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GENERALIZED NÖRLUND METHOD AND CONVERGENCE ACCELERATION

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Abstract. Some propositions on λ-boundedness for generalized Nörlund method of summability $(\mathcal{N}, P_n)$, where P_n are bounded linear operators from Banach space X into X, are proved. Using these results we have verified some propositions on convergence acceleration by $\mathcal{N}$ and several Tauberian remainder theorems for generalized Nörlund method.

Key words: convergence acceleration, summability methods, Tauberian remainder theorems

1. Introduction and Lemmas

Let X, Y be Banach spaces and $\mathcal{L}(X,Y)$ be the space of all bounded linear operators from Banach space X into Y. A sequence $x = (\xi_k)$ $(\xi_k \in X)$ is called $\lambda-$ bounded if

$$\exists \lim \xi_k = \xi \,\wedge\, \beta_k = \lambda_k(\xi_k - \xi) \,\wedge\, \beta_k = O(1),$$

where $\lambda = (\lambda_k)$ is a sequence of real numbers with $0 < \lambda_k \nearrow \infty$.

Let m_X^λ be the set of all $\lambda-$bounded sequences. A sequence $x = (\xi_k)$ is called summable (see [24] and [5]) by a generalized method $\mathcal{A} = (A_{nk})$, $A_{nk} \in \mathcal{L}(X,Y)$ if $y = (\eta_n)$ with

$$\eta_n = \sum_{k=0}^{\infty} A_{nk}\xi_k \tag{1.1}$$

is convergent. Let $\mu = (\mu_k)$ with $0 < \mu_k \nearrow \infty$. The transformation $\mathcal{A}$ is called preserving λ-boundedness (see [6, 15, 22]) if $\mathcal{A}m_X^\lambda \subset m_Y^\lambda$. The transformation $\mathcal{A}$ is called accelerating λ-boundedness if

$$\mathcal{A}m_X^\lambda \subset m_Y^\mu \tag{1.2}$$

with $\lim \mu_k/\lambda_k = \infty$. A method $\mathcal{A} = (A_{nk})$ with $A_{nk} \in \mathcal{L}(X,X)$ is called regular if $\mathcal{A}c_X \subset c_X$ and $\lim_n \eta_n = \lim_k \xi_k$, while c_X is a set of convergent sequences with $\xi_k \in X$ and η_n is defined by (1.1). We denote by I and θ the identity and zero operator on any Banach space, respectively.

Kornfeld (see [8]) proved that any regular numerical method of summability can not accelerate universally the convergence. It is proved in [22] that any regular triangular generalized method $\mathcal{A}$ can not accelerate the convergence. Regardless of this fact in applied mathematics linear methods are used to accelerate the convergence (see [19]). Such acceleration is possible in some subsets of m_X^λ. The present article is a sequel to the inquiries [19] – [23] and [10]. Main results of convergence acceleration using nonlinear methods are presented in [3].

Different mathematicians have studied various generalizations of Nörlund method, see for example [2, 4, 7, 9, 12, 16]. We denote by $(\mathcal{N}, P_n)$ or $\mathcal{N}$ the generalized Nörlund method of summability defined (see [1, 11] and [21]) by

$$A_{nk} = \begin{cases} R_n P_{n-k}, & (k = 0, 1, \dots, n), \\ \theta, & (k > n), \end{cases} \tag{1.3}$$

where $P_k,\ R_n \in \mathcal{L}(X,X)$, while R_n is defined by

$$R_n \sum_{k=0}^{n} P_{n-k}\zeta = \zeta \quad (\zeta \in X,\ n \in \mathbf{N}_0). \tag{1.4}$$

If $P_k = I \quad (k \in \mathbf{N}_0)$, then we get the method of arithmetical means $\mathcal{H}$. If $\alpha \in \mathbf{R}$ with $-\alpha \notin \mathbf{N}$ and $P_k = \binom{k+\alpha-1}{k} I$, then we get Cesàro method $\mathcal{C}^\alpha$ (see also [18]). If $\alpha \in \mathbf{R} \setminus \{0\}$, $P_0 = \alpha I$, $P_1 = (1-\alpha) I$ and $P_k = \theta \quad (k > 1)$, then we get Zweier method $\mathcal{Z}_\alpha$.

Let us define a generalized method $\mathcal{B} = (\mathcal{N}, B_{n+1} - B_n)$ using the sequence of operators $B_n \in \mathcal{L}(X,X)$ satisfying the conditions $B_n^{-1} \in \mathcal{L}(X,X)$,

$$(n+1)\,\|B_{n+1} - B_n\| = \mathcal{O}(\|B_n\|)\,, \quad B_0 = \theta, \quad B_n \neq \theta, \quad (n \in \mathbf{N}) \tag{1.5}$$

To prove our propositions we use next lemmas (see [11] and [22]).

Lemma 1. *Matrix transformation* (1.1) *with* $X = Y$ *is regular iff the conditions*

$$\exists \lim_n A_{nk}\zeta = A_k\zeta \qquad (\zeta \in X,\ k \in \boldsymbol{N}_0), \tag{1.6}$$

$$\exists \lim_n \sum_{k=0}^{\infty} A_{nk}\zeta = A\varsigma \qquad (\varsigma \in X), \tag{1.7}$$

$$\sup_{|\xi_k| \le 1} \|\sum_{k=0}^{p} A_{nk}\xi_k\| = O(1) \qquad (n, p \in \boldsymbol{N}_0), \tag{1.8}$$

while $A = I$ *and* $A_k = \theta$ $(k \in \boldsymbol{N}_0)$, *are satisfied.*

Lemma 2. *Let* $A_{nk} \in \mathcal{L}(X,Y)$, $\mathcal{A} = (A_{nk})$ *and* $e_X(\zeta) := (\zeta, \zeta, \dots)$ *with* $\zeta \in X$. *If*

$$\exists \lim_n A_{nk} = A_k \quad (k \in \boldsymbol{N}_0) \tag{1.9}$$

in the norm, then the conditions

$$\mathcal{A}e_X(\zeta) \in m_Y^\mu \quad (\zeta \in X), \tag{1.10}$$

$$\sum_k \lambda_k^{-1} \|A_k\| < \infty, \tag{1.11}$$

$$\mu_n \sum_k \lambda_k^{-1} \|A_{nk} - A_k\| = O(1) \tag{1.12}$$

are sufficient for the inclusion (1.2).

2. Convergence Preservation and Convergence Acceleration

Proposition 1. *The conditions*

$$\lim_n \|R_n\| \, \|P_{n-k}\| = 0, \quad (k = 0, 1, \dots, n), \tag{2.1}$$

$$\|R_n\| \sum_{k=0}^n \|P_k\| = \mathcal{O}(1) \tag{2.2}$$

imply that the method $(\mathcal{N}, P_n)$ *is regular.*

Proof. Let us use Lemma 1 to prove the assertion of Proposition 1. Using (1.4) we get

$$R_n \sum_{k=0}^n P_{n-k}\zeta = I\zeta \quad (\zeta \in X, n \in \mathbf{N}_0).$$

That means the condition (1.7) is satisfied. Using the properties of the norm and (2.1) we get

$$\left\| R_n P_{n-k}\zeta \right\| \le \|R_n\| \, \|P_{n-k}\| \, \|\zeta\| \to 0 \quad (\zeta \in X, \ k = 0, 1, \dots, n).$$

Therefore the condition (1.6) is satisfied. If $\|\xi_k\| \le 1$, then using the properties of the norm and (2.2) we get

$$\left\| \sum_{k=0}^p R_n P_{n-k}\xi_k \right\| \le \sum_{k=0}^n \|R_n\| \, \|P_{n-k}\| \, \|\xi_k\| \le \|R_n\| \sum_{k=0}^n \|P_k\| = \mathcal{O}(1),$$

while $n, p \in \mathbf{N}_0$. So the condition (1.8) is satisfied. That means all the conditions of Lemma 1 are satisfied and the assertion of Proposition 1 follows from Lemma 1. ■

As in [22] is proved that any regular triangular generalized method of summability $\mathcal{A} = (A_{nk})$, satisfying the condition $\sum_{k=0}^n A_{nk} = I$, can not accelerate the convergence, we get the Corollary 1.

Corollary 1. If the generalized method of summability $(\mathcal{N}, P_n)$ satisfies the conditions (2.1) and (2.2), then this method can not accelerate the convergence.

Corollary 2. The conditions (1.5) and

$$\lim_n \left\|B_{n+1}^{-1}\right\| \|B_{n-k+1} - B_{n-k}\| = 0 \quad (k = 0, 1, \ldots, n)\,, \tag{2.3}$$

$$\left\|B_{n+1}^{-1}\right\| \sum_{k=0}^{n} \|B_{k+1} - B_k\| = \mathcal{O}(1) \tag{2.4}$$

imply that the generalized method $\mathcal{B}$ is regular.

Proposition 2. *The conditions* (2.1), (2.2) *and*

$$\mu_n \sum_{k=0}^{n} \lambda_k^{-1} \|R_n P_{n-k}\| = \mathcal{O}(1) \tag{2.5}$$

are sufficient for the inclusion $\mathcal{N} m_X^\lambda \subset m_X^\mu$.

Proof. Let us use Lemma 2 and take A_{nk} by (1.3). Then condition (1.9) follows from (2.1). Condition (1.4) implies condition (1.10) and conditions (2.1) and (2.2) imply that the method $(\mathcal{N}, P_n)$ is regular. That is why $A_k = \theta$ and $A = I$. So condition (1.11) is satisfied. The condition (2.5) implies (1.12), thus the assertion of Proposition 2 follows from Lemma 2. ■

Remark 1. If the method $(\mathcal{N}, P_n)$ is accelerating the convergence, then condition (2.5) is satisfied.

Corollary 3. Conditions (1.5), (2.3), (2.4) and

$$\mu_n \sum_{k=0}^{n} \lambda_k^{-1} \left\|B_{n+1}^{-1} \left(B_{n+1-k} - B_{n-k}\right)\right\| = O(1) \tag{2.6}$$

are sufficient for the inclusion $\mathcal{B} m_X^\lambda \subset m_X^\mu$.

Remark 2. If the method $\mathcal{B}$ is accelerating the convergence, then condition (2.6) is satisfied.

3. Tauberian Remainder Theorems

In [13] and [15] several Tauberian theorems for the generalized Nörlund method of summability are proved. Sõrmus (see [14]) proved the first Tauberian theorem for the generalized methods of summability in Banach spaces. In [20]–[23] and [10] several Tauberian remainder theorems for the generalized methods of summability are proved. The Tauberian remainder theorems in form and content depend on the methods of the proof. We mainly use the method which bases on the summability with the given rapidity (see [6, 18]).

Proposition 3. *If* $\mathcal{N}x \in m_X^\lambda$ *and the conditions*

$$\|P_k\| \le M, \quad (k+1)\,\lambda_k \Delta\xi_k = \mathcal{O}\,(1)\,, \quad (\forall k \in \boldsymbol{N}_0)\,, \tag{3.1}$$

$$\lambda_n \|R_n\| \sum_{k=1}^{n} \frac{1}{\lambda_k} = \mathcal{O}\,(1) \tag{3.2}$$

are satisfied, then $x \in m_X^\lambda$.

Proof. We have

$$\lambda_n\,(\xi_n - \xi) = \lambda_n\left(\xi_n - \sigma_n^{(1)}\right) + \lambda_n\left(\sigma_{(n}^{(1)} - \xi\right), \tag{3.3}$$

while

$$\sigma_n^{(1)} = \sum_{k=0}^{n} R_n P_{n-k} \xi_k. \tag{3.4}$$

As $\sum_{k=0}^{n} R_n P_{n-k} = I$ we get

$$\lambda_n\left(\xi_n - \sigma_n^{(1)}\right) = \lambda_n\Big(\sum_{k=0}^{n} R_n P_{n-k} \xi_n - \sum_{k=0}^{n} R_n P_{n-k} \xi_k\Big)$$
$$= \lambda_n R_n \sum_{k=0}^{n} P_{n-k}\,(\xi_n - \xi_k)\,.$$

Therefore using Abel's partial summation formula (see [1])

$$\sum_{k=0}^{n} a_k b_k = -\sum_{k=0}^{n-1}\sum_{\nu=0}^{k} a_\nu\,(b_{k+1} - b_k) + \sum_{\nu=0}^{n} a_\nu b_n,$$

we get

$$\lambda_n\left(\xi_n - \sigma_n^{(1)}\right) = -\lambda_n R_n\Big(\sum_{k=0}^{n-1}\sum_{\nu=0}^{k} P_{n-\nu}\,(\xi_n - \xi_{k+1} - \xi_n + \xi_k)\Big)$$
$$= \lambda_n R_n \sum_{k=0}^{n-1}\sum_{\nu=0}^{k} P_{n-\nu} \Delta\xi_{k+1} = \lambda_n R_n \sum_{k=1}^{n}\sum_{\nu=0}^{k} P_{n-\nu} \Delta\xi_k$$

and

$$\lambda_n \left\|\xi_n - \sigma_n^{(1)}\right\| \le \lambda_n \|R_n\| \sum_{k=1}^{n}\sum_{\nu=0}^{k} \|P_{n-\nu}\|\,\|\Delta\xi_k\|\,.$$

That is why using (3.1)–(3.2) we get

$$\lambda_n \left\|\xi_n - \sigma_n^{(1)}\right\| = \mathcal{O}\,(1)\;\lambda_n \|R_n\| \sum_{k=1}^{n} \frac{1}{\lambda_k} = \mathcal{O}\,(1)\,.$$

Therefore

$$\lambda_n \left\| \xi_n - \sigma_n^{(1)} \right\| = \mathcal{O}(1). \tag{3.5}$$

As

$$\mathcal{N}x \in m_X^\lambda \Leftrightarrow \lambda_n \left(\sigma_n^{(1)} - \xi \right) = \mathcal{O}(1),$$

then using (3.3) and (3.5), we get that the assertion of the Proposition 3 is valid. ■

Let us define the operators $P_n^{(m)}$ $(m \in \mathbf{N}_0)$ with

$$P_n^{(0)} = P_n, \quad P_n^{(m)} = \sum_{k=0}^{n} P_k^{(m-1)} \quad (m \in \mathbf{N}). \tag{3.6}$$

Let us denote by $\mathcal{N}^{(m)}$ $(m \in \mathbf{N}_0)$ the generalized Nörlund method defined by

$$A_{nk} = \begin{cases} R_n^{(m)} P_{n-k}^{(m)}, & (k = 0, 1, \dots, n), \\ \theta, & (k > n), \end{cases}$$

where $P_n^{(m)}$, $R_n^{(m)} \in \mathcal{L}(X, X)$, while $R_n^{(m)}$ is defined by

$$R_n^{(m)} \sum_{k=0}^{n} P_{n-k}^{(m)} \zeta = \zeta \quad (\zeta \in X, \ n \in \mathbf{N}_0). \tag{3.7}$$

If $P_n = I$ $(n \in \mathbf{N}_0)$ the method $\mathcal{N}^{(m)}$ is a generalized Cesàro method $\mathcal{C}^m$. Therefore all the following results in certain sense are the generalizations of the statements (see [18]) proved for the Cesàro method C^m.

Lemma 3. *If the quantity $\sigma_n^{(m+1)}$ is defined by*

$$\sigma_n^{(m+1)} = R_n^{(m)} \sum_{k=0}^{n} P_{n-k}^{(m)} \xi_k \quad (m \in \boldsymbol{N}_0) \tag{3.8}$$

and the operators $P_n^{(m+1)}$ and $R_n^{(m)}$ $(m \in \boldsymbol{N}_0)$ are commuting, then

$$\sigma_n^{(m+1)} = R_n^{(m)} \sum_{k=0}^{n} P_k^{(m)} \sigma_k^{(m)} \quad (n \in \boldsymbol{N}_0, \, m \in \boldsymbol{N}). \tag{3.9}$$

Proof. As the operators $P_n^{(m+1)}$ and $R_n^{(m)}$ are commuting, then it follows from (3.6) and (3.7) that $P_k^m R_k^{m-1} = I$ and

$$\sum_{k=0}^{n} R_n^{(m)} P_k^{(m)} \sigma_k^{(m)} = \sum_{k=0}^{n} R_n^{(m)} P_k^{(m)} \sum_{\nu=0}^{k} R_k^{(m-1)} P_{k-v}^{(m-1)} \xi_\nu$$

$$= \sum_{\nu=0}^{n} \left(R_n^{(m)} \sum_{k=\nu}^{n} P_{k-\nu}^{(m-1)} \right) \xi_\nu = \sum_{\nu=0}^{n} \left(R_n^{(m)} \sum_{k=0}^{n-\nu} P_k^{(m-1)} \right) \xi_\nu = R_n^{(m)} \sum_{k=0}^{n} P_{n-k}^{(m)} \xi_k.$$

Therefore using (3.8) and (3.9) we get that the assertion of Lemma 3 is valid. ■

Lemma 4. *If operators P_n, P_{n+1} and $R_n^{(m)}$ with $n \in \boldsymbol{N}_0$, $m \in \boldsymbol{N}$ are commuting in pairs and the quantity $\sigma_n^{(m+1)}$ is defined by (3.8), then*

$$\Delta\sigma_n^{(m+1)} = R_n^{(m)} R_{n-1}^{(m)} P_n^{(m)} \sum_{k=1}^{n} P_{k-1}^{(m+1)} \Delta\sigma_k^{(m)}. \tag{3.10}$$

Proof. As operators P_n, P_{n+1} and $R_n^{(m)}$ with $n \in \mathbf{N}_0$, $m \in \mathbf{N}$ are commuting in pairs, then using (3.9) and Abel's partial summation formula we get

$$\begin{aligned}
P_{n-1}^{(m+1)} P_n^{(m+1)} \Delta\sigma_n^{(m+1)} &= P_{n-1}^{(m+1)} P_n^{(m+1)} \left(\sigma_n^{(m+1)} - \sigma_{n-1}^{(m+1)}\right) \\
&= P_{n-1}^{(m+1)} P_n^{(m+1)} \left(R_n^{(m)} \sum_{k=0}^{n} P_k^{(m)} \sigma_k^{(m)} - R_{n-1}^{(m)} \sum_{k=0}^{n-1} P_k^{(m)} \sigma_k^{(m)}\right) \\
&= P_{n-1}^{(m+1)} P_n^{(m+1)} \left(\left(R_n^{(m)} - R_{n-1}^{(m)}\right) \sum_{k=0}^{n-1} P_k^{(m)} \sigma_k^{(m)} + R_n^{(m)} P_n^{(m+1)} \sigma_n^{(m)}\right) \\
&= \left(P_{n-1}^{(m+1)} P_n^{(m+1)} R_n^{(m)} - P_{n-1}^{(m+1)} P_n^{(m+1)} R_{n-1}^{(m)}\right) \sum_{k=0}^{n-1} P_k^{(m)} \sigma_k^{(m)} \\
&\qquad + P_{n-1}^{(m+1)} P_n^{(m+1)} R_n^{(m)} P_n^{(m+1)} \sigma_n^{(m)} \\
&= -P_n^{(m)} \sum_{k=0}^{n} P_k^{(m)} \sigma_k^{(m)} + P_n^{(m)} P_n^{(m)} \sigma_n^{(m)} + P_{n-1}^{(m+1)} P_n^{(m)} \sigma_n^{(m)} \\
&= P_n^{(m)} \sum_{k=0}^{n-1} P_k^{(m+1)} \Delta\sigma_{k+1}^{(m)} = P_n^{(m)} \sum_{k=1}^{n} P_{k-1}^{(m+1)} \Delta\sigma_k^{(m)}
\end{aligned}$$

and

$$R_n^{(m)} R_{n-1}^{(m)} P_{n-1}^{(m+1)} P_n^{(m+1)} \Delta\sigma_n^{(m+1)} = R_n^{(m)} R_{n-1}^{(m)} P_n^{(m)} \sum_{k=0}^{n-1} P_k^{(m+1)} \Delta\sigma_{k+1}^{(m)}. \tag{3.11}$$

As $R_{n-1}^{(m)} P_{n-1}^{(m+1)} = I$ and $R_n^{(m)} P_n^{(m+1)} = I$, then (3.11) implies that the assertion (3.10) is valid. ■

Lemma 5. *If operators P_n, P_{n+1} and $R_n^{(m)}$ with $n \in \boldsymbol{N}_0$, $m \in \boldsymbol{N}$ are commuting in pairs and*

$$\left\|P_{k-1}^{(m)}\right\| = \mathcal{O}\left(\left\|P_k^{(m)}\right\|\right), \quad n\lambda_n \left\|\Delta\sigma_n^{(m)}\right\| = \mathcal{O}(1), \tag{3.12}$$

$$n \left\|R_{n-1}^{(m)}\right\| \left\|P_n^{(m)}\right\| = \mathcal{O}(1), \quad \lambda_n \left\|R_n^{(m)}\right\| \sum_{k=0}^{n} \frac{\left\|P_k^{(m)}\right\|}{\lambda_k} = \mathcal{O}(1), \tag{3.13}$$

then we have that

$$n\lambda_n \left\|\Delta\sigma_n^{(m+1)}\right\| = \mathcal{O}(1). \tag{3.14}$$

Proof. As by (3.10) we have

$$n\lambda_n \left\| \Delta\sigma_n^{(m+1)} \right\| \leq n\lambda_n \left\| R_n^{(m)} \right\| \left\| R_{n-1}^{(m)} \right\| \left\| P_n^{(m)} \right\| \sum_{k=1}^{n} \left\| P_{k-1}^{(m+1)} \right\| \left\| \Delta\sigma_k^{(m)} \right\|,$$

then using (3.6) and (3.12)–(3.13) we get

$$n\lambda_n \left\| \Delta\sigma_n^{(m+1)} \right\| = \lambda_n \left\| R_n^{(m)} \right\| \sum_{k=1}^{n} k \left\| P_k^{(m)} \right\| \mathcal{O}\Big(\frac{1}{k\lambda_k}\Big) =$$

$$= \mathcal{O}\left(1\right) \lambda_n \left\| R_n^{(m)} \right\| \sum_{k=1}^{n} \frac{\left\| P_k^{(m)} \right\|}{\lambda_k} = \mathcal{O}\left(1\right).$$

That means the assertion (3.14) is valid. ■

Corollary 4. If operators P_n, P_{n+1} and $R_n^{(m)}$ with $n \in \mathbf{N}_0$, $m \in \mathbf{N}$ are commuting in pairs,

$$n\lambda_n \Delta\sigma_n^{(1)} = \mathcal{O}\left(1\right) \tag{3.15}$$

and the conditions (3.12), (3.13) are satisfied, then the assertion (3.14) is valid.

Remark 3. If the quantity $\sigma_n^{(1)}$ is defined by (3.4), then

$$\Delta\sigma_n^{(1)} = \sum_{k=1}^{n} \left(R_n P_{n-k}^{(1)} - R_{n-1} P_{n-k-1}^{(1)} \right) \Delta\xi_k.$$

Proposition 4. *If operators P_n, P_{n+1} and $R_n^{(m)}$ with $n \in \boldsymbol{N}_0$, $m \in \boldsymbol{N}$ are commuting in pairs, $\mathcal{N}^{(m)}x \in m_X^{\lambda}$ $(m \in \boldsymbol{N})$ and*

$$\lambda_n \sum_{k=1}^{n} \left\| R_n^{(m)} P_{k-1}^{(m+1)} \right\| \left\| \Delta\sigma_k^{(m)} \right\| = \mathcal{O}\left(1\right), \tag{3.16}$$

then $\mathcal{N}^{(m-1)}x \in m_X^{\lambda}$.

Proof. Using (3.9), (3.7) and Abel's partial summation formula, we get

$$\begin{aligned}
\sigma_n^{(m+1)} - \sigma_n^{(m)} &= R_n^{(m)} \sum_{k=0}^{n} P_k^{(m)} \sigma_k^{(m)} - R_n^{(m)} \sum_{k=0}^{n} P_k^{(m)} \sigma_n^{(m)} \\
&= R_n^{(m)} \sum_{k=0}^{n} P_k^{(m)} \left(\sigma_k^{(m)} - \sigma_n^{(m)} \right) \\
&= R_n^{(m)} \Big(- \sum_{k=0}^{n-1} P_k^{(m+1)} \left(\sigma_{k+1}^{(m)} - \sigma_n^{(m)} - \sigma_k^{(m)} + \sigma_n^{(m)} \right) + P_n^{(m+1)} \cdot 0 \Big) \\
&= -R_n^{(m)} \sum_{k=0}^{n-1} P_k^{(m+1)} \Delta\sigma_{k+1}^{(m)} = -R_n^{(m)} \sum_{k=1}^{n} P_{k-1}^{(m+1)} \Delta\sigma_{k+1}^{(m)}.
\end{aligned}$$

So we have

$$\lambda_n \left(\sigma_n^{(m+1)} - \sigma_n^{(m)}\right) = -\lambda_n R_n^{(m)} \sum_{k=1}^{n} P_{k-1}^{(m+1)} \Delta\sigma_{k+1}^{(m)}$$

and using (3.10) we get

$$\lambda_n \left(\sigma_n^{(m+1)} - \sigma_n^{(m)}\right) = \mathcal{O}\,(1)\,.$$

As

$$\mathcal{N}^{(m)}x \in m_X^{\lambda} \Leftrightarrow \lambda_n \left(\sigma_n^{(m+1)} - \xi\right) = \mathcal{O}\,(1)\,,$$

while $\xi = \lim_n \sigma_n^{(m+1)}$, and

$$\lambda_n \left(\sigma_n^{(m)} - \xi\right) = \lambda_n \left(\sigma_n^{(m)} - \sigma_n^{(m+1)}\right) + \lambda_n \left(\sigma_n^{(m+1)} - \xi\right),$$

therefore we get

$$\lambda_n \left(\sigma_n^{(m)} - \xi\right) = \mathcal{O}\,(1)\,.$$

That means the statement of the Proposition 4 is valid. ∎

Corollary 5. If operators P_n, P_{n+1} and $R_n^{(m)}$ with $n \in \mathbf{N}_0$, $m \in \mathbf{N}$ are commuting in pairs, $\mathcal{N}^{(m)}x \in m_X^{\lambda}$ $(m \in \mathbf{N})$ and conditions (3.12) – (3.13) and (3.15) are satisfied, then $\mathcal{N}^{(m-1)}x \in m_X^{\lambda}$.

References

[1] J. Boos. *Classical and modern methods in summability.* Oxford University Press, 2000.

[2] D. Borwein and R. Kiesel. Weighted means and summability by generalized Nörlund and other methods. *J. Math. Anal. Appl.*, **184**, 607–619, 1994.

[3] C. Brezinski. Numerical analysis 2000. *J. Comput. Appl. Math.*, **122**, 1–357, 2000.

[4] J. DeFranza. A general inclusion theorem for Nörlund summability. *Canad. Math. Bull.*, **25**, 447–455, 1982.

[5] G. Kangro. On matrix transformations of sequences in Banach spaces. *Proc. Estonian Acad. Sci. Tech. Phys. Math.*, **5**, 108–128, 1956.

[6] G. Kangro. Summability factors for the series λ-bounded by the methods of Riesz and Cesàro. *Acta Comment. Univ. Tartuensis*, **277**, 136–154, 1971.

[7] R. Kiesel. Strong laws and summability for sequences of φ-mixing random variables in Banach spaces. *Electron. Comm. Probab.*, **2**, 27–41, 1997.

[8] I. Kornfeld. Nonexistence of an universally accelerating linear summability methods. *J. Comput. Appl. Math.*, **53**, 309–321, 1994.

[9] O. Meronen and A. Tali. Some summability methods b-equivalent to the Cesàro methods. *Proc. Est. Acad. Sci., Phys. Math.*, **51**, 137–148, 2002.

[10] O. Meronen and I. Tammeraid. Generalized Euler-Knopp method and convergence acceleration. *Math. Mod. and Anal.*, **11**, 87–94, 2006.

[11] A. Nappus and T. Sormus. Einige verallgemeinerte Matrixverfahren. *Proc. Estonian Acad. Sci. Phys. Math.*, **45**, 201–210, 1996.

[12] P. Natarajan and V. Srinivasan. On generalized Nörlund methods in non-Arhimedean fields. *Bull. Allahabad Math. Soc.*, **15**, 41–46, 2000.

[13] A. Rohtla and A. Tali. A Tauberian theorem for a family of generalized Nörlund methods. *Acta Comment. Univ. Tartuensis*, **770**, 80–87, 1987. in Russian

[14] T. Sormus. Tauberian theorems for generalized summability methods in Banach spaces. *Proc. Estonian Acad. Sci. Phys. Math.*, **49**, 170–182, 2000.

[15] U. Stadtmüller. On a family of summability methods and one-sided Tauberian conditions. *J. Math. Anal. Appl.*, **196**, 99–119, 1995.

[16] U. Stadtmüller and A. Tali. On certain families of generalized Nörlund methods and power series methods. *J. Math. Anal. Appl.*, **238**, 44–66, 1999.

[17] U. Stadtmüller and A. Tali. Comparison of certain summability metdods by speed of convergence. *Anal. Math.*, **29**, 277–242, 2003.

[18] I. Tammeraid. Tauberian remainder theorems for the Cesàro and Hölder methods of summability. *Acta Comment. Univ. Tartuensis*, **277**, 161–170, 1971.

[19] I. Tammeraid. Convergence rate of iterative process and weighted means. In: *Proc. of the OFEA'2001. Conference OFEA'2001 Saint Petersburg, Russia, 2001*, volume 2, 49–55, 2002.

[20] I. Tammeraid. Convergence acceleration and linear methods. *Math. Mod. and Anal.*, **8**, 87–92, 2003.

[21] I. Tammeraid. Generalized linear methods and convergence acceleration. *Math. Mod. and Anal.*, **8**, 329–335, 2003.

[22] I. Tammeraid. Several remarks on acceleration of convergence using generalized linear methods of summability. *J. Comput. Appl. Math.*, **159**, 365–373, 2003.

[23] I. Tammeraid. Generalized Riesz method and convergence acceleration. *Math. Mod. and Anal.*, **9**, 341–348, 2004.

[24] K. Zeller. Verallgemeinerte Matrix Transformationen. *Math. Z.*, **56**, 18–20, 1952.