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NON-NEWTONIAN STOKES FLOW WITH FRICTIONAL BOUNDARY CONDITIONS [1]

F. SAIDI

Laboratory of Mathematics of University of Saint-Etienne

23, rue Dr. Paul Michelon, Saint-Etienne 42 023, France
E-mail: fouad.saidi@free.fr

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Abstract. In this work we deal with the boundary value problem for the non-Newtonian fluid flow with boundary conditions of friction type, mostly by means of variational inequalities. Among others, theorems concerning existence and uniqueness or non-uniqueness of weak solutions are presented.

Key words: Non-Newtonian fluids, variational inequality, nonlinear boundary conditions of friction type

1. Introduction

This paper is concerned with the boundary value problem for the stationary power law Stokes flow with a certain nonlinear boundary condition to be specified in Section 4, which we call the boundary condition of friction type or the frictional boundary condition, see [9].

Extensive study has been done so far for the motion of incompressible fluid which is governed by the Stokes/Navier-Stokes equation, or by the non-Newtonian Stokes/Navier-Stokes equation in hydrodynamics as well as in mathematics. As to the boundary condition, almost all of these works have dealt with the adhesive boundary condition to the surface of a rigid body, namely, with the Dirichlet boundary condition. This is of course reasonable from or consistent with the nature of such fluids and walls. However, there are phenomena, whose mathematical analysis seems to require introduction of some non-routine boundary conditions which might allow non-trivial motion of fluid on or across the boundary, for instance, slip or leak of fluid at the boundary. Examples are flow through a drain or canal with its bottom covered by sherbet of mud and pebbles, flow of melted iron coming out from

[1] Corresponding to: Fouad Saidi, 6 rue Jean Baptiste Corot, Montbrison 42 600, France.

a melting furnace, flow through a net or sieve, flow through a filter, and water flow in a purification plant etc.

Furthermore, among these phenomena there are those cases where the non-trivial movements, say leak or slip, take place only when magnitude of the stress at boundary surpasses a threshold. Also, the boundary condition of friction type was frequently used in free boundary problems containing dynamic or static contact lines, see [4, 16, 19, 21]. Our intention to introduce the frictional boundary condition is to propose a way of modeling of these boundary phenomena and carry out its mathematical analysis.

Our formulation and analysis are based on the theory of variational inequalities for nonlinear operators, see [13]. However, in dealing with the frictional boundary conditions which allows the leak in some way or other, we have to apply a new argument with resort to the Hahn-Banach theorem, which seems worthy of some interest by its own right, namely, from the view point of theory of variational inequalities.

As a matter of fact, the key idea of this work, particularly that of the above-mentioned argument by use of the Hahn-Banach theorem was gotten originally when Fujita gave a series of lectures at Collège de France in October of 1993, see [5]. Since then in some papers and lectures at international conferences, the authors have dealt with several closely related problems, some being presented along with numerical examples, see [6, 7, 8, 10]. In these previous works, the authors have mainly considered the (pure) slip boundary condition or the (pure) leak boundary condition of friction type, although they have touched even those cases where the flow is governed by the Navier-Stokes equation. In this paper, however, we restrict our consideration to the flow governed by the non-Newtonian Stokes equation and also to those frictional boundary conditions where the transition from the trivial adhesive state to a non-trivial movement on the boundary depends on the magnitude of the total stress there. One reason of such restriction is our intention to focus on the characteristic difficulties caused solely by the frictional boundary condition. The case of the Newtonian fluid flow has been studied H. Fujita and H. Kawarada, see [9].

The plan of this paper is the following. In Section 2, we describe our problem. Some preliminaries are presented in Section 3. In Section 4 we include the definition of the boundary conditions of friction type. The PDE formulation of the boundary value problem is given in Section 5. We present in Section 6 the formulation in terms of variational inequalities. The final Section 7 is devoted to present the results concerning the equivalence between VI formulation and PDE formulation, existence of solutions, and the uniqueness or non-uniqueness of solutions.

2. Basic Equations and Assumptions

Let Ω be a bounded domain Ω in $\mathbb{R}^m$ ($m = 2$ or 3). The smooth boundary $\partial\Omega = \Gamma$ of Ω is assumed to be composed of two separate compact components Γ_0 and Γ_1, i.e. $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$. To fix the idea, one could imagine, for example, that Ω stands for the inside of a vessel filled by the liquid and Γ_0 the inner wall of the vessel, Γ_1 the outer wall.

As mentioned in Introduction, throughout the present paper we deal with the stationary equation for non-Newtonian and incompressible fluid which is written in a familiar form as follows.

$$\begin{cases} \operatorname{div}(T) + f = 0, \\ \operatorname{div}(u) = 0. \end{cases} \tag{2.1}$$

Here, u is the velocity field and f the external force. The stress tensor T is decomposed as follows, see [12]

$$T = T(u,p) = -pI + \nu |e(u)|^{r-2} e(u), \quad r > 1, \tag{2.2}$$

where p is the pressure, I is the m-dimensional identity matrix and $e(u)$ is the symmetric deformation velocity tensor of components

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The positive constant ν in (2.2) stands for the kinetic viscosity. Incidentally, by replacing f formally by $\rho(u.\nabla)u + f$ in (2.1) (ρ is the density of the fluid), our formulation is valid for the non-Newtonian Navier-Stokes equation below, and some of our results will be extended thereto in a forthcoming paper.

$$\begin{cases} \operatorname{div}(T) + \rho(u.\nabla)u + f = 0, \\ \operatorname{div}(u) = 0. \end{cases}$$

For the boundary condition, we impose the usual Dirichlet boundary condition on Γ_0

$$u = g \text{ on } \Gamma_0, \tag{2.3}$$

while on Γ_1 we are going to impose a nonlinear boundary condition of friction type of our main concern which will be specified below.

Concerning g in (2.3), we assume for simplicity, the restricted flux condition

$$\int_{\Gamma_0} g.n \, d\Gamma = 0,$$

where n means the unit outer normal to the boundary condition. The reason of our somewhat simplified setting is the following. Indeed, we could consider the case where Γ_1 is a portion of Γ with non-empty $\partial\Gamma_1$. However, then we meet (seemingly unexplored) difficulties concerning the regularity of solutions at $\partial\Gamma_1$, which should be discussed in a separate paper. On the other hand, if we consider the case where Γ_0 is void and hence Γ_1 is equal to Γ, then the solvability of the stationary problem which we are going to consider involves further technical complication which is not essential from our point of view in the present paper.

3. Preliminaries

Now we list up symbols and function spaces to be used below.

3.1. Symbols

We denote by $|a|$ the absolute value of a if a is a real number and the norm of a if $a = (a_i)$ is a numerical m-vector (m=2 or 3). If a and b are vectors, $a.b$ means the scalar product of a and b. We adopt the summation convention.

We also denote by $n = n(s) = (n_i)$ the outer unit normal to the boundary at the point $s \in \Gamma$. The stress vector at the boundary is $\sigma = \sigma(u,p) = (T_{ij}n_j)$.

3.2. Function spaces and related symbols

We consider only real functions and real vector functions. L^r-space and the Sobolev spaces like $W^{1,r}$, $W^{1/r',r}$, $W^{-1,r'}$ and $W^{-\frac{1}{r'},r'}$ are well known, see [14], where r' is the conjugate exponent of r defined by $r' = r/(r-1)$ if $1 < r < \infty$. Unless necessity arises, we do not distinguish real functions and real vector functions in notation, neither Sobolev spaces of functions and those of vector functions. In addition, we use the spaces

$$W_0^{1,r}(\Omega) = \left\{v \in W^{1,r}(\Omega) \ : \ v = 0 \text{ on } \Gamma\right\},$$
$$W_d^{1,r}(\Omega) = \left\{v \in W^{1,r}(\Omega) \ : \ \operatorname{div}(v) = 0 \text{ in } \Omega\right\},$$
$$W_{0,d}^{1,r}(\Omega) = \left\{v \in W_0^{1,r}(\Omega) \ : \ \operatorname{div}(v) = 0 \text{ in } \Omega\right\}.$$

Furthermore, we put

$$\gamma = \text{ the trace operator from } W^{1,r}(\Omega) \text{ to } W^{\frac{1}{r'},r}(\Gamma) \text{ or to } W^{\frac{1}{r'},r}(\Gamma_1).$$

However, when it is clear from the context that we are considering functions or vector functions, say, v on the boundary, we shall simply write v instead of γv. The norms and inner products of Sobolev spaces are denoted in a usual manner, for instance:

$$(u,v) = (u,v)_{r,r'} = \text{ the inner product of } u \in L^r \text{ and } v \in L^{r'}.$$

Sometimes, the pairing $< \tau, \xi >$ of $\tau \in W^{-\frac{1}{r'},r'}(\Gamma_1)$ and $\xi \in W^{\frac{1}{r'},r}(\Gamma_1)$ will be written in an intuitive way as

$$\int_{\Gamma_1} \tau.\xi \, d\Gamma.$$

Finally, we introduce the following nonlinear form:

$$a(u,v) = \int_\Omega |e(u)|^{r-2} e_{ij}(u) e_{ij}(v) \, dx.$$

4. Boundary Conditions of Friction Type

Now we fix a given function $k = k(s)$ on Γ_1, which might called the modulus function of friction or the barrier function of the frictional boundary condition that we are going to describe. We assume that k is continuous on Γ_1 and hence it is bounded and bounded away from 0.

DEFINITION 1. We say that (u, p) satisfies the boundary condition of friction type (or frictional boundary condition) on Γ_1, if the following conditions hold true almost everywhere on Γ_1

$$|\sigma(u,p)| \le k \tag{4.1}$$

and

$$\begin{cases} |\sigma(u,p)| < k \Longrightarrow u = 0, \\ |\sigma(u,p)| = k \Longrightarrow \begin{cases} u = 0 \text{ or } u \neq 0, \\ u \neq 0 \Longrightarrow \sigma(u,p) = -k\dfrac{u}{|u|}. \end{cases} \end{cases} \tag{4.2}$$

From the frictional boundary condition follows that

$$\sigma(u,p).u + k|u| = 0 \text{ a. e. on } \Gamma_1.$$

Actually, the whole frictional boundary condition is equivalent to the following system, which is theoretically more convenient for our later consideration:

$$\begin{cases} |\sigma(u,p)| \le k, \\ \sigma(u,p).u + k|u| = 0. \end{cases} \tag{4.3}$$

Remark 1. Let $\partial|.|$ denote the subdifferential of the non-smooth function $|.| : \mathbb{R}^3 \to \mathbb{R}_+$ (or $|.| : \mathbb{R}^2 \to \mathbb{R}_+$) where ξ is the length of the vector ξ. Then the frictional boundary condition can be written in a concise form as

$$-\sigma(u,p) \in k\partial|u| \text{ at almost every point on } \Gamma_1.$$

Here we recall that

$$\partial|\xi| = \begin{cases} \text{the unit vector in the direction of } \xi = \dfrac{\xi}{|\xi|} & \text{if } \xi \neq 0, \\ \text{the closed unit ball in the vector space} = \{\eta \ : \ |\eta| \le 1\} & \text{if } \xi = 0. \end{cases}$$

Remark 2. In a similar manner, one can formulate the slip boundary condition of friction type without leak on Γ_1 as well as the leak boundary condition of friction type without slip on Γ_1. As to formulations and results of these slip or leak boundary conditions of friction type, see [20].

5. PDE Formulation

Our problem is formally stated in a classical form as

Problem 1. (Problem PDE classical) Find (u, p) satisfying the stationary power law Stokes equation (2.1)-(2.2) in Ω, the Dirichlet boundary condition (2.3) on Γ_0, and the frictional boundary condition on Γ_1.

Remark 3. A particular difficulty of the boundary problem above lies in the fact that the pressure p can not contain an arbitrary additive constant because of the condition $|\sigma(u,p)| \leq k$, which differs from the usual situation of the Dirichlet boundary problem for the non-Newtonian Stokes equation. On the other hand, we shall see that, as long as the Reynolds number remains sufficiently small, there is no fluid motion on Γ_1, which means that the additive constant to p may not be free but is admitted of an allowance to some extent. Thus we have to check carefully whether we can choose the pressure (i. e., the additive constant in the pressure) so that the frictional boundary condition is satisfied.

Henceforth, we assume that

$$f \in L^{r'}(\Omega), \quad g \in W^{\frac{1}{r'},r}(\Gamma_1).$$

Concerning f, a weaker assumption that $f \in W^{-1,r'}(\Omega)$ can be used. However, for some argument below the assumption $f \in L^{r'}$ reduces technical arguments. Anyway, we are going to develop $W^{1,r}$-theory in regard to the velocity u. Therefore, we need a weak formulation of the given problem.

Definition 2. (weak power law Stokes equation) A vector function $u \in W_d^{1,r}(\Omega)$ and a scalar function $p \in L^{r'}(\Omega)$ are said to satisfy the weak power law Stokes equation if the following identity holds:

$$\nu a(u, \varphi) - (p, \operatorname{div}(\varphi)) = (f, \varphi), \quad \forall \varphi \in W_0^{1,r}(\Omega). \tag{5.1}$$

Then (u, p) is called a weak solution of the power law Stokes equation. Moreover, we say that p is an accompanying pressure of the velocity u.

When we restrict test functions in (5.1) to solenoidal ones, we obtain another weak version of the power law Stokes equation which does not involve p.

Definition 3. (solenoidal weak power law Stokes equation) A vector function $u \in W_d^{1,r}(\Omega)$ is said to be solution of the solenoidal weak power law Stokes equation if the following identity holds:

$$\nu a(u, \varphi) = (f, \varphi), \quad \forall \varphi \in W_{0,d}^{1,r}(\Omega). \tag{5.2}$$

We have

Lemma 1. *Let $u \in W_d^{1,r}(\Omega)$ be a solution of the solenoidal weak power law Stokes equation, then there exists $p \in L^{r'}(\Omega)$ such that (u, p) is a weak solution of the power law Stokes equation.*

For bounded Lipschitz domains this Lemma follows from Corollary 2 in [17] and Corollary 4 in [18]. See also [1, 2].

Before formulating a weak version of our boundary value problem, we introduce some admissible sets of vector functions, which are used for formulation of variational inequalities in the next section.

$$K^g = \left\{v \in W^{1,r}(\Omega) : v = g \text{ on } \Gamma_0\right\}, \quad K^0 = \left\{v \in W^{1,r}(\Omega) : v = 0 \text{ on } \Gamma_0\right\},$$
$$K^g_d = \left\{v \in W^{1,r}_d(\Omega) : v = g \text{ on } \Gamma_0\right\}, \quad K^0_d = \left\{v \in W^{1,r}_d(\Omega) : v = 0 \text{ on } \Gamma_0\right\}.$$

Problem 2. (weak problem) Find $u \in K^g_d$ and $p \in L^{r'}(\Omega)$ such that

1. (u,p) satisfies the weak power law Stokes equation,
2. (u,p) satisfies the frictional boundary condition on Γ_1.

Since the regularity assumed of (u,p) is not sufficient to take the traces of p and ∇u on Γ_1, the meaning of the stress vector $\sigma(u,p)$ involved in the frictional boundary condition of Problem 2 must be understood in the manner explained below.

First, we note that the totality of traces $\gamma\varphi$ of all $\varphi \in K^0$ is equal to $W^{\frac{1}{r'},r}(\Gamma_1)$. On the other hand, as well known, see [15], any $\eta \in W^{\frac{1}{r'},r}(\Gamma_1)$ can be extended to a $\varphi \in K^0$. Moreover, if we can assume that (u,p) is smooth up to the boundary, then the identity

$$\int_{\Gamma_1} \sigma(u,p).\varphi \, d\Gamma = \nu a(u,\varphi) - (p, \operatorname{div}(\varphi)) - (f,\varphi)$$

is immediately obtained by Green's formula for all $\varphi \in K^0$. In view of this identity, we intend to re-define $\sigma(u,p)$ as a functional over $W^{\frac{1}{r'},r}(\Gamma_1)$ through

$$< \sigma, \eta >= \nu a(u,\varphi_\eta) - (p, \operatorname{div}(\varphi_\eta)) - (f,\varphi_\eta), \quad \forall \eta \in W^{\frac{1}{r'},r}(\Gamma_1), \tag{5.3}$$

where φ_η is an extension of η to a vector function belongs to K^0. As a matter of fact, if (u,p) is a solution of the weak power law Stokes equation, the value of the right-hand side of (5.3) does not depend by virtue of (5.2) on the manner of extension of η.

Thus the first inequality $|\sigma(u,p)| \le k$ of the frictional boundary condition in Problem 2 should be understood as the requirement that $\sigma(u,p) \in W^{-\frac{1}{r'},r'}(\Gamma_1)$ is realized by a vector function which is bounded by k on Γ_1 almost everywhere. In this way, the frictional boundary condition can be well incorporated in our formulation in terms of solutions of less regularity, although proof of further regularity of the solution under nicer given data is an interesting (and seemingly open) mathematical problem.

Remark 4. In dealing with the power law Stokes equation as well as the power law Navier-Stokes equation, one could use the Dirichlet form

$$d(u,v) = \int_\Omega |\nabla u|^{r-2} \nabla u.\nabla v \, dx$$

in place of $a(u,v)$, provided that the whole boundary condition is the Dirichlet boundary condition. If the boundary condition is of the Neumann type or involves the stress vector $\sigma(u,p)$, one has to use $a(u,v)$ for the weak formulation of the problem. Naturally, this remark must be noted also when one deals with the hydrodynamical potential theory in the Newtonian case, see [11].

In analyzing problems with solenoidal admissible functions later, we need the lemma below, see [14], which concerned with the set

$$Y_0 = \left\{\gamma\varphi \text{ on } \Gamma_1 : \varphi \in K_d^0\right\}. \tag{5.4}$$

Lemma 2. *Let denote*

$$Y_0 = \left\{\eta \in W^{\frac{1}{r'},r}(\Gamma_1) : <\eta, n> = \int_{\Gamma_1} \eta.n \, d\Gamma = 0\right\}. \tag{5.5}$$

For any $\eta \in Y_0$, there exists its extension $\varphi \in K_d^0$ such that the inequality

$$\|\varphi\|_{1,r,\Omega} \le c_0 \|\eta\|_{\frac{1}{r'},r,\Gamma_1}$$

holds with a positive constant c_0 depending only on the domain.

By this lemma, if u is a solution of the weak solenoidal power law Stokes equation and p is any one of its accompanying pressure, then $\sigma(u,p)$ can be re-defined as a functional on Y_0 by

$$<\sigma, \eta> = \nu a(u, \varphi_\eta) - (f, \varphi_\eta), \quad \forall \eta \in Y_0.$$

Here φ_η is an extension of η to a vector function in K_d^0.

6. Formulation in Terms of Variational Inequalities

Let us define a non-negative functional (barrier functional) $j(v)$ for v in K^g by putting

$$j(v) = \int_{\Gamma_1} k|v| \, d\Gamma = \int_{\Gamma_1} k|\gamma v| \, d\Gamma.$$

We state two formulations in terms of variational inequalities.

Problem 3. (problem in variational inequalities) Find $u \in K_d^g$ and $p \in L^{r'}(\Omega)$ such that

$$\nu a(u, v-u) - \big(p, \operatorname{div}(v-u)\big) - (f, v-u) + j(v) - j(u) \ge 0, \quad \forall v \in K^g. \tag{6.1}$$

Problem 4. (problem in solenoidal variational inequalities) Find $u \in K_d^g$ such that

$$\nu a(u, v-u) - (f, v-u) + j(v) - j(u) \ge 0, \quad \forall v \in K_d^g. \tag{6.2}$$

If (u, p) is a solution of Problem 3, then u alone solves Problem 4 automatically. On the other hand, we shall see later that for any solution u of Problem 4 we can choose its accompanying pressure p so that (u, p) solves Problem 3.

Now, let (u, p) be a solution of Problem 3. Then, substituting into (6.1) $v = u \pm \varphi$ with $\varphi \in W_0^{1,r}(\Omega)$, we have

$$\nu a(u, \varphi) - (p, \mathrm{div}\varphi) - (f, \varphi) = 0,$$

and see that the weak power law Stokes equation (5.1) is satisfied. Also, noting that $v - u$ belongs to K^0, we can rewrite in terms of $\sigma(u, p)$ the variational inequality (6.1) of Problem 3 as follows;

$$< \sigma, \gamma(v - u) > + j(v) - j(u) \geq 0, \quad \forall v \in K^g. \tag{6.3}$$

Similarly, the variational inequality (6.2) can be reduced to

$$< \sigma, \gamma(v - u) > + j(v) - j(u) \geq 0, \quad \forall v \in K_d^g. \tag{6.4}$$

7. Main Results and their Proofs

7.1. Theorems

We claim the following theorems.

Theorem 1. *Problem 3 and Problem 2 are equivalent.*

Theorem 2. *Problem 4 has a unique solution u.*

Theorem 3. *Let u be a solution of Problem 4. Then the accompanying pressure p can be chosen so that the frictional boundary condition on Γ_1 is satisfied, namely, (u, p) solves Problem 3. Consequently Problem 3 as well as Problem 2 has a unique solution.*

Remark 5. The velocity part u of solution of Problem 3 is unique. However, the uniqueness of the pressure part p of the solution of Problem 3 depends on cases. In fact, suppose that f in Ω and g on Γ_0 are small and such that the classical boundary value problem for the power law Stokes equation with the Dirichlet boundary condition (2.3) on Γ_0 and $u = 0$ on Γ_1 has a smooth solution such that

$$C_m = \sup_{\Gamma_1} |\sigma(u, p)| < \inf_{\Gamma_1} k = C_M.$$

Then (u, p) is a solution of Problem 3 (with no movement on Γ_1), while replacement of p by $p' = p - \lambda$ with a constant λ subject to $|\lambda| \leq C_M - C_m$ still yields a solution of Problem 3. Actually, we see then

$$|\sigma(u, p')| < k \text{ almost everywhere on } \Gamma_1.$$

On the other hand, let (u,p) be a solution of Problem 3 with non-trivial movement, i. e., $u \neq 0$ on Γ_1. Then p is uniquely determined. To see this, suppose that $u \neq 0$ takes place on portion ω with a positive measure of Γ_1, and note that the equality

$$-\sigma(u,p) = k\frac{u}{|u|}$$

must hold on ω, which prevents addition of any non-zero constant to p.

7.2. Proof of Theorem 1

Let (u,p) be a solution of Problem 3. It satisfies the weak power law Stokes equation as noted above, while the Dirichlet boundary condition (2.3) on Γ_0 is obviously fulfilled since $u \in K^g$. Therefore, it remains to verify the frictional boundary conditions.

Let η be an arbitrary function on $W^{\frac{1}{r'},r'}(\Gamma_1)$ and let φ in K^0 be its extension. Putting $v = u + \varphi$ and substituting it into (6.1), we have

$$< \sigma, \eta > + \int_{\Gamma_1} k\left(|u+\eta| - |u|\right)\, d\Gamma \geq 0. \tag{7.1}$$

By using an elementary property of $|.|$ it follows that

$$- < \sigma, \eta > \leq \int_{\Gamma_1} k|\eta|\, d\Gamma. \tag{7.2}$$

In view of the inequality (7.2) with η replaced by $-\eta$ and of the original (7.2) also, we have eventually

$$| < \sigma, \eta > | \leq \int_{\Gamma_1} k|\eta|\, d\Gamma.$$

This implies that the functional σ on $W^{\frac{1}{r'},r}(S)$ can be extended by continuity to a bounded functional on Banach space

$$X \equiv L^1_k(\Gamma_1) = \left\{\xi \ : \ \int_{\Gamma_1} k|\xi|\, d\Gamma < +\infty\right\} \text{ with } \|\xi\|_X = \int_{\Gamma_1} k|\xi|\, d\Gamma$$

and that it functional norm ≤ 1. Since the dual space X^* of X can be identified with the Banach space

$$X^* \equiv L^\infty_k(\Gamma_1) = \left\{\xi \ : \ \text{ess} \sup_{\Gamma_1} \frac{|\xi(s)|}{k(s)} < +\infty\right\}$$

with

$$\|\xi\|_{X^*} = \text{ess} \sup_{\Gamma_1} \frac{|\xi(s)|}{k(s)},$$

we have $\sigma \in L^\infty_k(\Gamma_1)$ with its norm ≤ 1, namely, we have $|\sigma(u,p)| \leq k$ almost everywhere on Γ_1, obtaining (4.1). Then, putting $\eta = -\gamma u$ in (7.1), we have

$$-\int_{\Gamma_1} \sigma.u \, d\Gamma - \int_{\Gamma_1} k|u| \, d\Gamma \geq 0,$$

which gives in view of (4.2)

$$\int_{\Gamma_1} (\sigma.u + k|u|) \, d\Gamma = 0. \tag{7.3}$$

However, we have $|\sigma(u,p)| \leq k$ almost everywhere on Γ_1 and then

$$\sigma.u + k|u| \geq (k - |\sigma(u,p)|) \, |u| \geq 0 \text{ almost everywhere on } \Gamma_1.$$

Hence, (7.3) implies

$$\sigma.u + k|u| = 0 \text{ almost everywhere on } \Gamma_1,$$

as desired.

Let (u,p) be a solution of Problem 2. It suffices to verify (6.3). This is immediate in view of (4.2) and (4.3), as

$$\int_{\Gamma_1} \sigma.(v-u) \, d\Gamma + j(v) - j(u) = \int_{\Gamma_1} (\sigma.v + k|v|) \, d\Gamma$$
$$- \int_{\Gamma_1} (\sigma.u + k|u|) \, d\Gamma = \int_{\Gamma_1} (\sigma.v + k|v|) \, d\Gamma \geq 0.$$

7.3. Proof of Theorem 2

Application of a standard theorem in variational inequalities for the minimization of the convex functional

$$J(v) = \frac{\nu}{r} a(v,v) - (f,v) + j(v), \quad v \in K_d^g$$

immediately gives the results, see [13].

7.4. Proof of Theorem 3

Let u be a solution of Problem 4, and p be any accompanying pressure of u. First of all, by substituting $v = u \pm \varphi$ into (6.4), we see that the functional $\sigma(u,p)$ on the Banach space Y_0 (recall (5.4) and (5.5)) satisfies the following inequality

$$< \sigma, \pm\eta > + j(u \pm \eta) - j(u) \geq 0,$$

where η is an arbitrary element of Y_0 and $\varphi \in K_d^0$ is its extension. Hence, in the same way as before, we deduce

$$| < \sigma, \eta > | \leq \int_{\Gamma_1} k|\eta| \, d\Gamma, \quad \forall \eta \in Y_0.$$

Thus the functional σ can be regarded as a bounded linear functional on $\overline{Y_0}$, which is the closure Y_0 in the Banach space $L_k^1(\Gamma_1)$, and actually with its functional norm ≤ 1.

Here we should note that $\overline{Y_0}$ is not dense in $L^1_k(\Gamma_1)$. However, by virtue of the Hahn-Banach theorem concerning extension of linear functional, see [3, 22], there exists an extension $\sigma^* \in L^\infty_k(\Gamma_1) = X^*$ of σ with the property $\|\sigma^*\|_{X^*} \leq 1$. Furthermore, by the same argument as before, we see that σ^* is realized by a vector function on Γ_1 with its $L^\infty_k(\Gamma_1)$-norm ≤ 1. In this sense, σ^* is a vector function subject to

$$|\sigma^*| \leq k \text{ almost everywhere on } \Gamma_1. \tag{7.4}$$

Furthermore, from the fact that

$$< \sigma^*, \eta >=< \sigma, \eta >, \quad \forall \eta \in Y_0,$$

and by the definition of Y_0 given by (5.5) in Lemma 2, it is easily shown that there exists a real number λ such that

$$\sigma^* - \sigma = \lambda n \text{ almost everywhere on } \Gamma_1,$$

where n is the unit outer normal on Γ_1. This implies that σ^* is obtained from $\sigma = \sigma(u, p)$ by replacing the original pressure p by $p^* = p - \lambda$ and thus we can regard σ^* as the stress vector for u and p^*. In other words, $\sigma^* = \sigma(u, p^*)$. Hence, (7.4) implies (4.1), i. e., the first inequality of the frictional boundary conditions. After this has been obtained, the argument to show

$$\sigma(u, p^*).u + k|u| = 0 \text{ almost everywhere on } \Gamma_1$$

is quite similar to the corresponding one above. Consequently, together with (u, p^*) solves Problem 3, and therefore Problem 2 according to Theorem 1.

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