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HIGH-ACCURACY DIFFERENCE SCHEMES FOR THE NONLINEAR TRANSFER EQUATION $\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} = F(U)$

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Abstract. In the present paper, for the initial boundary value problem for the non-homogeneous nonlinear transport equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = f(u),$$

the basic principles for constructing difference schemes of any order of accuracy $O(\tau^M)$, $M \geq 1$, on characteristic grids with the minimal stencil were introduced. To construct a difference scheme the Steklov averaging idea for the right-hand side

$$f(u) \approx \Big(\frac{1}{u_i^{n+1} - u_i^n}\int_{u_i^n}^{u_i^{n+1}} \frac{du}{f(u)}\Big)^{-1},$$

was used. The case of $f(u) = \lambda u^2$ was investigated in detail. A strict analysis of the order of approximation, stability, and convergence in nonlinear case was made. The performed numerical experiments justify theoretical results.
Key words: High accuracy difference scheme, nonlinear transport equation, method of characteristics

1. Introduction

One of the main questions in constructing difference schemes for equations of mathematical physics is an accuracy, i.e., convergence of an approximate solution to an exact solution of an original problem. It is natural to desire the maximum order of convergence rate for a minimum stencil of the grid. In this connection, we note the recent papers by Tadmor [1], Goloviznin [6, 7], Galanin[2, 3], and Čiegis [20] concerning construction and analysis of computational methods of high-order accuracy for hyperbolic equations.

The following question is emerged: can we construct exact difference schemes (EDS), i.e., such schemes for that the truncation error is equal to zero or $y = u$ at the grid nodes? Here y is the approximate solution of the finite-difference scheme, u is the exact solution of the given differential problem. For the ODEs corresponding results are given in the papers of Samarskii [19] and Gavrilyuk [4].

It is much more difficult to construct EDSs for nonlinear PDEs. For certain classes of homogeneous hyperbolic systems the EDS can be constructed on the basis of the method of characteristics [10, 16, 18]. It is worth to note paper [17], in which an EDS for a semilinear PDE having linear advection and an odd-cubic reaction term were constructed. The author has used a method based on the works by Mickens [13, 14, 15]. In [12], exact difference schemes for the transfer equation with variable coefficients using the special Steklov averaging on moving grids have been constructed. Unfortunately, it is impossible to construct a similar method in the quasilinear and non-homogeneous case [5, 18].

In this paper, an initial boundary value problem for quasilinear transfer equation with non-homogeneous and nonlinear right-hand side is solved. Using the moving characteristic grids and minimal grid stencil difference schemes of any order of accuracy $O(\tau^M)$, $M \geq 1$ are constructed, here $\tau = t_{n+1} - t_n$ a discrete time step. All the theoretical aspects of difference schemes such as the approximation, the stability and the convergence in nonlinear case were also considered. To construct schemes with a given accuracy, we use the mathematical apparatus presented in [11, 12]. Some numerical results confirming that the error of the proposed finite–difference schemes is equal to $O(\tau^M)$ are presented.

2. Statement of the Problem

In the domain (see Fig.1)

$$\overline{Q}_T = \{(x,t) : 0 \leq x \leq x_l(t), 0 \leq t \leq T\},$$

the initial–boundary value problem is considered for the transport equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \lambda u^2, \quad 0 < x \leq x_l(t), \quad 0 < t \leq T,\ \lambda < 0, \tag{2.1}$$

$$u(0,t) = \mu(t), \quad u(x,0) = u_0(x),$$

$$0 < u_0 \leq u(x,t) \leq u_1, \quad u_0'(x) \geq 0 \quad u(x,t) \in C_1^1(\overline{Q}_T). \tag{2.2}$$

Equation (2.1) along the characteristic $\dfrac{dx}{dt} = u$ can be written in the following form [8, 10]:

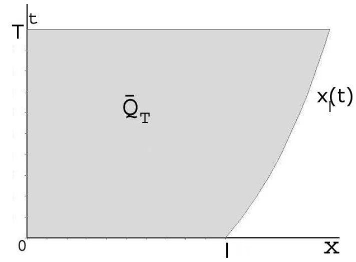

Figure 1. Domain of the problem definition

$$\begin{cases} \dfrac{dx}{dt} = u, \quad 0 \le t \le T, \\ \left.\dfrac{du}{dt}\right|_{\frac{dx}{dt}=u} = \lambda u^2, \quad 0 \le x \le x_l(t), \quad \lambda < 0, \\ \dfrac{dx_l}{dt} = u\left(x_l(t), t\right), \quad x_l(0) = l. \end{cases} \tag{2.3}$$

Selection of the domain $\overline{Q}_T$ of the problem with a moving right boundary is due to the characteristic method. In this case, some nodes of the moving grid will be located on the boundary of the domain.

3. Difference Scheme

Let us introduce the initial uniform grid on the segment $[0, l]$

$$\overline{\omega}_h^0 = \left\{x_i^0 = ih, \quad i = \overline{0, N}, \quad hN = l, \quad h = \frac{l}{N}\right\}$$

with a constant step $h > 0$. The moving grid is constructed in $\overline{Q}_T$ as

$$\overline{\omega}_{h\tau}^n = \left\{(x_{hi}^n, t_n), \ i = \overline{-n, N}, \ x_{h(-n)}^n = 0, \ t_n = n\tau, \ n = \overline{0, N_0}, \ \tau N_0 = T\right\}.$$

In this paper, the following notation is used:

$$x_i^n = x_i(t_n), \quad u_i^n = u\left(x_i^n, t_n\right), \quad u_{hi}^n = u\left(x_{hi}^n, t_n\right).$$

Applying the specific Steklov averaging [8, 12] the differential problem is approximated by the difference scheme

$$\frac{x_{hi}^{n+1} - x_{hi}^n}{\tau} = \sum_{k=0}^{M-1} \frac{(\lambda\tau)^k}{k+1} \left(u_{hi}^n\right)^{k+1}, \quad M \ge 1, \quad i = \overline{-n, N}, \tag{3.1}$$

$$x_{hi}^0 = x_i^0, \quad x_{h(-n)}^n = 0, \qquad i = \overline{0, N}, \ n = \overline{0, N_0},$$

$$\frac{u_{hi}^{n+1} - u_{hi}^n}{\tau} = \lambda u_{hi}^{n+1} u_{hi}^n, \qquad i = \overline{-n, N}, \ n = \overline{0, N_0 - 1}, \tag{3.2}$$

$$u_{hi}^0 = u_0\left(x_i^0\right), \ u_{h(-n)}^n = \mu(t_n), \qquad i = \overline{0, N}, \ n = \overline{0, N_0}. \tag{3.3}$$

4. Truncation Error

The discrete problem for the error of the discrete solution

$$\delta x_i^n = x_{hi}^n - x_i^n, \quad \delta u_i^n = u_{hi}^n - u_i^n, \quad \delta v_{t,i}^n = \frac{\delta v_i^{n+1} - \delta v_i^n}{\tau}$$

can be written in the following form:

$$\delta x_{t,i}^n - \sum_{k=0}^{M-1} \frac{(\lambda\tau)^k}{k+1} \sum_{j=0}^{k} \frac{(k+1)!}{(k+1-j)!j!} \left(\delta u_i^n\right)^{k+1-j} \left(u_i^n\right)^j = \psi_{0i}^n, \quad i = \overline{-n, N},$$

$$\delta x_i^0 = 0, \quad \delta x_{(-n)}^n = 0, \quad i = \overline{0, N}, \ n = \overline{0, N_0},$$

$$\delta u_{t,i}^n - \lambda\left(u_i^n \delta u_i^n + u_i^{n+1} \delta u_i^n + \delta u_i^n \delta u_i^{n+1}\right) = \psi_{1i}^n, \ \ i = \overline{-n, N}, \ \ n = \overline{0, N_0 - 1},$$

$$\delta u_i^0 = 0, \quad \delta u_{(-n)}^n = 0, \quad i = \overline{0, N}, \ \ n = \overline{0, N_0}.$$

The truncation error is expressed by the equality $\psi_i^n = \psi_{0i}^n + \psi_{1i}^n$, where

$$\psi_{0i}^n = -x_{t,i}^n + \sum_{k=0}^{M-1} \frac{(\lambda\tau)^k}{k+1} \left(u_i^n\right)^{k+1},$$

$$\psi_{1i}^n = -u_{t,i}^n + \lambda u_i^{n+1} u_i^n, \quad i = \overline{-n, N}, \quad n = \overline{0, N_0 - 1}.$$

Using equation (2.3), the following equalities hold:

$$\frac{dx}{dt} = u, \ \ \frac{d^2x}{dt^2} = \frac{du}{dt} = \lambda u^2, \ \ \frac{d^3x}{dt^3} = 2\lambda u \frac{du}{dt} = 2\lambda^2 u^3, \ldots$$

Taking into account the above equalities and substituting them into the Taylor formula we obtain

$$x_{t,i}^n = \frac{x_i^{n+1} - x_i^n}{\tau} = \frac{dx_i^n}{dt} + \frac{\tau}{2!}\frac{d^2 x_i^n}{dt^2} + \cdots + \frac{\tau^{M-1}}{M!}\frac{d^M x_i^n}{dt^M} + \frac{\lambda^M \tau^M}{M+1} \left(\widetilde{u}_i^n\right)^{M+1}$$
$$= \sum_{k=0}^{M-1} \frac{(\lambda\tau)^k}{k+1} \left(u_i^n\right)^{k+1} + \frac{(\lambda\tau)^M}{M+1} \left(\widetilde{u}_i^n\right)^{M+1}.$$

From here the equality follows

$$\psi_{0i}^n = -\frac{\lambda^M \tau^M}{M+1} \left(\widetilde{u}_i^n\right)^{M+1} = O\left(\tau^M\right), \tag{4.1}$$

where $\widetilde{u}_i^n = \left(x_i\left(\widetilde{t_n}\right), \widetilde{t_n}\right)$, $\widetilde{t_n} = t_n + \Theta_n \tau$, $0 < \Theta_n < 1$.

In [12], it was shown that for the Cauchy problem for a nonlinear ordinary differential equation

$$\frac{du}{dt} = f_1(u) f_2(t), \quad f_1(u) \neq 0, \quad t > 0, \quad u(0) = u_0,$$

the difference scheme with the specific Steklov averaging

$$\frac{y^{n+1}-y^{n}}{\tau}=\Big(\frac{1}{y^{n+1}-y^{n}}\int_{y^{n}}^{y^{n+1}}\frac{du}{f_1\left(u\right)}\Big)^{-1}\frac{1}{\tau}\int_{t_n}^{t_{n+1}}f_2\left(t\right)\,dt,\quad y^{0}=u_0, \quad (4.2)$$

is exact. In our case,

$$f_1(u)=\lambda u^2,\quad f_2(t)=1.$$

Consequently, from (4.2) we obtain

$$\psi_{1i}^{n}=-u_{ti}^{n}+\lambda u_i^{n+1}u_i^{n}=0. \quad (4.3)$$

Taking into account (4.3), the expression for ψ_i^n can be written in the form

$$\psi_i^n=\psi_{0i}^n+\psi_{1i}^n=O\left(\tau^M\right).$$

From here it follows that the initial differential problem is approximated by the difference scheme (3.1), (3.2) with any order of $M\geq 1$.

5. Stability of the Difference Scheme

We introduce the following notation for the grid norms:

$$\|u_h^n\|_{\overline{C}_n}=\max_{-n\leq i\leq N}|u_{hi}^n|,\quad \|u_h^n\|_{C_n}=\max_{-n+1\leq i\leq N}|u_{hi}^n|.$$

Perturbing the initial and boundary conditions in (3.1), (3.2) we arrive at the problem

$$\widetilde{x}_{ht,i}^n=\sum_{k=0}^{M-1}\frac{(\lambda\tau)^k}{k+1}\left(\widetilde{u}_i^n\right)^{k+1},\quad \widetilde{u}_{ht,i}^n=\lambda\widetilde{u}_{hi}^{n+1}\widetilde{u}_{hi}^n,\ \ i=\overline{-n,N},\ \ n=\overline{0,N_0-1},$$

$$\widetilde{x}_{hi}^0=x_i^0,\quad \widetilde{x}_{h(-n)}^n=0,\quad \widetilde{u}_{hi}^0=\widetilde{u}_i\left(x_i^0\right),\ \widetilde{u}_{h(-n)}^n=\widetilde{\mu}\left(t_n\right), \quad (5.1)$$

$$0<\widetilde{u}_0\leq\widetilde{u}_{hi}^n\leq\widetilde{u}_1,\quad \widetilde{u}_0'(x)\geq 0\quad i=\overline{-n,N},\quad n=\overline{0,N_0}.$$

Definition 1. The difference scheme is uniformly stable with respect to the initial data and the boundary condition if there exist constants $M_1>0,\ M_2>0$ independent of τ and n, such that for any n the inequality

$$\left\|\widetilde{x}_{h,i}^{n+1}-x_{h,i}^{n+1}\right\|_{\overline{C}_{n+1}}+\left\|\widetilde{u}_{hi}^{n+1}-u_{hi}^{n+1}\right\|_{\overline{C}_{n+1}}\leq M_1\left\|\widetilde{x}_h^0-x_h^0\right\|_{\overline{C}_0} \quad (5.2)$$

$$+M_2\max\left\{\max_{1\leq k\leq n+1}\left|\widetilde{\mu}\left(t_k\right)-\mu\left(t_k\right)\right|,\left\|\widetilde{u}_h^0-u_h^0\right\|_{\overline{C}_0}\right\}$$

is valid.

Let $\Delta x_i^n=\widetilde{x}_{hi}^n-x_{hi}^n$, $\Delta u_i^n=\widetilde{u}_{hi}^n-u_{hi}^n$, $i=\overline{-n,N}$, $n=\overline{0,N_0}$. Subtracting from (5.1) the equations (3.1), (3.2), we arrive at the perturbation problem

$$\Delta x_{t,i}^n=\Delta u_i^n\sum_{k=0}^{M-1}\frac{(\lambda\tau)^k}{k+1}\sum_{j=0}^{k}\left(\widetilde{u}_{hi}^n\right)^{k+1-j}\left(u_{hi}^n\right)^j, \quad (5.3)$$

$$\Delta u^n_{t,i} = \lambda u^{n+1}_{hi} \Delta u^n_i + \lambda \widetilde{u}^n_{hi} \Delta u^{n+1}_i, \; i = \overline{-n, N}, \; n = \overline{0, N_0 - 1},$$
$$\Delta x^0_i = 0, \quad \Delta x^n_{(-n)} = 0, \quad \Delta u^0_i = \widetilde{u}_i \left(x^0_i\right) - u_i \left(x^0_i\right),$$
$$\Delta u^n_{(-n)} = \widetilde{\mu}\left(t_n\right) - \mu\left(t_n\right), \qquad i = \overline{0, N}, \; n = \overline{0, N_0}.$$

In order to get the corresponding estimates expressing the stability of the difference scheme, it is necessary to show that u^n_{hi} and $\widetilde{u}^n_{hi}$ are bounded.

Lemma 1. *Let the conditions*

$$0 < u_0 \le u(x,t) \le u_1, \quad 0 < \widetilde{u}_0 \le \widetilde{u}(x,t) \le \widetilde{u}_1 \tag{5.4}$$

be met, where $\widetilde{u}(x,t)$ is the solution of the perturbed problem

$$\frac{\partial \widetilde{u}}{\partial t} + \widetilde{u} \frac{\partial \widetilde{u}}{\partial x} = \lambda \widetilde{u}^2, \quad 0 < x \le x_l(t), \quad 0 < t \le T, \; \lambda < 0,$$
$$\widetilde{u}(0,t) = \mu(t), \quad \widetilde{u}(x,0) = \widetilde{u}_0(x), \quad \widetilde{u}_0'(x) \ge 0.$$

Then for the solutions u^n_{hi}, $\widetilde{u}^n_{hi}$ the estimates

$$0 < u_0 \le \|u^n_h\|_{\overline{C}_n} \le u_1, \quad 0 < \widetilde{u}_0 \le \|\widetilde{u}^n_h\|_{\overline{C}_n} \le \tilde{u}_1,$$

are fulfilled.

The proof of Lemma 1 follows from the conditions of the exact approximation:

$$u_h\left(x_h, t_n\right) = u\left(x, t_n\right), \quad \widetilde{u}_h\left(x_h, t_n\right) = \widetilde{u}\left(x, t_n\right).$$

Using formula (3.2), we can write that

$$u^{n+1}_{hi} = \frac{u^n_{hi}}{1 - \lambda \tau u^n_{hi}}, \quad i = \overline{-n, N}, \quad n = \overline{0, N_0 - 1}.$$

We rewrite the above identity in the form

$$\frac{1}{u^{n+1}_{hi}} = \frac{1}{u^n_{hi}} - \lambda \tau = \ldots = \frac{1}{u^0_{hi}} - n \lambda \tau = \frac{1 - \lambda t_n u_0\left(x^0_i\right)}{u_0\left(x^0_i\right)}.$$

Hence, the following formula holds

$$u^{n+1}_{hi} = \frac{u_0\left(x^0_i\right)}{1 - \lambda t_n u_0\left(x^0_i\right)}, \quad i = \overline{-n, N}, \quad n = \overline{0, N_0 - 1}.$$

Now we will prove estimate (5.2). In view of (5.3) we can write Δu^{n+1}_i in the following form

$$\Delta u^{n+1}_i = \Delta u^n_i \frac{1 + \lambda \tau \widetilde{u}^{n+1}_{hi}}{1 - \lambda \tau \widetilde{u}^n_{hi}}, \quad i = \overline{-n, N}, \quad n = \overline{0, N_0 - 1}.$$

Hence for all corresponding indexes i we have the inequality

$$\left|\Delta u_i^{n+1}\right| \leq \left|\Delta u_i^n\right| \left|\frac{1+\lambda\tau\widetilde{u}_{hi}^{n+1}}{1-\lambda\tau\widetilde{u}_{hi}^{n}}\right| \leq \left|\Delta u_i^n\right| \left|1+\lambda\tau\widetilde{u}_{hi}^{n+1}\right|$$
$$\leq \left|\Delta u_i^n\right| \left|1+\tau\left|\lambda\right|\widetilde{u}_{hi}^{n+1}\right| \leq \left|\Delta u_i^n\right| \left|1+\tau\left|\lambda\right| u_2\right| \leq e^{\tau|\lambda|u_2}\left|\Delta u_i^n\right|.$$

From here we can find

$$\left\|\Delta u^{n+1}\right\|_{C_{n+1}} \leq e^{\tau|\lambda|u_2}\left\|\Delta u^n\right\|_{\overline{C}_n} = e^{\tau|\lambda|u_2}\max_{-n\leq i\leq N}\left|\Delta u_i^n\right|$$
$$= e^{\tau|\lambda|u_2}\max\left\{\left|\Delta u_{(-n)}^n\right|, \max_{-n+1\leq i\leq N}\left|\Delta u_i^n\right|\right\}$$
$$= e^{\tau|\lambda|u_2}\max\left\{\left|\Delta u_{(-n)}^n\right|, \left\|\Delta u^n\right\|_{C_n}\right\}.$$

Therefore,

$$\left\|\Delta u^{n+1}\right\|_{C_{n+1}} \leq e^{\tau|\lambda|u_2}e^{\tau|\lambda|u_2}\max\left\{\left|\Delta u_{(-n)}^n\right|, \left|\Delta u_{(-n+1)}^{n-1}\right|, \left\|\Delta u^{n-1}\right\|_{\overline{C}_{n-1}}\right\}$$
$$\leq \ldots \leq e^{T|\lambda|u_2}\max\left\{\max_{0\leq k\leq n}\left|\Delta u_{(-k)}^k\right|, \left\|\Delta u_0\right\|_{C_0}\right\}.$$

On the basis of the above estimate we conclude that

$$\left\|\Delta u^{n+1}\right\|_{\overline{C}_{n+1}} \leq e^{T|\lambda|u_2}\max\left\{\left|\Delta u_{(-n-1)}^{n+1}\right|, \left\|\Delta u^{n+1}\right\|_{C_{n+1}}\right\}$$
$$\leq e^{T|\lambda|u_2}\max\left\{\max_{1\leq k\leq n+1}\left|\widetilde{\mu}\left(t_k\right)-\mu\left(t_k\right)\right|, \left\|\widetilde{u}_0-u_0\right\|_{C_0}\right\}. \tag{5.5}$$

Now we consider the problem for Δx_i^{n+1}

$$\Delta x_i^{n+1} = \Delta x_i^n + \tau\Delta u_i^n\sum_{k=0}^{M-1}\frac{(\lambda\tau)^k}{k+1}\sum_{j=0}^{k}\left(\widetilde{u}_{hi}^n\right)^{k+1-j}\left(u_{hi}^n\right)^j.$$

Hence the following estimate is obtained

$$\left|\Delta x_i^{n+1}\right| \leq \left|\Delta x_i^n\right| + \tau\left|\Delta u_i^n\right|\left|\sum_{k=0}^{M-1}\frac{(\lambda\tau)^k}{k+1}\sum_{j=0}^{k}\left(\widetilde{u}_{hi}^n\right)^{k+1-j}\left(u_{hi}^n\right)^j\right|$$
$$\leq \left|\Delta x_i^n\right| + \tau\left|\Delta u_i^n\right|\left|\sum_{k=0}^{M-1}(\lambda\tau)^k d_1^{k+1}\right|$$
$$\leq \left|\Delta x_i^n\right| + \tau d_1\left|\Delta u_i^n\right|\left|\sum_{k=0}^{M-1}(\lambda\tau d_1)^k\right|,$$

where $d_1 = \max\{u_1, u_2\}$. On the other hand we can write

$$\sum_{k=0}^{M-1}(\lambda\tau d_1)^{k+1} = \frac{1-(\lambda\tau d_1)^{M-1}}{1-\lambda\tau d_1} \leq 1-(\lambda\tau d_1)^{M-1} \leq c,$$

because $1-\lambda\tau d_1 > 1$, where $c = 1+(|\lambda|\,Td_1)^{M-1}$.

Hence we can write

$$\left|\Delta x_i^{n+1}\right| \le \left|\Delta x_i^n\right| + \tau d_1 c \left|\Delta u_i^n\right|.$$

Let denote

$$\Phi_0 = e^{T|\lambda|\widetilde{d}_0} \max\left\{\max_{1\le k\le n+1} \left|\widetilde{\mu}(t_k) - \mu(t_k)\right|, \|\widetilde{u}_0 - u_0\|_{\overline{C}_0}\right\}.$$

Bearing in mind inequality (5.5), we conclude that

$$\left\|\Delta x^{n+1}\right\|_{\overline{C}_{n+1}} \le \|\Delta x^n\|_{\overline{C}_n} + \tau d_1 \Phi_0.$$

Iterating this inequality, we get the estimate

$$\begin{aligned}\left\|\Delta x^{n+1}\right\|_{\overline{C}_{n+1}} &\le \|\Delta x^n\|_{\overline{C}_n} + \tau d_1 c\Phi_0 \le \dots \\ &\le \left\|\Delta x^0\right\|_{\overline{C}_0} + d_1 c\,(n+1)\,\tau\Phi_0 \le \left\|\Delta x^0\right\|_{\overline{C}_0} + d_1 c T\Phi_0.\end{aligned}$$

Hence for the solution of the perturbation problem the a priori estimate

$$\begin{aligned}&\left\|\widetilde{x}_{h,i}^{n+1} - x_{h,i}^{n+1}\right\|_{\overline{C}_{n+1}} + \left\|\widetilde{u}_{hi}^{n+1} - u_{hi}^{n+1}\right\|_{\overline{C}_{n+1}} \le \left\|\widetilde{x}^0 - x^0\right\|_{\overline{C}_0} \\ &\quad + (1 + d_1 c T)\, e^{T|\lambda|\widetilde{d}_0} \max\left\{\max_{1\le k\le n+1} \left|\widetilde{\mu}(t_k) - \mu(t_k)\right|, \|\widetilde{u}_0 - u_0\|_{\overline{C}_0}\right\} \qquad (5.6)\end{aligned}$$

is obtained. Inequality (5.6) expresses the *stability* of the difference scheme with respect to the initial data and the boundary condition.

6. Convergence

In this section we investigate the convergence of the solution of the difference scheme (3.1), (3.2). We consider the discrete problem for the global errors of the discrete solution

$$\begin{aligned}&\delta x_{ti}^n = \sum_{k=0}^{M-1}\sum_{j=0}^{k} \frac{(k+1)!}{(k+1-j)!j!} (\delta u_i^n)^{k+1} (u_i^n)^j + \psi_0,\ i = \overline{-n, N},\ n = \overline{0, N_0 - 1}, \\ &\delta x_i^0, \quad \delta x_{(-n)}^n = 0, \quad i = \overline{0, N},\ n = \overline{0, N_0}, \qquad (6.1) \\ &\delta u_{ti}^n = \lambda \delta u_i^{n+1} \delta u_i^n + \lambda \left(u_i^{n+1} \delta u_i^n + \delta u_i^{n+1} u_i^n\right), \quad i = \overline{-n, N},\ n = \overline{0, N_0 - 1}, \\ &\delta u_i^0, \quad \delta u_{(-n)}^n = 0, \qquad i = \overline{0, N},\ n = \overline{0, N_0}.\end{aligned}$$

In Section 3, we proved that the error of the approximation for the solution u_{hi}^n is equal to zero, i.e. $\left\|\delta u^{n+1}\right\|_{\overline{C}_{n+1}} = 0$. Thus we can rewrite the first equation (6.1) in the following form:

$$\delta x_i^{n+1} = \delta x_i^n + \tau\psi_0, \quad i = -\overline{n, N}, \quad n = \overline{0, N_0 - 1}. \qquad (6.2)$$

Equation (4.1) yields the following estimate

$$\left|\delta x_i^{n+1}\right| \le \left|\delta x_i^n\right| + \tau \left|\frac{(\lambda\tau)^M}{M+1}\left(\widetilde{u}_i^n\right)^{M+1}\right| \le \left|\delta x_i^n\right| + \tau\left|\frac{(\lambda\tau)^M}{M+1}d_0^{M+1}\right| \le \dots$$
$$\le \left|\delta x_i^0\right| + (n+1)\tau\left|\frac{(\lambda\tau)^M}{M+1}d_0^{M+1}\right| \le \left|\delta x_i^0\right| + \tau^M\left|\lambda^M\right|\frac{Td_0^{M+1}}{M+1}. \tag{6.3}$$

On the basis of this relation we obtain the following inequality:

$$\left\|\delta x^{n+1}\right\|_{\overline{C}_{n+1}} \le \left\|\delta x^0\right\|_{\overline{C}_0} + |\lambda|^M\tau^M\frac{Td_0^{M+1}}{M+1} \le |\lambda|^M\tau^M\frac{Td_0^{M+1}}{M+1} = O\left(\tau^M\right),$$

which expresses the convergence of the difference scheme with order $O(\tau^M)$.

7. Numerical Experiments

In this section, we present the numerical results to confirm that the error of the method is of order $O\left(\tau^M\right)$. We consider problem (2.1), (2.2) with $l = 2, \quad T = 1$,

$$u(0,t) = \mu(t) = \frac{1}{2t+1}, \quad u(x,0) = u_0(x) = 2 - e^{-x}.$$

Then the function $u(x,t) = (2-e^{-x})/(2t+1)$ satisfies problem (2.1), (2.2). From the equation for characteristic

$$\frac{dx}{dt} = u = \frac{2}{(2t+1)} - \frac{e^{-x}}{2t+1},$$

we can find the explicit form for the characteristics

$$x(t) = \ln\left(c(2t+1) + 0.5\right),$$

where the constant c is defined by

$$c = \begin{cases} e^{x_i^0} - 0.5, & \text{for nodes} \quad x_i^0,\ i = \overline{0,N}, \\ \dfrac{1}{2\left(2t_n+1\right)}, & \text{for nodes} \quad x_{(-n)}^n,\ n = \overline{1,N_0}. \end{cases}$$

From here it follows that

$$x_i(t) = \begin{cases} \ln\left(e^{x_i^0}(2t+1) - t\right), & \text{for nodes} \quad x_i^0, \quad i = \overline{0,N}, \\ \ln\left(\frac{1}{2(2t_n+1)}(2t+1) + \frac{1}{2}\right), & \text{for nodes} \quad x_{(-n)}^n,\ n = \overline{1,N_0}. \end{cases} \tag{7.1}$$

Equality (7.1) expresses the characteristic curves of equation (2.1).

In the numerical experiments, we consider only the global error for δx_i^n, because we have proved in Section 4 that $\delta u_i^n = 0$. Let us use the following notation of the norm $z^N = \|\delta x\|_{\overline{C}_n}$. To show that the error of the method is $O(\tau^M)$, we will use the solutions of the difference scheme constructed on the following meshes:

$$\overline{\omega}_{h\tau} = \Big\{(x_{hi}^n, t_n),\ i = \overline{-n, N},\ x_{h(-n)}^n = 0,\ t_n = n\tau,\ n = \overline{0, N_0},\ \tau N_0 = T\Big\},$$

$$\overline{\omega}_{h\tau/2} = \Big\{(x_{hi}^{n_1}, t_{n_1}),\ i = \overline{-n_1, N},\ x_{h(-n_1)}^{n_1} = 0,\ t_{n_1} = n_1\tau/2,$$
$$n_1 = \overline{0, 2N_0},\ \tau/2N_{01} = T,\ N_{01} = 2N_0\Big\},$$

$$\overline{\omega}_{h\tau/4} = \Big\{(x_{hi}^{n_2}, t_{n_2}),\ i = \overline{-n_2, N},\ x_{h(-n_2)}^{n_2} = 0,\ t_{n_2} = n_2\tau/4,$$
$$n_2 = \overline{0, 4N_0},\ \tau/4N_{02} = T,\ N_{02} = 2N_{01}\Big\}.$$

To estimate the maximum errors, we use solutions computed on two different discrete grids

$$D^N = \frac{1}{2^M - 1} \max_{(x,t)\in\overline{\omega}_{h\tau}} \left| x_{hi}^n - x_{hi}^{2n_1} \right|,$$

where $x_{hi}^n \in \overline{\omega}_{h\tau}$, $x_{hi}^{n_1} \in \overline{\omega}_{h\tau/2}$. The value of $D^{N/2}$ is defined similarly on $\overline{\omega}_{h\tau/2}$ and $\overline{\omega}_{h\tau/4}$. The numerical orders of convergence are calculated by

$$p^N = \log_2 \left(D^{N/2}/D^N \right).$$

The results of the numerical experiment for various values of the parameter M are presented in Table 1 and Figure 2.

Table 1. Results for $M = 1$, $M = 3$, $M = 5$ and $M = 9$

τ	0,1	0,05	0,025	0,0125
$z_N(1)$	$5 \cdot 10^{-2}$	$2,6 \cdot 10^{-2}$	$1,3 \cdot 10^{-2}$	$6,4 \cdot 10^{-3}$
$D^N(1)$	$4.8 \cdot 10^{-2}$	$2,58 \cdot 10^{-2}$	$1,29 \cdot 10^{-2}$	$6,3 \cdot 10^{-3}$
$P^N(1)$	0,98	1	1	1
$z_N(3)$	$4,3 \cdot 10^{-4}$	$5 \cdot 10^{-5}$	$6 \cdot 10^{-6}$	$7,5 \cdot 10^{-7}$
$D^N(3)$	$4.9 \cdot 10^{-4}$	$5,6 \cdot 10^{-5}$	$6,4 \cdot 10^{-6}$	$7,7 \cdot 10^{-7}$
$P^N(3)$	3,1	3,1	3	3
$z_N(5)$	$6,9 \cdot 10^{-6}$	$1,9 \cdot 10^{-7}$	$5,6 \cdot 10^{-9}$	$1,7 \cdot 10^{-10}$
$D^N(5)$	$7.6 \cdot 10^{-6}$	$2,2 \cdot 10^{-8}$	$6,1 \cdot 10^{-9}$	$1,74 \cdot 10^{-10}$
$P^N(5)$	5,2	5,1	5	5
$z_N(9)$	$3,7 \cdot 10^{-9}$	$5,6 \cdot 10^{-12}$	$9,4 \cdot 10^{-15}$	
$D^N(9)$	$4.8 \cdot 10^{-9}$	$7,0 \cdot 10^{-12}$	$1,1 \cdot 10^{-14}$	
$P^N(9)$	9,4	9,2	9,1	

The presented results confirm the theoretical results that the difference solution $x_{h,i}^n$ converges to the exact solution of the initial problem with order $M \geq 1$.

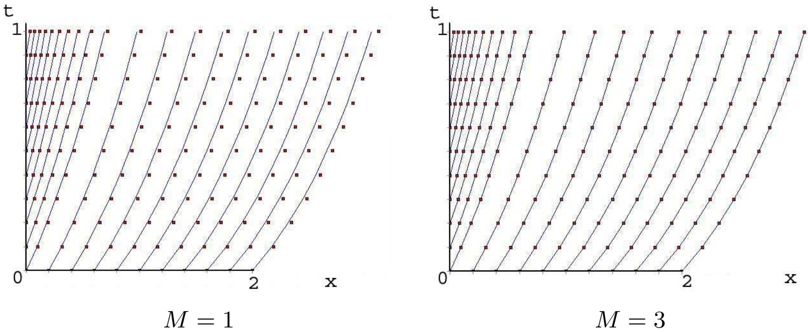

Figure 2. Exact $x_i\,(t)$ and approximate x_{hi}^n values of characteristics.

Remark 1. Analogously we can construct difference schemes of any order of accuracy for the more general equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = f_1\,(x,t)\,f_2\,(u)\,, \quad f_2\,(u) \neq 0, \tag{7.2}$$

which along the characteristic $x = x\,(t)$, satisfying $dx/dt = u$, can be written in the form

$$\left.\frac{du}{dt}\right|_{\frac{dx}{dt}=u} = f_1\,(x,t)\,f_2\,(u)\,.$$

Following [12], we can show that for the characteristic grid

$$\frac{x_{hi}^{n+1} - x_{hi}^n}{\tau} = u_{hi}^n + \frac{\tau}{2!}\frac{d^2x_{hi}^n}{dt^2} + \frac{\tau^2}{3!}\frac{d^3x_{hi}^n}{dt^3} + \ldots + \frac{\tau^{M-1}}{M!}\frac{d^Mx_{hi}^n}{dt^M} \tag{7.3}$$

the difference scheme

$$\frac{u_{hi}^{n+1} - u_{hi}^n}{\tau} = \frac{1}{\tau}\int_{t_n}^{t_{n+1}} f_1\Big(x_i\,(\xi)\,,\xi\Big)d\xi\Big(\frac{1}{u_{hi}^{n+1} - u_{hi}^n}\int_{u_{hi}^n}^{u_{hi}^{n+1}}\frac{d\eta}{f_2\,(\eta)}\Big)^{-1} \tag{7.4}$$

exactly approximates the differential equation (7.2). To apply formula (7.3) having the order of accuracy $O(\tau^M)$, $M \geq 1$, we should express the derivatives $d^\alpha x/dt^\alpha$, $\alpha = 1, 2, \ldots, M$ as the values of functions f_1 and f_2 by using equation (7.2)

$$\frac{d^2x}{dt^2} = \frac{du}{dt} = f_1\,(x(t),t)\,f_2(u); \quad \frac{d^3x}{dt^3} = f_2(u)\Big[f_1^2f_{2u}^{'} + \frac{\partial f_1}{\partial t} + u\frac{\partial f_1}{\partial x}\Big]; \quad \ldots$$

Remark 2. One of the most interesting points in constructing high-accuracy difference schemes for equation (7.2) is the case where integrals from the right-hand side of (7.4) cannot be computed exactly. In this case, we recommend to apply the Euler—Maclaurin formula based on trapezoid rule

$$\int_{x_0}^{x_{N_1}} f(x)dx \approx h_1\Big(\frac{f_0}{2} + \sum_{j=1}^{N_1-1} f_{0+jh_1} + \frac{f_{N_1}}{2}\Big)$$
$$+ \sum_{m=1}^{M_1} (-1)^m a_m h_1^{2m} \left[f_{N_1}^{(2m-1)} - f_0^{(2m-1)}\right]. \quad (7.5)$$

The approximation error of this formula is $O\left(h_1^{2M_1+2}\right)$, $h_1 = \frac{x_{N_1}-x_0}{N_1}$. For $M_1 = 6$ the accuracy of this quadrature formula equals to $O(h_1^{14})$. In paper [9], the simple technique is proposed for calculation of coefficients a_m

$$\frac{1}{2M+1} = \frac{1}{2} + \sum_{m=1}^{M_1}(-1)^m \frac{(2M_1)!}{(2M_1-2m+1)} a_m,$$

independent of the step h_1 and of function $f(x)$. Application of formula (7.5) to problem (2.1)–(2.2) leads to the following difference scheme:

$$\frac{x_{hi}^{n+1} - x_{hi}^n}{\tau} = \sum_{k=0}^{M-1} \frac{(\lambda\tau)^k}{k+1} (u_{hi}^n)^{k+1}, \ M \geq 1, \ i = \overline{-n, N}, \ n = \overline{0, N_0}; \quad (7.6)$$
$$x_{hi}^0 = x_i^0, \quad x_{h(-n)}^n = 0, \quad i = \overline{0,N}, \ n = \overline{0,N_0};$$
$$\frac{u_{hi}^{n+1} - u_{hi}^n}{\tau} = \lambda N_1 \Big\{ \Big(\frac{1}{2\left(u_{hi}^{n+1}\right)^2} + \sum_{j=1}^{N_1-1} \frac{1}{\left(u_{hi}^n + jh_1\right)^2} + \frac{1}{2\left(u_{hi}^n\right)^2} \Big) \quad (7.7)$$
$$+ \sum_{m=1}^{M_1} (-1)^m a_m h_1^{2m-1} \Big[\Big(\frac{1}{\left(u_{hi}^{n+1}\right)^2} \Big)^{(2m-1)} - \Big(\frac{1}{\left(u_{hi}^n\right)^2} \Big)^{(2m-1)} \Big] \Big\}^{-1},$$
$$h_1 = \frac{u_{hi}^{n+1} - u_{hi}^n}{N_1} = u_{ht,i}\frac{\tau}{N_1}, \quad i = \overline{-n,N}, \ n = \overline{0, N_0 - 1};$$
$$u_{hi}^0 = u_0\left(x_i^0\right), \ u_{h(-n)}^n = \mu(t_n), \quad i = \overline{0,N}, \qquad n = \overline{0,N_0}. \quad (7.8)$$

The truncation error of difference scheme (7.6)–(7.8) has the order $O(\tau^M + (\tau/N_1)^{2M_1+2})$.

In Tables 2–5 the results of numerical calculations by this difference scheme are presented for $N_1 = 10, M_1 = 3$. Here $\|\cdot\|_C = \max_{0\leq n\leq N_0} \|\cdot\|_{C_n}$

Table 2. $M = 1$

τ	0,1	0,05	0,025	0,0125
$\|x_h - x\|_C$	$5 \cdot 10^{-2}$	$2,6 \cdot 10^{-2}$	$1,3 \cdot 10^{-2}$	$6,4 \cdot 10^{-3}$
$\|u_h - u\|_C$	$5,2 \cdot 10^{-16}$	$2,3 \cdot 10^{-18}$	$7,6 \cdot 10^{-19}$	$7,6 \cdot 10^{-19}$

Comparing these results with results presented in Table 1, we see a high performance of the proposed methods.

Table 3. $M = 3$

τ	0,1	0,05	0,025	0,0125
$\|x_h - x\|_C$	$4,3 \cdot 10^{-4}$	$5 \cdot 10^{-5}$	$6 \cdot 10^{-6}$	$7,5 \cdot 10^{-7}$
$\|u_h - u\|_C$	$5,2 \cdot 10^{-16}$	$2,3 \cdot 10^{-18}$	$7,6 \cdot 10^{-19}$	$7,6 \cdot 10^{-19}$

Table 4. $M = 5$

τ	0, 1	0,05	0,025	0,0125
$\|x_h - x\|_C$	$6,9 \cdot 10^{-6}$	$1,9 \cdot 10^{-7}$	$5,6 \cdot 10^{-9}$	$1,7 \cdot 10^{-10}$
$\|u_h - u\|_C$	$5,2 \cdot 10^{-16}$	$2,3 \cdot 10^{-18}$	$7,6 \cdot 10^{-19}$	

Table 5. $M = 9$

τ	0,1	0,05	0,025
$\|x_h - x\|_C$	$3,7 \cdot 10^{-9}$	$5,6 \cdot 10^{-12}$	$9,4 \cdot 10^{-15}$
$\|u_h - u\|_C$	$5,2 \cdot 10^{-16}$	$2,3 \cdot 10^{-18}$	$7,6 \cdot 10^{-19}$

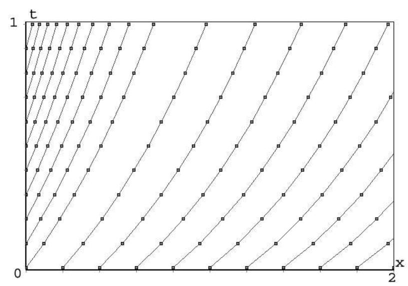

Figure 3. Exact $x_i(t)$ and approximate x_{hi}^n values of the characteristics.

Remark 3. Instead of the domain $\overline{Q}_T$ with moving right boundary (see Fig.1), we may use the rectangle

$$Q_T = \{(x, t) : 0 \le x \le l,\ 0 \le t \le T\}.$$

However, in this case, the nodes of the computational moving mesh will be placed outside the domain. The exact and approximate values of the moving mesh (characteristics) are presented in Fig. 3. These results were obtained by using scheme (3.1)–(3.2) with $\tau = 0,1$ and $M = 3$.

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