

# A SEPARATION PRINCIPLE OF TIME-VARYING DYNAMICAL SYSTEMS: A PRACTICAL STABILITY APPROACH

I. ELLOUZE and M. A. HAMMAMI

*Faculty of Sciences of Sfax, Department of Mathematics*

BP 802, 3018 Sfax Tunisia

E-mail: MohamedAli.Hammami@fss.rnu.tn

Received February 21, 2006; revised December 8, 2006; published online September 15, 2007

**Abstract.** In this paper we treat the problem of practical feedback stabilization for a class of nonlinear time-varying systems by means of an observer. A separation principle is given under a restriction about the perturbed term that the perturbation is bounded by an integrable function where the nominal system is supposed to be globally asymptotically stabilizable by a linear feedback. A practical stability approach is obtained. Furthermore, we give an example to show the applicability of our result.

**Key words:** time-varying systems, feedback, Lyapunov function, practical stabilization, state detection

## 1. Introduction

The analysis of stability of dynamic control time-varying systems is an important problem both in theory and practice [7, 8, 9]. When the states are not available the usual techniques is to build an observer. Many authors studied the problem of the conception of the observer. An observer is a dynamical system which is expected to produce an estimation of the state [1, 11, 12, 13].

The separation principle involves the design of a state observer and a state feedback stabilizing controller independently. For linear systems this problem is completely solved, but if the system contains some nonlinearities as a perturbation or disturbances, the problem in observer design still remains a difficult task. A separation principle is established if the closed-loop system remains stable when the state feedback controller is implied using state estimates [3, 4, 6]. However, global practical asymptotic stability by output feedback does not hold in general. Some results on semi-global stability have been reported [10].

In this paper, we give a separation principle for a certain class of time-varying systems

$$\begin{cases} \dot{x} = Ax + Bu + f(t, x), \\ y = Cx \end{cases}$$

in the practical sense based on results of analysis for cascaded systems [2, 5, 8, 9]. We give sufficient conditions to guarantee the practical global uniform stability of the closed-loop system by an estimated state feedback given by an observer design. This observer is constructed in such a way that the solutions of the error equation converge to a certain ball. We will estimate the radius of such a ball.

Given a time-varying input-output system

$$\begin{cases} \dot{x} = F(t, x, u), \\ y = Cx \end{cases} \quad (1.1)$$

where  $t \in \mathbb{R}_+$ ,  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^p$  and  $y \in \mathbb{R}^q$ . The function

$$F : [0, +\infty[ \times \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$$

is piecewise continuous in  $t$  and globally lipschitz in  $x$  on  $[0, +\infty[ \times \mathbb{R}^n$  and  $C$  is a constant matrix ( $q \times n$ ).

We first give the definition of uniform stability and uniform attractivity of (1.1) towards  $B_r = \{x \in \mathbb{R}^n / \|x\| \leq r\}$ .

**DEFINITION 1** [Uniform stability of  $B_r$ ].  $B_r$  is uniformly stable, if for all  $\varepsilon > r$ , there exists  $\delta = \delta(\varepsilon) > 0$ , such that

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0.$$

**DEFINITION 2** [Region of attraction of  $B_r$ ]. Let  $\phi(t; x)$  be the solution of (1.1) that starts at initial state  $x$  at time  $t = t_0$ . The region of attraction of  $B_r$ , denoted by  $R$ , is defined by

$$R = \{x \in \mathbb{R}^n / \|\phi(t, x)\| \rightarrow r \text{ as } t \rightarrow +\infty\}.$$

**DEFINITION 3** [Uniform attractivity of  $B_r$ ].  $B_r$  is uniformly attractive, if for all  $\varepsilon > r$ ,  $x_0 \in R$  and any  $t_0 \geq 0$ , there exists  $T(\varepsilon, x_0) > 0$  such that

$$\|x(t)\| < \varepsilon, \quad \forall t \geq t_0 + T(\varepsilon, x_0).$$

**DEFINITION 4.** *i)* The system (1.1) is said to be uniformly practically asymptotically stable with region of attraction  $R$ , if there exists  $B_r \subset \mathbb{R}^n$  such that  $B_r$  is uniformly stable and uniformly attractive.

*ii)* The system (1.1) is globally uniformly practically asymptotically stable, if it is practically stable with  $\mathbb{R}^n$  as a region of attraction.

DEFINITION 5. The system (1.1) is uniformly exponentially convergent to  $B_r$ , if there exists  $\gamma > 0$ , such that for all  $t_0 \in \mathbb{R}_+$ ,  $x_0 \in R$ , there exists  $k \geq 0$ , such that

$$\|x(t)\| \leq k(\|x_0\|) \exp(-\gamma(t - t_0)) + r. \tag{1.2}$$

We say that the system is strongly practically stable if for  $r > 0$ , system (1.1) is uniformly exponentially convergent to  $B_r$ . The system is globally practically uniformly exponentially stable if it is strongly practically stable with  $\mathbb{R}^n$  as a region of attraction.

DEFINITION 6. The system (1.1) is uniformly exponentially convergent to zero, if there exists  $\gamma > 0$ , such that for all  $t_0 \in \mathbb{R}_+$ ,  $x_0 \in R$ , there exists  $k \geq 0$ , such that

$$\|x(t)\| \leq k(\|x_0\|) \exp(-\gamma(t - t_0)) + r(t),$$

with  $\lim_{t \rightarrow +\infty} r(t) = 0$ .

For the concept of observer, we aim at simplifying the design of this system by exploiting the linear form of the nominal system.

DEFINITION 7 [Practical exponential observer]. A practical exponential observer for (1.1) is a dynamical system which has the following form

$$\dot{\hat{x}} = F(t, \hat{x}, u) - L(C\hat{x} - y),$$

where  $L$  is the gain matrix and  $e = \hat{x} - x$  is the origin of the error equation, which is given by

$$\dot{e} = F(t, \hat{x}, u) - F(t, x, u) - LCe$$

and it is globally practically exponentially stable. It means that it is globally uniformly practically asymptotically stable and the following estimation holds

$$\|e(t)\| \leq \lambda_1(\|e(t_0)\|) e^{-\lambda_2(t-t_0)} + r, \quad \forall t \geq t_0$$

with  $\lambda_1, \lambda_2, r > 0$ .

Note that, the origin  $x = 0$  may not be an equilibrium point of the system (1.1). We can no longer study stability of the origin as an equilibrium point nor should we expect the solution of the system to approach the origin as  $t \rightarrow \infty$ . The inequality (1.2) implies that  $x(t)$  will be ultimately bounded by a small bound  $r > 0$ , that is,  $\|x(t)\|$  will be small for sufficiently large  $t$ . If in (1.2)  $r$  can be replaced by a smooth map  $r(t)$  as a function of  $t$  which tends to zero as  $t$  tends to  $+\infty$ , the ultimate bound approaches zero. This can be viewed as a robustness property of convergence to the origin provided that  $F$  satisfies  $F(t, 0, 0) = 0, \forall t \geq 0$ . In this case the origin becomes an equilibrium point.

## 2. Stabilization

We consider the following dynamical system

$$\begin{cases} \dot{x} = Ax + Bu + f(t, x), \\ y = Cx, \end{cases} \quad (2.1)$$

where  $A$  is a  $(n \times n)$ ,  $B$  is a  $(n \times p)$ ,  $C$  is a  $(q \times n)$  constant matrices and  $f(t, x)$  is continuous, globally Lipschitz in  $x$ , uniformly in  $t$ .

We consider the following assumptions.

( $H_1$ ) The pair  $(A, B)$  is controllable, so there exists a constant matrix  $K$  of dimension  $(p \times n)$  such that for all positive definite symmetric matrix  $Q_1$ ,

$$Q_1 \geq c_1 I, \quad c_1 > 0,$$

there exists a positive definite symmetric matrix  $P_1$ ,

$$c_2 I < P_1 < c_3 I, \quad c_2 > 0, \quad c_3 > 0,$$

which satisfies

$$P_1 A_K + A_K^T P_1 = -Q_1, \quad (2.2)$$

where  $A_K = A + BK$ . The matrices inequalities mean that, we have for all  $x \in \mathbb{R}^n$ ,

$$c_1 \|x\|^2 \leq x^T Q_1 x, \quad c_2 \|x\|^2 \leq x^T P_1 x \leq c_3 \|x\|^2.$$

( $H_2$ ) There exists a function  $\psi(t)$ , such that for all  $t \geq 0$

$$\|f(t, x)\| \leq \psi(t), \quad (2.3)$$

with  $\int_0^{+\infty} \psi(s) ds \leq M < +\infty$ .

**Theorem 1.** *Under assumptions ( $H_1$ ) and ( $H_2$ ), the system (2.1) in closed-loop with the linear feedback  $u(x) = Kx$ , is globally uniformly practically exponentially stable.*

*Proof.* We consider a quadratic Lyapunov function  $V(t, x) = x^T P_1 x$ . Taking into account (3.3) and (4.1), the derivative of  $V$  along the trajectories of system (2.1) is given by

$$\begin{aligned} \dot{V}(t, x) &= \dot{x}^T P_1 x + x^T P_1 \dot{x} \\ &= (A_K x + f(t, x))^T P_1 x + x^T P_1 (A_K x + f(t, x)) \\ &\leq -x^T Q_1 x + 2\|P_1\| \|f(t, x)\| \|x\| \leq -c_1 \|x\|^2 + 2c_3 \psi(t) \|x\| \\ &\leq -\frac{c_1}{c_3} V(t, x) + 2\frac{c_3}{\sqrt{c_2}} \psi(t) \sqrt{V(t, x)}. \end{aligned}$$

Let  $v(t) = \sqrt{V(t, x)}$ . The derivative of  $v$  is given by  $\dot{v}(t) = \frac{\dot{V}(t, x)}{2\sqrt{V(t, x)}}$ , which implies that  $\dot{v}(t) \leq -\frac{c_1}{2c_3}v(t) + \frac{c_3}{\sqrt{c_2}}\psi(t)$ . Integrating between  $t_0$  and  $t$ , one obtains  $\forall t \geq t_0$ ,

$$v(t) \leq v(t_0)e^{-\frac{c_1}{2c_3}(t-t_0)} + \frac{c_3}{\sqrt{c_2}} \int_{t_0}^t \psi(s)e^{-\frac{c_1}{2c_3}(t-s)} ds,$$

which implies that

$$v(t) \leq v(t_0)e^{-\frac{c_1}{2c_3}(t-t_0)} + \frac{c_3}{\sqrt{c_2}}M.$$

It follows that,

$$\|x(t)\| \leq \sqrt{\frac{c_3}{c_2}}\|x_0\|e^{-\frac{c_1}{2c_3}(t-t_0)} + \frac{c_3}{c_2}M.$$

Hence, the above estimation shows the global uniform exponential stability of  $B_\alpha$ , with  $\alpha = \frac{c_3}{c_2}M$ . We have proved that system (2.1) in closed-loop with the linear feedback  $u(x) = Kx$  is globally strongly practically stable. ■

*Example 1.* We consider the system

$$\dot{x} = Ax + Bu + f(t, x)$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -4 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \quad f(t, x) = \frac{1}{1+t^2}.$$

The system  $\dot{x} = Ax + Bu$  is globally uniformly exponentially stabilizable, we can take a linear feedback law  $u(x) = Kx$  with  $K = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$  such that  $A + BK$  is the Hurwitz matrix.

The solution of the Lyapunov equation  $PA + A^T P = -I$  is given by

$$P = \begin{pmatrix} 5/16 & -1/4 \\ -1/4 & 7/16 \end{pmatrix}.$$

Let  $V(t, x) = x^T P x$ , which satisfies the assumption  $(H_1)$  where

$$c_1 = 1, \quad c_2 = \frac{6 - 2\sqrt{7}}{16} > 0, \quad c_3 = \frac{6 + 2\sqrt{7}}{16} > 0.$$

Moreover, the function  $f(t, x)$  is continuous and satisfies the assumption  $(H_2)$ , because  $\int_0^{+\infty} \frac{dt}{1+t^2} = \frac{\pi}{2} < \infty$ . Thus, all the assumptions of previous theorem are satisfied. We conclude that the system  $(S)$  is globally uniformly practically exponentially stable by  $u(x) = Kx$ .

Now, we will treat another class of systems by taking instead of the assumption  $(H_2)$  the following one.

$(H'_2)$  We assume that there exists a constant  $M' > 0$ , such that

$$\|f(t, x)\| \leq M'. \quad (2.4)$$

Note that, if we replace  $M'$  by  $M'(t) \rightarrow 0$  as  $t \rightarrow +\infty$ , we shall suppose that  $f(t, 0) = 0$ ,  $\forall t \geq 0$ , in such a way the origin becomes an equilibrium point.

**Theorem 2.** *Under assumptions  $(H_1)$  and  $(H'_2)$ , the system (2.1) in closed-loop with the linear feedback  $u(x) = Kx$  is globally uniformly practically exponentially stable.*

*Proof.* We consider a quadratic Lyapunov function candidate  $V(t, x) = x^T P_1 x$ . Taking into account (2.2) and (2.3), the derivative of  $V$  along the trajectories of system (2.1) is given by

$$\begin{aligned} \dot{V}(t, x) &= \dot{x}^T P_1 x + x^T P_1 \dot{x} \\ &= (A_K x + f(t, x))^T P_1 x + x^T P_1 (A_K x + f(t, x)) \\ &\leq -x^T Q_1 x + 2\|P_1\| \|f(t, x)\| \|x\| \leq -c_1 \|x\|^2 + 2c_3 M' \|x\| \\ &\leq -\frac{c_1}{c_3} V(t, x) + 2\frac{c_3}{\sqrt{c_2}} M' \sqrt{V(t, x)} \end{aligned}$$

Let  $v(t) = \sqrt{V(t, x)}$ . The derivative of  $v$  is given by  $\dot{v}(t) = \frac{\dot{V}(t, x)}{2\sqrt{V(t, x)}}$ , which implies that

$$\dot{v}(t) \leq -\frac{c_1}{2c_3} v(t) + \frac{c_3}{\sqrt{c_2}} M'.$$

Integrating between  $t_0$  and  $t$ , one obtains  $\forall t \geq t_0$ ,

$$v(t) \leq v(t_0) e^{-\frac{c_1}{2c_3}(t-t_0)} + \frac{2c_3^2}{\sqrt{c_2}c_1} M'.$$

It follows that,

$$\|x(t)\| \leq \sqrt{\frac{c_3}{c_2}} \|x_0\| e^{-\frac{c_1}{2c_3}(t-t_0)} + \frac{2c_3^2}{c_2 c_1} M'.$$

This yields, the global uniform exponential stability of  $B_\nu$ , with  $\nu = \frac{2c_3^2}{c_2 c_1} M'$ . We have proved that system (2.1) in closed-loop with the linear feedback  $u(x) = Kx$  is globally strongly practically stable. ■

### 3. Observer Design

We consider the system (2.1) satisfying the following assumptions.

( $H_3$ ) The pair  $(A, C)$  is observable, hence there exists a gain matrix  $L(n \times q)$  such that for all positive definite symmetric matrix  $Q_2$ ,

$$Q_2 \geq b_1 I, \quad b_1 > 0$$

there exists a positive definite symmetric matrix  $P_2$ ,  $b_2 I < P_2 < b_3 I$ ,  $b_2 > 0$ ,  $b_3 > 0$ , which satisfies

$$P_2 A_L + A_L^T P_2 = -Q_2, \quad \text{where } A_L = A - LC. \tag{3.1}$$

To design an observer, we consider the dynamical system

$$\dot{\hat{x}} = A\hat{x} + Bu + f(t, \hat{x}) - L(C\hat{x} - y), \tag{3.2}$$

where  $\hat{x} \in \mathbb{R}^n$  is the state estimate of  $x(t)$  in the sense that  $e(t) = \hat{x}(t) - x(t)$  satisfies the following estimation

$$\|e(t)\| \leq \|e(t_0)\| e^{-\lambda(t-t_0)} + r, \quad \forall t \geq t_0.$$

**Proposition 1.** *Under assumptions ( $H_2$ ) and ( $H_3$ ), the system (3.2) is a practical exponential observer for the system (2.1).*

*Proof.* We consider now the error equation with  $e = \hat{x} - x$ ,

$$\dot{e} = \dot{\hat{x}} - \dot{x} = (A - LC)e + f(t, \hat{x}) - f(t, x). \tag{3.3}$$

We consider the quadratic Lyapunov function candidate,  $W(t, e) = e^T P_2 e$ . Taking into account (2.4), the derivative of  $W$  along the trajectories of system (3.3) is given by

$$\begin{aligned} \dot{W}(t, e) &= \dot{e}^T P_2 e + e^T P_2 \dot{e} \\ &= (A_L e + f(t, \hat{x}) - f(t, x))^T P_2 e + e^T P_2 (A_L e + f(t, \hat{x}) - f(t, x)) \\ &\leq -e^T Q_2 e + 2\|P_2\| \|f(t, \hat{x}) - f(t, x)\| \|e\| \\ &\leq -b_1 \|e\|^2 + 4b_3 \psi(t) \|e\| \\ &\leq -\frac{b_1}{b_3} W(t, e) + 4\frac{b_3}{\sqrt{b_2}} \psi(t) \sqrt{W(t, e)}. \end{aligned}$$

Let  $w(t) = \sqrt{W(t, e)}$ . The derivative of  $w$  is given by  $\dot{w}(t) = \frac{\dot{W}(t, e)}{2\sqrt{W(t, e)}}$ ,

which implies that

$$\dot{w}(t) \leq -\frac{b_1}{2b_3} w(t) + 2\frac{b_3}{\sqrt{b_2}} \psi(t).$$

Integrating between  $t_0$  and  $t$ , one obtains  $\forall t \geq t_0$ ,

$$w(t) \leq w(t_0) e^{-\frac{b_1}{2b_3}(t-t_0)} + \frac{2b_3}{\sqrt{b_2}} \int_{t_0}^t \psi(s) e^{-\frac{b_1}{2b_3}(t-s)} ds.$$

We get,

$$w(t) \leq w(t_0)e^{-\frac{b_1}{2b_3}(t-t_0)} + \frac{2b_3}{\sqrt{b_2}}M.$$

It follows that,

$$\|e(t)\| \leq \sqrt{\frac{b_3}{b_2}}\|e(t_0)\|e^{-\frac{b_1}{2b_3}(t-t_0)} + \frac{2b_3}{b_2}M.$$

The last estimation shows the global uniform exponential stability of  $B_\kappa$  respecting the error equation with  $\kappa = \frac{2b_3}{b_2}M$ . Therefore, we can deduce that, (3.3) is globally practically exponentially stable. We conclude that, the origin of system (3.2) is a practical exponential observer for the system (2.1). ■

For the second class of systems treated above, one has an analogue result as the one obtained in Proposition 1.

**Proposition 2.** *Under assumptions  $(H'_2)$ , and  $(H_3)$ , the system (3.2) is a practical exponential observer for the system (2.1).*

*Proof.* Consider the error equation with  $e = \hat{x} - x$ ,

$$\dot{e} = \hat{\dot{x}} - \dot{x} = (A - LC)e + f(t, \hat{x}) - f(t, x). \quad (3.4)$$

The quadratic Lyapunov function candidate, can be taken as  $W(t, e) = e^T P_2 e$ . Taking into account (2.4), the derivative of  $W$  along the trajectories of system (3.4) is given by:

$$\begin{aligned} \dot{W}(t, e) &= \dot{e}^T P_2 e + e^T P_2 \dot{e} \\ &= (A_L e + f(t, \hat{x}) - f(t, x))^T P_2 e + e^T P_2 (A_L e + f(t, \hat{x}) - f(t, x)) \\ &\leq -e^T Q_2 e + 2\|P_2\|\|f(t, \hat{x}) - f(t, x)\|\|e\| \\ &\leq -b_1\|e\|^2 + 4b_3 M'\|e\| \leq -\frac{b_1}{b_3}W(t, e) + 4\frac{b_3}{\sqrt{b_2}}M'\sqrt{W(t, e)}. \end{aligned}$$

Let  $w(t) = \sqrt{W(t, e)}$ . The derivative of  $w$  is given by,  $\dot{w}(t) = \frac{\dot{W}(t, e)}{2\sqrt{W(t, e)}}$

which implies that

$$\dot{w}(t) \leq -\frac{b_1}{2b_3}w(t) + 2\frac{b_3}{\sqrt{b_2}}M'.$$

Integrating between  $t_0$  and  $t$ , one obtains  $\forall t \geq t_0$ ,

$$w(t) \leq w(t_0)e^{-\frac{b_1}{2b_3}(t-t_0)} + \frac{4b_3^2}{\sqrt{b_2}b_1}M'.$$

It follows that,

$$\|e(t)\| \leq \sqrt{\frac{b_3}{b_2}}\|e(t_0)\|e^{-\frac{b_1}{2b_3}(t-t_0)} + \frac{4b_3^2}{b_2b_1}M'.$$

The last inequality implies the global uniform exponential stability of  $B_\eta$  with  $\eta = \frac{4b_3^2}{b_2b_1}M'$ . Hence, we can deduce that, the origin of system (3.3) is globally practically exponentially stable. It follows that, the system (3.2) is a practical exponential observer for the system (2.1). ■

Note that, if we suppose that  $M' = M'(t)$  which goes to zero when  $t$  tends to  $+\infty$ , then the solutions of the error equation converge exponentially to zero.

### 4. Separation Principle

We consider the system (2.1) controlled by the linear feedback law  $u(\hat{x}) = K\hat{x}$  and estimated with the observer (3.2).

**Theorem 3.** *Under assumptions  $(H_1)$ ,  $(H_2)$  and  $(H_3)$ , the system*

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + BK\hat{x} + f(t, \hat{x}) - LCe \\ \dot{e} = (A - LC)e + f(t, \hat{x}) - f(t, x) \end{cases} \tag{4.1}$$

*is practically globally uniformly exponentially stable.*

*Proof.* In order to study the stabilization problem via an observer, we consider the cascaded system

$$\begin{cases} \dot{\hat{x}} = \varphi(t, \hat{x}) + g(t, \hat{x})e, \\ \dot{e} = h(t, \hat{x}, e), \end{cases} \tag{4.2}$$

where  $\varphi(t, \hat{x}) = A\hat{x} + BK\hat{x} + f(t, \hat{x})$ ,  $g(t, \hat{x}) = -LC$ ,

$$h(t, \hat{x}, e) = (A - LC)e + f(t, \hat{x}) - f(t, x).$$

One has,  $\dot{\hat{x}} = \varphi(t, \hat{x})$  is globally strongly practically stable with the Lyapunov function associated can be taken as  $v(t, \hat{x}) = \sqrt{\hat{x}^T P_1 \hat{x}}$ . This Lyapunov function satisfies

$$\sqrt{c_2}\|\hat{x}\| \leq v(t, \hat{x}) \leq \sqrt{c_3}\|\hat{x}\|, \tag{4.3}$$

$$\frac{\partial v}{\partial t}(t, \hat{x}) + \frac{\partial v}{\partial \hat{x}}\varphi(t, \hat{x}) \leq -c_4v(t, \hat{x}) + \frac{c_3}{\sqrt{c_2}}\psi(t),$$

$\|\frac{\partial v}{\partial \hat{x}}\| \leq \frac{c_3}{\sqrt{c_2}}$ , where  $c_4 = \frac{c_1}{2c_3}$ . Also,  $\dot{e} = h(t, \hat{x}, e)$  is globally strongly practically stable and the following estimation can be obtained

$$\|e(t)\| \leq \sqrt{\frac{b_3}{b_2}}\|e(t_0)\|e^{-\frac{b_1}{2b_3}(t-t_0)} + \frac{2b_3}{b_2}M.$$

Therefore, the derivative of  $v$  along the trajectories of system (3.2) is given by using (4.2) and (4.3).

$$\begin{aligned} \dot{v}(t, \hat{x}) &= \frac{\partial v}{\partial t}(t, \hat{x}) + \frac{\partial v}{\partial \hat{x}}\varphi(t, \hat{x}) + \frac{\partial v}{\partial \hat{x}}g(t, \hat{x})e \\ &\leq -c_4v(t, \hat{x}) + \frac{c_3}{\sqrt{c_2}}\psi(t) + \frac{c_3}{\sqrt{c_2}}\|LC\|\|e\| \\ &\leq -c_4v(t, \hat{x}) + \frac{c_3}{\sqrt{c_2}}\psi(t) + \frac{c_3}{\sqrt{c_2}}\|LC\|\left(\sqrt{\frac{b_3}{b_2}}\|e(t_0)\|e^{-\frac{b_1}{2b_3}(t-t_0)} + \frac{2b_3}{b_2}M\right), \end{aligned}$$

which implies that

$$\dot{v}(t, \hat{x}) \leq -c_4v(t, \hat{x}) + \lambda\|e(t_0)\|e^{-\gamma(t-t_0)} + \frac{c_3}{\sqrt{c_2}}\psi(t) + R,$$

where

$$\lambda = \frac{c_3}{\sqrt{c_2}}\|LC\|\sqrt{\frac{b_3}{b_2}}, \quad \gamma = -\frac{b_1}{2b_3}, \quad R = \frac{2b_3c_3}{b_2\sqrt{c_2}}M\|LC\|.$$

Integrating between  $t_0$  and  $t$ , one obtains  $\forall t \geq t_0$ ,

$$\begin{aligned} v(t, \hat{x}) &\leq v(t_0, \hat{x}_0)e^{-c_4(t-t_0)} \\ &\quad + \int_{t_0}^t \left( \lambda\|e(t_0)\|e^{-\gamma(s-t_0)} + \frac{c_3}{\sqrt{c_2}}\psi(s) + R \right) e^{-c_4(t-s)} ds \\ &\leq v(t_0, \hat{x}_0)e^{-c_4(t-t_0)} + \lambda\|e(t_0)\| \int_{t_0}^t e^{-c_4(t-s)} e^{-\gamma(s-t_0)} ds \\ &\quad + \int_{t_0}^t \left( R + \frac{c_3}{\sqrt{c_2}}\psi(s) \right) e^{-c_4(t-s)} ds \\ &\leq v(t_0, \hat{x}_0)e^{-c_4(t-t_0)} + \lambda\|e(t_0)\| \int_{t_0}^t e^{-c_4t} e^{c_4s} e^{-\gamma s} e^{\gamma t_0} ds \\ &\quad + \frac{c_3}{\sqrt{c_2}}M + R \left[ \frac{1}{c_4} e^{-c_4(t-s)} \right]_{t_0}^t \leq v(t_0, \hat{x}_0)e^{-c_4(t-t_0)} \\ &\quad + \lambda\|e(t_0)\|e^{\gamma t_0} e^{-c_4t} \int_{t_0}^t e^{c_4s} e^{-\gamma s} ds + \frac{c_3}{\sqrt{c_2}}M + \frac{R}{c_4} \left[ 1 - e^{-c_4(t-t_0)} \right]. \end{aligned}$$

So, there exist some positive constants  $\beta$  and  $\mu$ , such that

$$v(t, \hat{x}) \leq v(t_0, \hat{x}_0)e^{-c_4(t-t_0)} + \lambda\beta\|e(t_0)\|e^{-\mu(t-t_0)} + \frac{c_3}{\sqrt{c_2}}M + \frac{R}{c_4} \left[ 1 - e^{-c_4(t-t_0)} \right].$$

Letting  $\delta = \min(c_4, \mu)$  and  $l = \max(\sqrt{c_3}, \lambda\beta)$ , we get

$$v(t, \hat{x}) \leq l(\|\hat{x}_0\| + \|e_0\|)e^{-\delta(t-t_0)} + \frac{c_3}{\sqrt{c_2}}M + \frac{R}{c_4}.$$

It follows that

$$\|\hat{x}(t)\| \leq \frac{l}{\sqrt{c_2}} \|(\hat{x}_0, e_0)\| e^{-\delta(t-t_0)} + \frac{c_3}{c_2} M + \frac{R}{c_4 \sqrt{c_2}}.$$

Then, the cascaded system (4.2) is practically globally exponentially stable. ■

We will use the same argument as in Theorem 3 to prove an analogue result when  $f$  is a bounded function.

**Theorem 4.** *Under assumptions  $(H_1)$ ,  $(H'_2)$ , and  $(H_3)$ , the system (4.1) is practically globally uniformly exponentially stable.*

*Proof.* In order to study the stabilization problem via an observer, we consider the system (4.2) with  $k(t, \hat{x}) = A\hat{x} + BK\hat{x} + f(t, \hat{x})$ , which satisfy

$$\frac{\partial v}{\partial t}(t, \hat{x}) + \frac{\partial v}{\partial \hat{x}} k(t, \hat{x}) \leq -c_4 v(t, \hat{x}) + r_1,$$

where  $r_1 = \frac{c_3}{\sqrt{c_2 c_1}}$ . Second,  $\dot{e} = h(t, \hat{x}, e)$  is globally strongly practically stable and we have the following estimation

$$\|e(t)\| \leq \sqrt{\frac{b_3}{b_2}} \|e(t_0)\| e^{-\frac{b_1}{2b_3}(t-t_0)} + \frac{4b_3^2}{b_2 b_1} M'.$$

Taking into account (4.2) and (4.3), the derivative of  $v$  along the trajectories of system (3.3) is given by:

$$\begin{aligned} \dot{v}(t, \hat{x}) &= \frac{\partial v}{\partial t}(t, \hat{x}) + \frac{\partial v}{\partial \hat{x}} k(t, \hat{x}) + \frac{\partial v}{\partial \hat{x}} g(t, \hat{x}) e \leq -c_4 v(t, \hat{x}) + r_1 + \frac{c_3}{\sqrt{c_2}} \|LC\| \|e\| \\ &\leq -c_4 v(t, \hat{x}) + r_1 + \frac{c_3}{\sqrt{c_2}} \|LC\| \left( \sqrt{\frac{b_3}{b_2}} \|e(t_0)\| e^{-\frac{b_1}{2b_3}(t-t_0)} + \frac{4b_3^2}{b_2 b_1} M' \right). \end{aligned}$$

Which implies

$$\dot{v}(t, \hat{x}) \leq -c_4 v(t, \hat{x}) + \lambda \|e(t_0)\| e^{-\gamma(t-t_0)} + R_1$$

with  $R_1 = r_1 + \frac{4b_3^2 c_3}{b_2 b_1 \sqrt{c_2}} M \|LC\|$ . Integrating between  $t_0$  and  $t$ , one obtains

$$\begin{aligned} v(t, x_1) &\leq v(t_0, \hat{x}_0) e^{-c_4(t-t_0)} + \int_{t_0}^t \left( \lambda \|e(t_0)\| e^{-\gamma(s-t_0)} + R_1 \right) e^{-c_4(t-s)} ds \\ &\leq v(t_0, \hat{x}_0) e^{-c_4(t-t_0)} + \lambda \|e(t_0)\| \int_{t_0}^t e^{-c_4(t-s)} e^{-\gamma(s-t_0)} ds \\ &\quad + R_1 \int_{t_0}^t e^{-c_4(t-s)} ds \leq v(t_0, \hat{x}_0) e^{-c_4(t-t_0)} \end{aligned}$$

$$\begin{aligned}
& + \lambda \|e(t_0)\| \int_{t_0}^t e^{-c_4 t} e^{c_4 s} e^{-\gamma s} e^{\gamma t_0} ds + R_1 \left[ \frac{1}{c_4} e^{-c_4(t-s)} \right]_{t_0}^t \leq v(t_0, \hat{x}_0) e^{-c_4(t-t_0)} \\
& + \lambda \|e(t_0)\| e^{\gamma t_0} e^{-c_4 t} \int_{t_0}^t e^{c_4 s} e^{-\gamma s} ds + \frac{R_1}{c_4} \left[ 1 - e^{-c_4(t-t_0)} \right].
\end{aligned}$$

Which implies that there exist some positive constants  $\beta$  and  $\mu$ , such that

$$v(t, x_1) \leq v(t_0, \hat{x}_0) e^{-c_4(t-t_0)} + \lambda \beta \|e(t_0)\| e^{-\mu(t-t_0)} + \frac{R_1}{c_4} \left[ 1 - e^{-c_4(t-t_0)} \right].$$

We get  $v(t, x_1) \leq l (\|\hat{x}_0\| + \|e_0\|) e^{-\delta(t-t_0)} + \frac{R_1}{c_4}$  for a certain  $l > 0$ . Which implies that

$$\|\hat{x}(t)\| \leq \frac{l}{\sqrt{c_2}} \|(\hat{x}_0, e_0)\| e^{-\delta(t-t_0)} + \frac{R_1}{c_4 \sqrt{c_2}}.$$

Then, cascaded system (4.2) is practically globally exponentially stable. ■

## References

- [1] D. M. Dawson and Z. Qu and J. C. Caroll. On the state observation and output feedback problems for nonlinear uncertain dynamic systems. *Systems & Control letters*, **18**, 217–222, 1992.
- [2] A. Ferfera and M. A. Hammami. Sur la stabilisation des systèmes non linéaires en cascade. *Bulletin of the Belgian Mathematical society*, **7**, 97–105, 2000.
- [3] J. P. Gauthier and Y. Kupka. A separation principle for bilinear systems with dissipative drift. *IEEE Trans. on Automat. Contr. AC-37*, **12**, 1970–1974, 1992.
- [4] M. A. Hammami. Global convergence of a control system by means of an observer. *Journal of Optimization Theory and Applications*, **108**(2), 377–388, 2001.
- [5] M. Jankovic, R. Sepulchre and P. V. Kokotovic. Constructive Lyapunov stabilization of non linear cascaded systems. *IEEE Trans. on Automat. Contr.*, **41**, 1723–1736, 1996.
- [6] N.H. Jo and J.H. Seo. Local separation principle for non linear systems. *International journal of control*, **73**(4), 292–302, 2000.
- [7] H. K. Khalil. *Nonlinear systems, 2nd edition*. Mac-Millan, 1996.
- [8] E. Panteley and A. Loria. On global uniform asymptotic stability of nonlinear time-varying systems in cascade. *Systems & Control letters*, **33**, 131–138, 1998.
- [9] E. Panteley and A. Loria. Growth rate conditions for uniform asymptotic stability of cascaded time-varying systems. *Automatica*, **37**(3), 453–460, 2001.
- [10] A. Teel and L. Praly. Global stabilizability and observability imply semi global stabilizability by output feedback. *Systems & Control letters*, **22**, 313–325, 1994.
- [11] F. Thau. Observing the states of nonlinear dynamic systems. *International journal of control*, **17**, 471–479, 1993.
- [12] B. L. Walcott, M. Corless and S. H. Zak. Comparative study of non linear state observation techniques. *International journal of control*, **45**(6), 2109–2132, 1987.
- [13] B. L. Walcott and S. H. Zak. State observation of nonlinear uncertain dynamical systems. *IEEE Trans. on Automat. Contr.*, **32**(2), 166–170, 1987.