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# MULTIPLICITY RESULTS FOR THE NEUMANN BOUNDARY VALUE PROBLEM

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**Abstract.** We provide multiplicity results for the Neumann boundary value problem, when the second order differential equation is of the form $x'' = f(x)$.

**Key words:** critical points, multiple solutions, homoclinic solutions

## 1. Introduction

We consider the following boundary value problem

$$\begin{cases} x'' = f(x), \\ x'(0) = 0, \quad x'(1) = 0, \end{cases} \tag{1.1}$$

where $f(x)$ is a continuously differentiable function which has simple zeros. Our goal is to prove the multiplicity results for problem (1.1). They can be generalized to the case of $f(x)$ being a function with $n$ simple zeros. Similar results for $f(x) = -x + x^2$ with the Dirichlet boundary conditions were obtained in [3].

## 2. Simple Cases

Let us consider problem (1.1). Our assumptions on a function $f(x)$ are the following: (C1) $f \in C^1(R)$, (C2) $f(x)$ has simple zeros at $p_1 < p_2 < p_3 < p_4 < p_5$, (C3) $f(-\infty) = -\infty$ and respectively $f(+\infty) = +\infty$.

An example of such a function is shown in Fig. 1.

Let us consider the primitive function $F(x) = \int_0^x f(s)\,ds$, which has exactly three local minimums at the points $p_1 < p_3 < p_5$ and consequently two local maximums at the points $p_2 < p_4$ as is shown in Fig. 2.

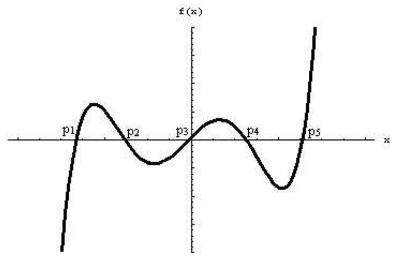



**Figure 1.** The function $f(x)$.

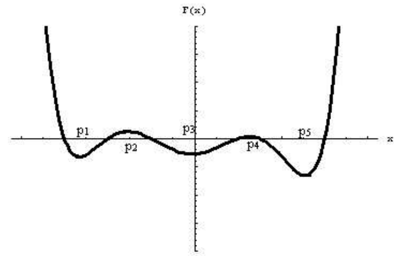



**Figure 2.** The primitive $F(x)$.

The phase portrait of the equivalent system

$$\begin{cases} x' = y, \\ y' = f(x) \end{cases} \tag{2.1}$$

depends on properties of the function $f(x)$ and its primitive $F(x)$. Let us consider the cases:

$$1)\ F(p_1) < F(p_3) < F(p_5), \quad 2)\ F(p_5) < F(p_3) < F(p_1),$$
$$3)\ F(p_3) < F(p_1) < F(p_5), \quad 4)\ F(p_3) < F(p_5) < F(p_1).$$

System (2.1) has three critical points of the type "saddle" at $(p_1; 0)$, $(p_3; 0)$, $(p_5; 0)$ and two critical points of the type "center" at $(p_2; 0)$, $(p_4; 0)$.

The following phase portraits describe periodical solutions, the phase trajectories of which go around one of critical points of the type "center". There are two homoclinic solutions (bold lines in Fig. 3 and Fig. 4) which go around the points $(p_2; 0)$, $(p_4; 0)$.

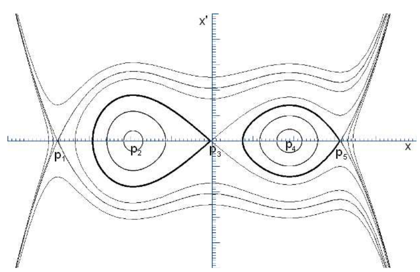



**Figure 3.** The phase portrait of the case 1.

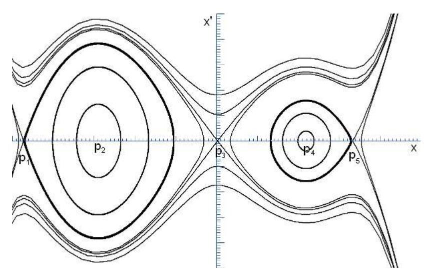



**Figure 4.** The phase portrait of the case 3.

**Theorem 1.** *Let the conditions*

$$n^2\pi^2 < |f_x(p_2)| < (n+1)^2\pi^2,$$

$$m^2\pi^2 < |f_x(p_4)| < (m+1)^2\pi^2$$

*hold. Then the Neumann boundary value problem* (1.1) *has at least* $2n+2m$ *nonconstant solutions.*

The proof is given in [1].

## 3. More Complicated Cases

Let us consider the following cases

$$1)\ F(p_1) < F(p_5) < F(p_3), \quad 2)\ F(p_5) < F(p_1) < F(p_3).$$

These cases are symmetrical. There exist the other periodical solutions with the property that the respective phase trajectories go around both of critical points of the type "center" (see Figure 5).

Consider the function

$$T(x_0) = \frac{1}{\sqrt{2}} \int_{x_0}^{x_1(x_0)} \frac{ds}{\sqrt{F(s) - F(x_0)}},$$

which is defined in the interval $(x_0^*; x_0^{**})$, where $x_1(x_0)$ is the first zero to the right of $x_0$ of the function $F(s) - F(x_0)$. Let us define $x_0^*$ to as the first zero to the left of the function $F_5 = \int_{p_5}^{x} f(s)ds$. Similarly define $x_0^{**}$ be the first zero to the left of the function $F_3 = \int_{p_3}^{x} f(s)ds$. Obviously $p_1 < x_0^* < x_0^{**} < p_2$.

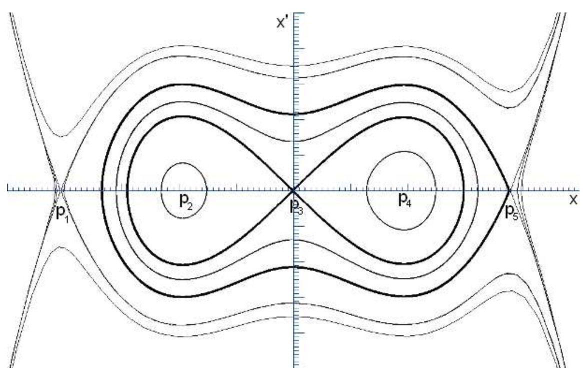



**Figure 5.** The phase portrait for the case $F(p_1) < F(p_5) < F(p_3)$.

**Theorem 2.** *Let* $T_{min} = \min\{T(x) : x \in (x_0^*, x_0^{**})\}$, *where* $x_1 > x_0$ *is the first zero of the function* $(F(s) - F(x_0))$. *Suppose that there exists an integer* $k$ *such that* $kT_{min} < 1 < (k+1)T_{min}$. *Then there are at least* $4k$ *solutions of the Neumann boundary value problem, with trajectories going around the two singular points of the type "center".*

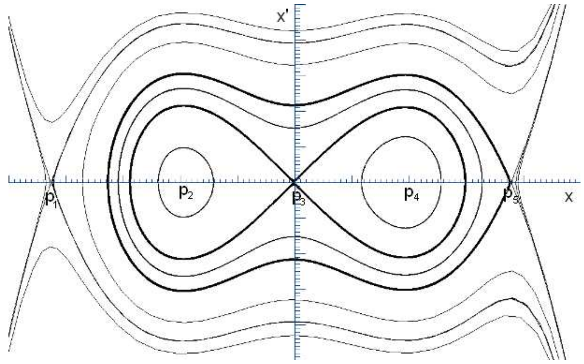



**Figure 6.** Phase portrait for the (4.1).

*Proof.* Let $T(z) = \min\{T(x_0),\ x_0^* < x_0 < x_0^{**}\}$. Consider the Cauchy problem (1.1), $x(0) = x_0$, $x'(0) = 0$, $x_0 \in (x_0^*; z)$. When $x_0$ is close to $z$, then the half period $T(x_0)$ satisfies the condition $kT_{min} < 1 < (k+1)T_{min}$. On the other hand, $T(x_0) \to +\infty$, when $x_0 \to x_0^*$. Hence there exist at least $k$ solutions of the problem. A similar result is valid for the case $z < m < x_0^{**}$. Hence additionally at least $k$ solutions exist.

Define $z_1$ as the first zero of $F(s) - F(z)$ to the right of $z$. Notice that $z_1 \in (z_2; p_5)$. Consider the case of $x(0) = n$ for $n \in (z_2; z_1)$. When $n$ is close to $z_1$, the condition is satisfied, and when $n$ is close to $z_2$, then $T(n) \to +\infty$. Hence there exist at least $k$ solutions of the problem. A similar result is valid for $n \in (z_1; p_5)$. Thus we have proved that totally at least $4k$ solutions exist. ■

*Remark 1.* "Small-amplitude" solutions of the given boundary value problem can exist in neighborhoods of the critical points of the type "center". The conditions for existence of such solutions are given in Theorem 1.

## 4. Example I

Consider the second-order nonlinear boundary value problem

$$\begin{cases} x'' = 6x^5 + 2.5x^4 - 76x^3 - 7.5x^2 + 148x + 2, \\ x'(0) = x'(1) = 0. \end{cases} \tag{4.1}$$

Function $f(x) = 6x^5 + 2.5x^4 - 76x^3 - 7.5x^2 + 148x + 2$ has exactly five simple zeros, where $f'(x) > 0$. The equivalent two-dimensional system has three critical points of the type "saddle". There are also two zeros where $f'(x) < 0$, and a system has two critical points of the type "center".

Respectively the function

$$F(x) = x^6 + 0.5x^5 - 19x^4 - 2.5x^3 + 74x^2 + 2x - 56$$

has three local minimums and consequently two local maximums as is shown in Fig. 7.

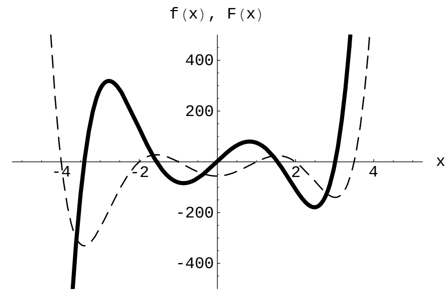
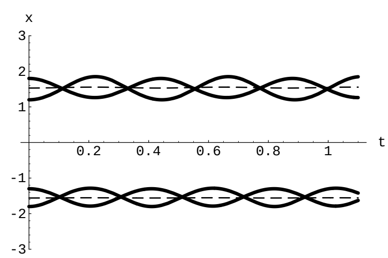



**Figure 7.** Function $f(x)$ (bold) and its primitive (dashed).

We notice that the condition $4^2\,\pi^2 < 242 < 5^2\,\pi^2$ holds, it follows from $f_x(p_2) = f_x(-1.56) = -242$. Then the boundary value problem (4.1) has at least eight solutions. Similarly, if $f_x(p_4) = f_x(1.54) = -211$, then the condition $4^2\,\pi^2 < 211 < 5^2\,\pi^2$ holds. So the boundary value problem (4.1) has at least eight solutions (see Fig. 8).



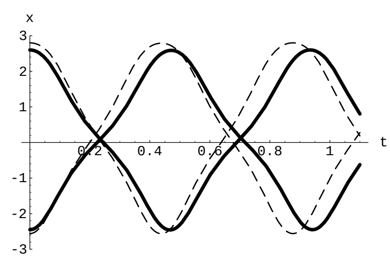




**Figure 8.** Small amplitudes solutions. **Figure 9.** Large amplitudes solutions.

In Table 1 values of $T_{min}$ are presented.
The boundary value problem (4.1) also has at least eight "large amplitudes" solutions as is shown in Fig. 9. Hence the boundary value problem (4.1) has at least 24 solutions.

## 5. Example II

Consider the second-order nonlinear boundary value problem

$$\begin{cases} x'' = x^7 - 0.5x^6 - 13.5x^5 + 6.5x^4 + 42.5x^3 - 18x^2 - 18x, \\ x'(0) = 0, \quad x'(1) = 0. \end{cases} \tag{5.1}$$

**Table 1.** Results of numerical experiments: values of $T_{min}$.

| $x_0$ | $x_1(x_0)$ | $T_0$ |
|---|---|---|
| -2.58 | 2.84 | 0.44 |
| -2.56 | 2.79 | 0.4383 |
| -2.54 | 2.76 | 0.4364 |
| -2.53 | 2.72 | 0.4327 |
| -2.51 | 2.69 | 0.44 |
| -2.47 | 2.63 | 0.46 |

The function

$$f(x) = x^7 - 0.5x^6 - 13.5x^5 + 6.5x^4 + 42.5x^3 - 18x^2 - 18x$$

has seven simple zeros. A respective two-dimensional system has exactly four critical points of the type "saddle" at the points $p_1 = -3$, $p_3 = -0.5$, $p_5 = 1$, $p_7 = 3$ and three critical points of the type "center" at the points $p_2 = -2$, $p_4 = 0$, $p_6 = 2$. The function

$$F(x) = \frac{1}{8}x^8 - \frac{1}{14}x^7 - \frac{9}{4}x^6 + \frac{13}{10}x^5 + \frac{85}{8}x^4 - 6x^3 - 9x^2$$

has 4 local minimums and consequently 3 local maximums as is shown in Fig. 10 and Fig. 11.

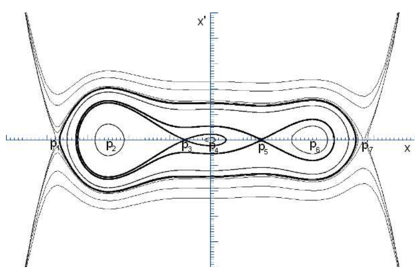



**Figure 10.** Phase portrait for the (5.1).

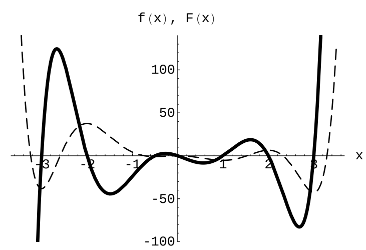



**Figure 11.** Function $f(x)$ (bold) and its primitive (dashed).

By computing of $f_x(p_2) = -180$, $f_x(p_4) = -18$, $f_x(p_6) = -100$ we get that the conditions

$$4^2\,\pi^2 < 180 < 5^2\,\pi^2, \quad 1^2\,\pi^2 < 18 < 2^2\,\pi^2, \quad 3^2\,\pi^2 < 100 < 4^2\,\pi^2$$

hold. Then the boundary value problem (5.1) has at least 16 solutions.

In Table 2 values of $T_{min}$ are presented. We get that boundary value

**Table 2.** Results of numerical experiments: values of $T_{min}$.

| $x_0$ | $x_1(x_0)$ | $T_0$ |
|---|---|---|
| -2.70 | 2.54 | 0.971 |
| -2.80 | 2.69 | 0.792 |
| -2.90 | 2.81 | 0.750 |
| -2.95 | 2.84 | 0.746 |
| -2.97 | 2.85 | 0.760 |
| -2.98 | 2.86 | 0.783 |

problem (5.1) has at least 4 "large" solutions due to Theorem 1, hence in total it has at least 20 solutions (see Fig. 12 and Fig. 13).

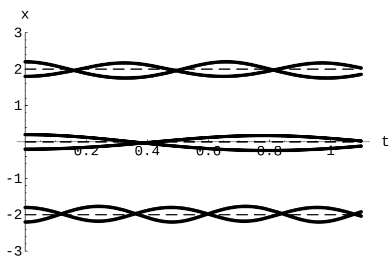



**Figure 12.** Solutions in the neighborhoods of constants $p_2$, $p_4$ and $p_6$.

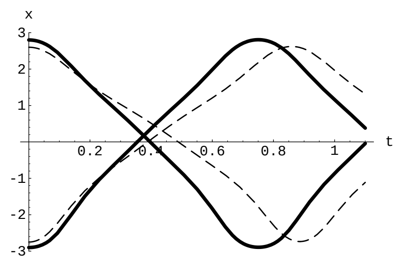



**Figure 13.** "Large amplitudes" solutions.

*Remark 2.* As can be viewed from Fig. 10 that "very large amplitude" solutions with closed orbits containing three "centers" exist. The respective solution of the Neumann BVP in our example exists on larger $t$-interval (computations show that the half-period of these solutions is greater than 1).

*Remark 3.* Let us mention that considering functions $f(x)$ with arbitrary large numbers of simple zeros we can obtain a hierarchy of solutions with phase trajectories going around numerous critical points.

*Remark 4.* The results for $f(x)$ with $n$ simple zeros are given in [2].

## Acknowledgment

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