) (b) there exists a constant $c' > 0$ such that

$$\|\sigma_\nu(u_{g_1}) - \sigma_\nu(u_{g_2})\|_{H'(\Gamma_3)} \le c' \|u_{g_1} - u_{g_2}\|_V .$$

Therefore we deduce that

$$\|\Psi(g_1) - \Psi(g_2)\|_{L^2(\Gamma_3)} \le cc' \|u_{g_1} - u_{g_2}\|_V .$$

On the other hand set $v = u_{g_1}$ in $Q^i_{ng_2}$ and $v = u_{g_2}$ in $Q^i_{ng_1}$ and add them, we obtain by using (2.13) (b), (2.9) and (2.12) that there exists a constant $d' > 0$ such that

$$\|u_{g_1} - u_{g_2}\|_V \le d' \|\mu\|_{L^\infty(\Gamma_3)} \|g_1 - g_2\|_{L^2(\Gamma_3)} .$$

Hence we find that

$$\|\Psi(g_1) - \Psi(g_2)\|_{L^2(\Gamma_3)} \le cc'd'L_p \|\mu\|_{L^\infty(\Gamma_3)} \|g_1 - g_2\|_{L^2(\Gamma_3)} ,$$

and when $\mu_0 = \dfrac{1}{cc'd'L_p}$, we have for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, that the mapping Ψ is a contraction. Thus, it has a fixed point g^* and u_{g*} is the solution of problem Q^i_n. As $u_{g*} \in H(div;\Omega)$ it follows that $u^{i+1} \in V_0$ and so $u^{i+1} \in K \cap V_0$. ■

Lemma 4. *We have the following estimates: there exists a constant* $\mu_1 > 0$ *such that for* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$*, there exist* $d_i > 0$, *i=1, 2, such that*

$$\left\|u^{i+1}\right\|_V \le d_1 \left\|\phi^{i+1}\right\|_V , \quad \left\|\Delta u^i\right\|_V \le d_2 \left\|\Delta\phi^i\right\|_V . \tag{3.4}$$

Proof. By setting $w = 0$ in the inequality (3.1) and by using hypothesis (2.13) (b) on $\mathcal{F}$ and the properties of j, there exists $c_1 > 0$ such that for $\|\mu\|_{L^\infty(\Gamma_3)} < c_1$, we deduce that there exists $d_1 > 0$ such that the first inequality (3.4) is satisfied. To show the second inequality (3.4) we consider the translated inequality of (3.1) at the time t_i that is

$$\begin{aligned}\left\langle \mathcal{F}\varepsilon(u^i), \varepsilon(w) - \varepsilon\left(u^i\right)\right\rangle_Q + j\left(u^i, v - u^{i-1}\right) - j\left(u^i, u^i - u^{i-1}\right)& \\ \ge \left\langle \phi^i, w - u^i\right\rangle_V \quad \forall\, w \in K.& \end{aligned} \tag{3.5}$$

Setting $w = u^i$ in (3.1) and $w = u^{i+1}$ in (3.5) and adding them, we obtain the inequality

$$\begin{aligned}-\left\langle \mathcal{F}\varepsilon\left(u^{i+1}\right) - \mathcal{F}\varepsilon\left(u^i\right), \varepsilon\left(\Delta u^i\right)\right\rangle_Q - j\left(u^{i+1}, \Delta u^i\right) + j\left(u^i, u^{i+1} - u^{i-1}\right)& \\ - j\left(u^i, u^i - u^{i-1}\right) \ge \left\langle -\Delta\phi^i, \Delta u^i\right\rangle_V .&\end{aligned}$$

Furthermore by using the inequality

$$\left|\left|u^{i+1}_\tau - u^{i-1}_\tau\right| - \left|u^i_\tau - u^{i-1}_\tau\right|\right| \le \left|u^{i+1}_\tau - u^i_\tau\right| ,$$

we have

$$j\left(u^i, u^{i+1} - u^{i-1}\right) - j\left(u^i, u^i - u^{i-1}\right) \le j\left(u^i, \Delta u^i\right) ,$$

and also the inequality

$$-\left\langle \mathcal{F}\varepsilon\left(u^{i+1}\right) - \mathcal{F}\varepsilon\left(u^i\right), \varepsilon\left(\Delta u^i\right)\right\rangle_Q + j\left(u^i, \Delta u^i\right) - j\left(u^{i+1}, \Delta u^i\right) \ge \left\langle -\Delta\phi^i, \Delta u^i\right\rangle_V .$$

Using the relation (2.11), there exists a constant $c_3 > 0$ such that

$$\left\|\sigma_\nu\left(u^{i+1}\right)-\sigma_\nu\left(u^i\right)\right\|_{H'(\Gamma_3)} \le c_3\left(\left\|\Delta u^i\right\|_V+\left\|\Delta\phi^i\right\|_V\right),$$

and then by using (2.9), (2.12), (2.13) (b) and the properties of j we deduce that there exist two positive constants d_3 and d_4 such that

$$L_2\left\|\Delta u^i\right\|_V^2 \le d_3 L_p \|\mu\|_{L^\infty(\Gamma_3)}\left\|\Delta u^i\right\|_V^2+d_4\left\|\Delta\phi^i\right\|_V\left\|\Delta u^i\right\|_V.$$

By setting $c_2=\frac{L_2}{2d_3L_p}$, we deduce that if $\|\mu\|_{L^\infty(\Gamma_3)}<c_2$, there exists $d_2>0$ such that

$$\left\|\Delta u^i\right\|_V \le d_2\left\|\Delta\phi^i\right\|_V.$$

Now, by taking $\mu_1=\min(c_1,c_2)$ we prove the lemma. ■

4. Existence of Solutions

The main result of this paragraph is to show the existence of a solution obtained as a limit of the interpolate function of the discrete solution.

We define the continuous function u^n in $[0,T]\to V$ by

$$u^n(t)=u^i+\frac{(t-t_i)}{\Delta t}\Delta u^i \quad \text{on } [t_i,t_{i+1}],\quad i=0,\dots,n-1.$$

As in [8] we have the following result.

Lemma 5. *There exists a function $u\in W^{1,\infty}(0,T;V)$ such that passing to a subsequence still denoted (u^n) we have*

$$u^n\to u \textit{ weak}*\ \textit{in } W^{1,\infty}(0,T;V).$$

Proof. From (4.1) we deduce that the sequence (u^n) is bounded in $C([0,T];V)$ and there exists a constant $c_3>0$ such that

$$\max_{t\in[0,T]}\|u^n(t)\|_V \le c_3\|\phi\|_{C([0,T];V)}.$$

From (4.2) we deduce that the sequence $(\dot u^n)$ is bounded in $L^\infty(0,T;V)$ and there exists a constant $c_4>0$ such that

$$\|\dot u^n(t)\|_V=\max_{0\le i\le n-1}\left\|\frac{\Delta u^{t_i}}{\Delta t}\right\| \le c_4\left\|\dot\phi\right\|_{L^\infty(0,T;V)}.$$

Consequently each one of the functions u^n belongs to the space $W^{1,\infty}(0,T;V)$ and the sequence (u^n) is bounded there. Then there exists a function $u\in W^{1,\infty}(0,T;V)$ and a subsequence still denoted (u^n) such that

$$u^n\to u \textit{ weak}*\ \text{in } W^{1,\infty}(0,T;V).$$

■

As in [10] let us introduce the following piecewise constant functions

$$\widetilde{u}^n : [0,T] \to V, \quad \tilde{\varphi}_1^n : [0,T] \to H, \quad \widetilde{\phi}^n : [0,T] \to V,$$
$$\widetilde{u}^n(t) = u^{i+1}, \quad \tilde{\varphi}_1^n(t) = \varphi_1(t_{i+1}), \quad \widetilde{\phi}^n(t) = \phi(t_{i+1})$$

defined for $t \in (t_i, t_{i+1}]$ and $i = 0, \ldots, n-1$.

Lemma 6. *There exists a subsequence still denoted* $(\tilde{u}_n)$ *such that*

$$(i) \quad \widetilde{u}^n \to u \textit{ weak} * \textit{ in } L^\infty(0,T;V),$$
$$(ii) \quad \widetilde{u}^n(t) \to u(t) \textit{ weakly in } V \textit{ a.e. } t \in [0,T],$$
$$(iii) \quad u(t) \in K \textit{ for all } t \in [0,T].$$

Proof. From (3.4) we deduce that the sequence $(\tilde{u}^n)$ is bounded in $L^\infty(0,T;V)$. Thus, we can extract from it a subsequence still denoted $(\tilde{u}^n)$ which converges weakly $*$ in $L^\infty(0,T;V)$. On the other hand, from [8] we deduce that for almost every $t \in (0,T)$

$$\|\tilde{u}^n(t) - u^n(t)\|_V \le \frac{T}{n} \|\dot{u}^n(t)\|_V, \tag{4.1}$$

and, since $(\dot{u}^n)$ is bounded in $L^\infty(0,T;V)$, we deduce from (4.1) that $\tilde{u}^n \to u$ *weak* $*$ in $L^\infty(0,T;V)$ as $n \to +\infty$, whence (i).

For the proof of (ii), since $W^{1,\infty}(0,T;V) \hookrightarrow C([0,T];V)$, we have $u^n(t) \to u(t)$ weakly in V, for all $t \in [0,T]$, and from (4.1) we conclude immediately.

For the proof of (iii) it suffices to remark that $\tilde{u}^n(t) \in K$ *a.e.* $t \in [0,T]$ and using the continuity of u. ■

Remark 2. As $\phi \in W^{1,\infty}(0,T;V)$ we have

$$\widetilde{\phi}^n \to \phi \text{ strongly in } L^2(0,T;V). \tag{4.2}$$

Lemma 7. *The sequence* $(\tilde{u}^n)$ *converges strongly to* u *in* $L^2(0,T;V)$.

Proof. From inequality (3.1) we deduce the inequality $\forall\, w \in K$

$$\left\langle \mathcal{F}\varepsilon\left(u^{i+1}\right), \varepsilon(w) - \varepsilon\left(u^{i+1}\right)\right\rangle_Q + j\left(u^{i+1}, w - u^{i+1}\right) \ge \left\langle \phi^{i+1}, w - u^{i+1}\right\rangle_V.$$

Therefore

$$\begin{aligned}&\langle \mathcal{F}\varepsilon(\tilde{u}^n(t)), \varepsilon(w) - \varepsilon(\tilde{u}^n(t))\rangle_Q + j(\tilde{u}^n(t), w - \tilde{u}^n(t)) \\ &\qquad \ge \left\langle \tilde{\phi}^n(t), w - \tilde{u}^n(t)\right\rangle_V \quad \forall\, w \in K, \textit{ a.e. } t \in [0,T].\end{aligned} \tag{4.3}$$

To show the strong convergence, we take $v = \tilde{u}^{n+m}(t)$ in (4.3) and $v = \tilde{u}^n(t)$ in the same inequality satisfied by $\tilde{u}^{n+m}(t)$ and adding them, we obtain the inequality

$$\begin{aligned}&\left\langle \mathcal{F}\varepsilon\left(\tilde{u}^{n+m}(t)\right) - \mathcal{F}\varepsilon(\tilde{u}^n(t)), \varepsilon(\tilde{u}^n(t)) - \varepsilon\left(\tilde{u}^{n+m}(t)\right)\right\rangle_Q \\ &\quad + j\left(\tilde{u}^{n+m}(t), \tilde{u}^{n+m}(t) - \tilde{u}^n(t)\right) + j\left(\tilde{u}^n(t), \tilde{u}^{n+m}(t) - \tilde{u}^n(t)\right) \\ &\quad \ge -\left\langle \tilde{\phi}^{n+m}(t) - \tilde{\phi}^n(t), \tilde{u}^{n+m}(t) - \tilde{u}^n(t)\right\rangle_V.\end{aligned}$$

Then using (2.12) (b) and the continuity of the mapping R, we deduce that

$$\begin{aligned}\|p(|R\sigma_\nu(u)|)\|_{L^2(\Gamma_3)} &\le L_p \|R\sigma_\nu(u)\|_{L^2(\Gamma_3)}\\ &\le L_p C\|\sigma_\nu(u)\|_{H'(\Gamma_3)} \le L_p C_1\Big(\sup_{t\in(0,T)}\|\tilde{u}^n(t)\|_V + \sup_{t\in(0,T)}\Big\|\tilde{\phi}^n(t)\Big\|_V\Big).\end{aligned}$$

As there exists a constant $C' > 0$ such that

$$\sup_{t\in(0,T)}\|\tilde{u}^n(t)\|_V \le C'\|\phi\|_{W^{1,\infty}(0,T;V)},$$

we deduce that there exists a constant $C_2 > 0$ such that

$$\begin{aligned}\left\|\tilde{u}^{n+m}(t)-\tilde{u}^n(t)\right\|_V^2 \le C_2\Big(&\|\mu\|_{L^\infty(\Gamma_3)}\|\phi\|_{W^{1,\infty}(0,T;V)}\\ &\times\left\|\tilde{u}^{n+m}(t)-\tilde{u}^n(t)\right\|_{L^2(\Gamma_3)^d} + \left\|\tilde{\phi}^{n+m}(t)-\tilde{\phi}^n(t)\right\|_V^2\Big).\end{aligned}$$

Keeping in mind that

$$\begin{aligned}\left\|\tilde{u}_\tau^{n+m}(t)-\tilde{u}_\tau^n(t)\right\|_{L^2(\Gamma_3)^d} &\le \left\|\tilde{u}_\tau^{n+m}(t)-u_\tau^{n+m}(t)\right\|_{L^2(\Gamma_3)^d}\\ &+\left\|u_\tau^{n+m}(t)-u_\tau^n(t)\right\|_{L^2(\Gamma_3)^d} + \|u_\tau^n(t)-\tilde{u}_\tau^n(t)\|_{L^2(\Gamma_3)^d},\end{aligned}$$

as (u^n) is bounded in $W^{1,\infty}(0,T;V)$, from the continuity of the trace map, we obtain that the sequence $(u^n|_{\Gamma_3})$ is bounded in $W^{1,\infty}\left(0,T;\left(L^2(\Gamma_3)\right)^d\right)$. It follows from the Arzela-Ascoli theorem that it is relatively compact in $C\left([0,T];\left(L^2(\Gamma_3)\right)^d\right)$ and therefore there exists a subsequence, still denoted by (u^n) such that

$$\forall\eta>0, \exists n_1\in\mathbf{N} \text{ such that} : \forall n\ge n_1,\quad \forall m\in\mathbf{N},\quad \forall t\in[0,T]:$$

$$\|u_\tau^{n+m}(t)-u_\tau^n(t)\|_{L^2(\Gamma_3)^d}\le\eta.$$

On the other hand using (2.9), we have for almost every $t\in(0,T)$

$$\|u_\tau^n(t)-\tilde{u}_\tau^n(t)\|_{(L^2(\Gamma_3))^d}\le d_\Omega\|u^n(t)-\tilde{u}^n(t)\|_V\le d_\Omega\frac{T}{n}\|\dot{u}^n(t)\|_V.$$

Combining these results we obtain that there exists a positive constant C_1 such that

$$\int_0^T\left\|\tilde{u}_\tau^{n+m}(t)-\tilde{u}_\tau^n(t)\right\|_{L^2(\Gamma_3)^d}^2 dt\le C_1\left(\frac{1}{n^2}+\eta^2\right).$$

On the other hand from (4.2) we have

$$\forall\eta>0, \exists n_2\in\mathbf{N}, \text{ such that} : \forall n\ge n_2, \forall m\in\mathbf{N}:$$

$$\int_0^T\left\|\tilde{\phi}^{n+m}(t)-\tilde{\phi}^n(t)\right\|_V^2 dt\le\eta.$$

Then we obtain that there exists a constant $C_2 > 0$ such that

$$\forall \eta > 0,\ \exists n_3 \in \mathbf{N}, \text{ such that} : \forall n \geq n_3 = \max(n_1, n_2),$$
$$\forall m \in N : \int_0^T \left\| \widetilde{u}^{n+m}(t) - \widetilde{u}^n(t) \right\|_V^2 dt \leq C_2 \Big(2\eta + \frac{1}{n} \Big).$$

On the other hand

$$\forall \eta > 0,\ \exists n_4 \in \mathbf{N}, \text{ such that} : \forall n \geq n_4 : \frac{1}{n} \leq \eta.$$

We thus deduce that

$$\forall \eta > 0,\ \exists n_5 = \max(n_4, n_3),\ \text{such that } \forall n \geq n_5 :$$
$$\int_0^T \left\| \widetilde{u}^{n+m}(t) - \widetilde{u}^n(t) \right\|_V^2 dt \leq 3C_2 \eta.$$

So we conclude that

$$\widetilde{u}^n \to u \text{ strongly in } L^2(0, T; V). \tag{4.4}$$

■

Proposition 2. *The function u is a solution of problem P_2.*

Proof. To prove that u is a solution of problem P_2, in the first inequality of problem P_n^i and for $v \in V$, set $w = u^i + v\Delta t$ and divide by Δt; we obtain the inequality

$$\Big\langle \mathcal{F}\varepsilon\left(u^{i+1}\right), \varepsilon(v) - \varepsilon\Big(\frac{\Delta u^i}{\Delta t}\Big) \Big\rangle_Q + j\left(u^{i+1}, v\right) - j\Big(u^{i+1}, \frac{\Delta u^i}{\Delta t}\Big)$$
$$\geq \Big\langle \phi(t_{i+1}), v - \frac{\Delta u^i}{\Delta t} \Big\rangle_V + \Big\langle \sigma_\nu\left(u^{i+1}\right), v_\nu - \frac{\Delta u^i_\nu}{\Delta t} \Big\rangle, \quad \forall\, v \in V.$$

Whence for any $v \in L^2(0, T; V)$, we have

$$\langle \mathcal{F}\varepsilon(\widetilde{u}^n(t)), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t)) \rangle_Q + j(\widetilde{u}^n(t), v(t)) - j(\widetilde{u}^n(t), \dot{u}^n(t))$$
$$\geq \Big\langle \widetilde{\phi}^n(t), v(t) - \dot{u}^n(t) \Big\rangle_V + \langle \sigma_\nu(\widetilde{u}^n(t)), v_\nu(t) - \dot{u}^n_\nu(t) \rangle \text{ for } a.a.\ t \in [0, T].$$

Integrating both sides of the previous inequality on $(0, T)$, we obtain the inequality

$$\int_0^T \langle \mathcal{F}\varepsilon(\widetilde{u}^n(t)), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t)) \rangle_Q \, dt + \int_0^T j(\widetilde{u}^n(t), v(t))\, dt$$
$$- \int_0^T (j(\widetilde{u}^n(t), \dot{u}^n(t)))\, dt \geq \int_0^T \Big\langle \widetilde{\phi}^n(t), v(t) - \dot{u}^n(t) \Big\rangle_V dt$$
$$+ \int_0^T \langle \sigma_\nu(\widetilde{u}^n(t)), v_\nu(t) - \dot{u}^n_\nu(t) \rangle \, dt. \tag{4.5}$$

The statement is proved by passing to the limit in the previous inequality and by using the following result of Lemma 8. ■

Lemma 8. *We have that* $u(t) \in V_0$ *for all* $t \in [0,T]$.

Proof. As $u(t) \in K$ *for all* $t \in [0,T]$ we have $u(t) \in H_1$ *for all* $t \in [0,T]$. It suffices to show that $div\sigma(u(t)) \in H$ *for all* $t \in [0,T]$. From inequality (4.3) we obtain that

$$div\sigma(\widetilde{u}^n(t)) + \tilde{\varphi}_1^n(t) = 0 \quad \textit{for all } t \in [0,T],$$

and then we have

$$\|div\sigma(\widetilde{u}^n(t))\|_H = \|\tilde{\varphi}_1^n(t)\|_H \leq \sup_{t\in(0,T)} \|\varphi_1(t)\|_H .$$

We deduce that for a fixed $t \in (0,T)$ the sequence $(div\sigma(\widetilde{u}^n(t)))$ is bounded in H, then there exists a function $\chi_t \in H$ such that after passing to a subsequence still denoted $(div\sigma(\widetilde{u}^n(t)))$ we have

$$(div\sigma(\widetilde{u}^n(t))) \rightarrow \chi_t \text{ weakly in } H.$$

As $div\sigma(\widetilde{u}^n(t)) \rightarrow div\sigma(u(t))$ in the sense of distributions, we deduce

$$div\,\sigma(u(t)) = \chi_t.$$

So that $u(t) \in V_0$ *for all* $t \in [0,T]$. ■

Remark 3. As $u(t) \in K$ for all $t \in [0,T]$ and $u(t) \in V_0$ for all $t \in [0,T]$, it follows that $u(t) \in K \cap V_0$ for all $t \in [0,T]$.

Lemma 9. *We have*

$$\liminf_{n\to\infty} \int_0^T j(\widetilde{u}^n(t), \dot{u}^n(t))\,dt \geq \int_0^T j\big(u(t), \dot{u}(t)\big)\,dt.$$

Proof. We write

$$j(\widetilde{u}^n(t), \dot{u}^n(t)) = (j(\widetilde{u}^n(t), \dot{u}^n(t)) - j(u(t), \dot{u}^n(t))) + j(u(t), \dot{u}^n(t)),$$

then we have

$$\left|\int_0^T (j(\widetilde{u}^n(t), \dot{u}^n(t)) - j(u(t), \dot{u}^n(t)))\,dt\right|$$
$$\leqslant c\,\|\mu\|_{L^\infty(\Gamma_3)} L_p \|R(\sigma_\nu(\widetilde{u}^n) - \sigma_\nu(u))\|_{L^2(0,T;L^2(\Gamma_3))} \|\dot{u}_\tau^n\|_{L^2\left(0,T;L^2(\Gamma_3)^d\right)} .$$

Using the continuity of the mapping R, we get that there exists a constant $c > 0$ such that

$$\|R(\sigma_\nu(\widetilde{u}^n(t)) - \sigma_\nu(u(t)))\|_{L^2(\Gamma_3)} \leqslant c\,\|\sigma_\nu(\widetilde{u}^n(t)) - \sigma_\nu(u(t))\|_{H'(\Gamma_3)} \quad a.e.\ t \in (0,T).$$

On the other hand using the definition of σ_ν, there exists a constant $c' > 0$ such that

$$\|\sigma_\nu(\widetilde{u}^n(t)) - \sigma_\nu(u(t))\|_{H'(\Gamma_3)} \leqslant c'\left(\|\widetilde{u}^n(t) - u(t)\|_V + \left\|\tilde{\phi}^n(t) - \phi(t)\right\|_V\right)$$
$$a.e.\ t \in (0,T).$$

Whence

$$\|R(\sigma_\nu(\widetilde{u}^n(t)) - \sigma_\nu(u(t)))\|_{L^2(\Gamma_3)}$$
$$\leqslant cc'\left(\|\widetilde{u}^n(t) - u(t)\|_V + \left\|\tilde{\phi}^n(t) - \phi(t)\right\|_V\right) \quad a.e.\ t \in (0,T),$$

and so there exists a constant $C' > 0$ such that

$$\|R(\sigma_\nu(\widetilde{u}^n)) - \sigma_\nu(u))\|_{L^2(0,T;L^2(\Gamma_3))}$$
$$\leqslant C'\Big(\|\widetilde{u}^n - u\|_{L^2(0,T;V)} + \left\|\widetilde{\phi}^n - \phi\right\|_{L^2(0,T;V)}\Big).$$

From (4.2) and (4.4) we deduce that

$$\lim_{n\to\infty} \|R(\sigma_\nu(\widetilde{u}^n) - \sigma_\nu(u))\|_{L^2(0,T:L^2(\Gamma_3))} = 0, \tag{4.6}$$

and we have

$$\liminf_{n\to\infty} \int_0^T j(u(t), \dot{u}^n(t))\,dt \geq \int_0^T j\big(u(t), \dot{u}(t)\big)\,dt$$

by Mazur's lemma. ■

Lemma 10. *For any $v \in L^2(0,T;V)$ we have the following properties:*

$$\lim_{n\to\infty} \int_0^T \langle \mathcal{F}\varepsilon(\widetilde{u}^n(t)), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t))\rangle_Q\,dt$$
$$= \int_0^T \big\langle \mathcal{F}\varepsilon(u(t)), \varepsilon(v(t)) - \varepsilon\big(\dot{u}(t)\big)\big\rangle_Q\,dt, \tag{4.7}$$

$$\lim_{n\to\infty} \int_0^T j(\widetilde{u}^n(t), v(t))\,dt = \int_0^T j(u(t), v(t))\,dt, \tag{4.8}$$

$$\lim_{n\to\infty} \int_0^T \left\langle \widetilde{\phi}^n(t), v(t) - \dot{u}^n(t)\right\rangle_V dt = \int_0^T \langle \phi(t), v(t) - \dot{u}(t)\rangle_V\,dt. \tag{4.9}$$

Proof. For the proof of (4.7), we write:

$$\int_0^T \langle \mathcal{F}\varepsilon(\widetilde{u}^n(t)), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t))\rangle_Q\,dt$$
$$= \int_0^T \langle \mathcal{F}\varepsilon(\widetilde{u}^n(t)) - \mathcal{F}\varepsilon(u(t)), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t))\rangle_Q\,dt$$
$$+ \int_0^T \langle \mathcal{F}\varepsilon(u(t)), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t))\rangle_Q\,dt.$$

Using (4.4) and (2.12)(a) we have

$$\left|\int_0^T \langle \mathcal{F}\varepsilon\left(\widetilde{u}^n(t)\right) - \mathcal{F}\varepsilon\left(u(t)\right), \varepsilon\left(v(t)\right) - \varepsilon\left(\dot{u}^n(t)\right)\rangle_Q \, dt\right|$$
$$\leq c\left\|\widetilde{u}^n - u\right\|_{L^2(0,T;V)} \left(\|v\|_{L^2(0,T;V)} + \left\|\dot{u}^n\right\|_{L^2(0,T;V)}\right) \to 0.$$

We deduce that

$$\lim_{n\to\infty}\int_0^T \langle \mathcal{F}\varepsilon\left(\widetilde{u}^n(t)\right) - \mathcal{F}\varepsilon\left(u(t)\right), \varepsilon\left(v(t)\right) - \varepsilon\left(\dot{u}^n(t)\right)\rangle_Q \, dt = 0,$$

and, we have

$$\int_0^T \langle \mathcal{F}\varepsilon\left(u(t)\right), \varepsilon\left(v(t)\right) - \varepsilon\left(\dot{u}^n(t)\right)\rangle_Q \, dt = \int_0^T \left(Au(t), v(t) - \dot{u}^n(t)\right)_V dt$$
$$\to \int_0^T \left(Au(t), v(t) - \dot{u}(t)\right)_V dt = \int_0^T \left\langle \mathcal{F}\varepsilon\left(u(t)\right), \varepsilon\left(v(t)\right) - \varepsilon\left(\dot{u}(t)\right)\right\rangle_Q dt.$$

Using (4.6) and (4.2), it is straightforward to prove (4.8) and (4.9) respectively. Passing now to the limit in inequality (4.5), we obtain the inequality

$$\int_0^T \left\langle \mathcal{F}\varepsilon\left(u(t)\right), \varepsilon\left(v(t)\right) - \varepsilon\left(\dot{u}(t)\right)\right\rangle_Q dt + \int_0^T j\left(u(t), v(t)\right) dt$$
$$- \int_0^T \left(j\left(u(t), \dot{u}(t)\right)\right) dt \geq \int_0^T \left\langle \phi(t), v(t) - \dot{u}(t)\right\rangle_V dt$$
$$+ \int_0^T \langle \sigma_\nu\left(u(t)\right), v_\nu(t) - \dot{u}_\nu(t)\rangle \, dt.$$

In this inequality we set

$$v(s) = \begin{cases} z & \text{for } s \in (t, t+\lambda), \\ \dot{u}(s) & \text{elsewhere,} \end{cases}$$

and obtain the inequality

$$\frac{1}{\lambda}\int_t^{t+\lambda} \left(\left\langle \mathcal{F}\varepsilon(u(s)), \varepsilon(z) - \varepsilon\left(\dot{u}(s)\right)\right\rangle_Q + j\left(u(s), z\right) - j\left(u(s), \dot{u}(s)\right)\right) ds$$
$$\geq \frac{1}{\lambda}\int_t^{t+\lambda} \left\langle \phi(s), z - \dot{u}(s)\right\rangle_V ds + \frac{1}{\lambda}\int_t^{t+\lambda} \left\langle \sigma_\nu\left(u(s)\right), z_\nu - \dot{u}_\nu(s)\right\rangle ds.$$

Passing to the limit, we obtain that u satisfies the inequality (2.16). To complete the proof we integrate on $(0,T)$ both sides of (4.3), that is

$$\int_0^T \langle \mathcal{F}\varepsilon\left(\widetilde{u}^n(t)\right), \varepsilon\left(v(t)\right) - \varepsilon\left(\widetilde{u}^n(t)\right)\rangle_Q \, dt + \int_0^T j(\widetilde{u}^n(t), v(t) - \widetilde{u}^n(t)) dt$$
$$\geq \int_0^T \left\langle \widetilde{\phi}^n(t), v(t) - \widetilde{u}^n(t)\right\rangle_V dt, \quad \forall v \in L^2(0,T;V)$$

such that $v(t) \in K$ a.e. $t \in [0,T]$. Passing to the limit in the above inequality, and using (4.2), (4.4), we obtain the inequality

$$\int_0^T \Big(\langle \mathcal{F}\varepsilon(u(t)), \varepsilon(v(t)) - \varepsilon(u(t)) \rangle_Q + j(u(t), v(t) - u(t)) \Big) dt$$
$$\geq \int_0^T \langle \phi(t), v(t) - u(t) \rangle_V \quad \forall\, v \in L^2(0,T;V);\, v(t) \in K,\ a.e.\ t \in [0,T].$$

Proceeding in a similar way, we deduce that u satisfies the inequality

$$\langle \mathcal{F}\varepsilon(u(t)), \varepsilon(w) - \varepsilon(u(t)) \rangle_Q + j(u(t), w - u(t))$$
$$\geq \langle \phi(t), w - u(t) \rangle_V \ \ \forall w \in K,\ a.e.\ t \in [0,T]. \quad (4.10)$$

By using Green's formula in inequality (4.10) as in [4], we obtain that u satisfies the inequality (2.18) and consequently u is a solution of problem P_2. ■

Remark 4. We can consider another variational formulation of problem P_1 defined as follows.

Problem P_3. *Find a displacement field* $u \in W^{1,\infty}(0,T;V)$ *such that* $u(0) = u_0$ *in* Ω, $u(t) \in K \cap V_0$, *for all* $t \in [0,T]$, *and for almost all* $t \in (0,T)$,

$$\langle \mathcal{F}\varepsilon(u(t)), \varepsilon(v) - \varepsilon(\dot u(t)) \rangle_Q + j(u(t), v) - j(u(t), \dot u(t)) \geq$$
$$\langle \phi(t), v - \dot u(t) \rangle_V + \langle \theta\sigma_\nu(u(t)), v_\nu - \dot u_\nu(t) \rangle_\Gamma \qquad \forall v \in V,$$
$$\langle \theta\sigma_\nu(u(t)), z_\nu - u_\nu(t) \rangle_\Gamma \geq 0 \ \ \forall\, z \in K,$$

where $R: H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma_3)$ is a linear and continuous mapping and $\langle .,. \rangle_\Gamma$ denotes the duality pairing on $H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$. The cutt-off function $\theta \in C_0^\infty(\mathbf{R}^d)$ has the property that $\theta = 1$ on $\bar\Gamma_3$ and $\theta = 0$ on $\bar S_2$ with S_2 an open subset such that for all $t \in [0,T]$ $\mathrm{supp}\varphi_2(t) \subset S_2 \subset \bar S_2 \subset \Gamma_2$.

5. Conclusion

In this article we have shown the existence of a weak solution to the quasistatic unilateral contact problem with friction for nonlinear elastic materials under a smallness assumption of the friction coefficient. The uniqueness of the solution represents, as far as we know, an open question.

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