Mathematical Modelling and Analysis
Volume 13 Number 4, 2008, pages 483–492
Doi:10.3846/1392-6292.2008.13.483-492
© 2008 Technika ISSN 1392-6292 print, ISSN 1648-3510 online

FINITE DIFFERENCE SCHEME FOR A SINGULARLY PERTURBED PARABOLIC EQUATIONS IN THE PRESENCE OF INITIAL AND BOUNDARY LAYERS*

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Received November 25, 2007; revised December 28, 2007; published online December 1, 2008

Abstract. The grid approximation of an initial-boundary value problem is considered for a singularly perturbed parabolic reaction-diffusion equation. The second-order spatial derivative and the temporal derivative in the differential equation are multiplied by parameters ε_1^2 and ε_2^2, respectively, that take arbitrary values in the open-closed interval $(0,1]$. The solutions of such parabolic problems typically have boundary, initial layers and/or initial-boundary layers. *A priori* estimates are constructed for the regular and singular components of the solution. Using such estimates and the condensing mesh technique for a tensor-product grid, piecewise-uniform in x and t, a difference scheme is constructed that converges $\overline{\varepsilon}$-uniformly at the rate $\mathcal{O}(N^{-2}\ln^2 N + N_0^{-1}\ln N_0)$, where $(N+1)$ and (N_0+1) are the numbers of mesh points in x and t respectively.

Key words: initial-boundary value problem, parabolic reaction-diffusion equation, perturbation vector-parameter $\overline{\varepsilon}$, finite difference approximation, boundary layer, initial layer, initial-boundary layer, piecewise-uniform grids, $\overline{\varepsilon}$-uniform convergence.

* This research was supported by the Russian Foundation for Basic Research under grants Nos. 07–01–00729, the Boole Centre for Research in Informatics at the National University of Ireland in Cork, and by the Mathematics Applications Consortium for Science and Industry in Ireland (MACSI) under the Science Foundation Ireland (SFI) Mathematics Initiative.

1 Introduction

When studying heat-transfer and mass-transfer processes where their duration and the coefficients of heat conduction/diffusion are small or large we must deal with boundary value problems for singularly perturbed parabolic equations, i.e., equations with small parameters multiplying the highest-order derivatives in space and/or the derivative in time. The smallness of these parameters induces boundary, initial and initial-boundary layers in the solution of the problem. As a consequence, for such problems the errors in the discrete solutions that are obtained using classical difference schemes are commensurable with the solutions of the boundary value problem itself (see, e.g., [16]) just as for parabolic problems where the highest-order spatial derivatives are multiplied by a small parameter [2, 3, 7, 9, 10, 15].

In this paper, a Dirichlet problem is considered for a singularly perturbed parabolic reaction-diffusion equation. The second-order spatial derivative and the temporal derivative in the differential equation are multiplied by parameters ε_1^2 and ε_2^2, respectively. These parameters are components of the perturbation vector-parameter $\overline{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, and $\varepsilon_1, \varepsilon_2 \in (0, 1]$. As one or both of the parameters ε_1 and ε_2 tend to zero, initial, boundary and/or initial-boundary layers arise in the solution of the Dirichlet problem.

For such problem, the standard difference schemes converge in the discrete maximum norm to the true solution under the condition $N^{-1} = o(\varepsilon_1)$, $N_0^{-1} = o(\varepsilon_2^2)$, where $(N+1)$ and (N_0+1) are the numbers of mesh points in x and t respectively; for each fixed value of the parameter $\overline{\varepsilon}$, this scheme on uniform meshes is convergent of order $\mathcal{O}(N^{-2} + N_0^{-1})$. Using the method of fitted meshes, i.e., piecewise-uniform meshes condensing in a neighbourhood of the initial and boundary layers, a difference scheme is constructed that converges $\overline{\varepsilon}$-uniformly at the rate $\mathcal{O}(N^{-2}\ln^2 N + N_0^{-1}\ln N_0)$; à definition of the $\overline{\varepsilon}$-uniform convergence see in Section 2. A description of the condensing mesh technique can be found, e.g., in [1, 3, 9, 10, 15].

Grid approximations of boundary value problems for parabolic equations that are two-dimensional in space, i.e., on a strip and on a rectangle, similar to the boundary value problem of the present paper have been studied in [16]. Sufficiently coarse *a priori* estimates that were derived in [16] allowed to establish the $\overline{\varepsilon}$-uniform convergence of the schemes constructed on piecewise-uniform meshes but only with order $\mathcal{O}(N^{-1}\ln N)^{2/3} + (N_0^{-1}\ln N_0)^{1/2})$ on a rectangular grid where N characterizes the number of nodes in x_1 and x_2 on studied spatial domains, and (N_0+1) is the number of nodes in time t.

2 Problem Formulation

2.1. On the domain $\overline{G}$, where

$$G = D \times (0, T], \quad \overline{G} = G \bigcup S, \quad D = (0,\, d), \tag{2.1}$$

we consider the initial–boundary value problem for the singularly perturbed parabolic equation [1]

$$
\begin{aligned}
L_{(2.2)}\, u(x,t) &= f(x,t), \quad (x,t) \in G, && (2.2a)\\
u(x,t) &= \varphi(x,t), \quad (x,t) \in S, && (2.2b)\\
L_{(2.2)} &\equiv \varepsilon_1^2\, a(x,t)\, \frac{\partial^2}{\partial x^2} - \varepsilon_2^2\, p(x,t)\, \frac{\partial}{\partial t} - c(x,t).
\end{aligned}
$$

The parameters ε_1 and ε_2 are components of a vector-parameter $\overline{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ and take arbitrary values in the open-closed interval $(0,1]$. Assume that the coefficients $a(x,t)$, $c(x,t)$, $p(x,t)$ and the right-hand side $f(x,t)$ are sufficiently smooth on the set $\overline{G}$ and that

$$
\begin{aligned}
&0 < a_0 \le a(x,t) \le a^0, \quad 0 < c_0 \le c(x,t) \le c^0,\\
&0 < p_0 \le p(x,t) \le p^0, \quad (x,t) \in \overline{G}.
\end{aligned}
\tag{2.3}
$$

Define $S^L = \mathit{\Gamma} \times (0,T]$, $S_0 = \overline{D} \times \{t = 0\}$, $S = S^L \bigcup S_0$, and $\mathit{\Gamma} = \overline{D} \setminus D$. The boundary function $\varphi(x,t)$ is assumed to be sufficiently smooth on the sets $\overline{S}^L$ and S_0 and to be continuous on S.

By a solution of the initial-boundary value problem, we mean a function $u \in C(\overline{G}) \bigcap C^{2,1}(G)$ that satisfies the differential equation G and the boundary condition on S.

Problems similar to (2.2), (2.1) arise in modelling heat transfer in a liquid flowing through a narrow channel [12], including the entrance. The parameters ε_1^2 and ε_2^2 characterize the heat-conduction coefficient and the flow velocity along the channel, and the coefficient $c(x,t)$ describes heat emission from the channel wall to the liquid flow. In modelling heat transfer in fluidized-bed reactors, the parameters ε_1^2 and ε_2^2 characterize the diffusion of particles and their convective velocity along the reactor, and the coefficient $c(x,t)$ describes, e.g., water evaporation heat and/or heat released by the reaction [5].

2.2. Let us discuss the behaviour of the solution u of the initial-boundary value problem. When one or both of the parameters ε_1 and ε_2 tend to zero, the solution of the boundary value problem exhibits layers which are initial, boundary and initial-boundary (see Remark 1 in Section 3). The initial-boundary layers are parabolic while the initial and boundary layers are regular or parabolic depending on the relation between the parameters ε_1 and ε_2.

The errors of discrete solutions to classical difference schemes that approximate the problem (2.2), (2.1) depend on the parameters ε_i and may be commensurable with the exact solutions of the boundary value problems for small values of the parameters ε_1 and ε_2; see the estimates (4.3), (4.6) in Section 4. When the error in the discrete solution is independent of the parameters ε_i and tends to zero as the number of mesh points grows, we say that the solution (or the difference scheme) converges $\overline{\varepsilon}$-uniformly. We are interested in numerical methods with this valuable property.

Our aim for the boundary value problem (2.2), (2.1) is to construct a difference scheme that converges ε-uniformly with convergence rates that are close to the first order in time and close to the second order in space.

[1] Throughout the paper, the notation $L_{(j.k)}$ ($M_{(j.k)}$, $G_{h(j.k)}$) means that these operators (constants, grids) are introduced in formula ($j.k$).

3 *A priori* Estimates

Let us give estimates for the solution u and its derivatives. They are derived using techniques from the papers [15, 6, 8, 4].

Using comparison theorems, we find that

$$|u(x,t)| \le M\big[\max_{\overline{G}} |f(x,t)| + \max_{S} |\varphi(x,t)|\big], \quad (x,t) \in \overline{G}. \tag{3.1}$$

Assume that the problem data satisfy the condition

$$\begin{aligned} &a,\, c,\, p,\, f \in C^{l+\alpha,(l+\alpha)/2}(\overline{G}), && (3.2a)\\ &\varphi \in C^{l+2+\alpha}(S_0) \bigcap C^{l+2+\alpha,(l+2+\alpha)/2}(\overline{S}^{\,L}) \bigcap C(S), \quad l \ge 0, \quad \alpha > 0, \end{aligned}$$

and that on the set $S^c = \overline{S}^{\,L} \bigcap S_0$, the data satisfy compatibility conditions (see, e.g., in [8]), that ensure

$$u \in C^{l+2+\alpha,(l+2+\alpha)/2}(\overline{G}) \tag{3.2b}$$

for each fixed value of the parameters ε_i. Additional conditions will be given later.

3.1. We find bounds for the solution of the problem (2.2), (2.1), using a classical technique [4, 8]. Set $\xi = \xi(x) = \varepsilon_1^{-1}\, x$ and $\tau = \tau(t) = \varepsilon_2^{-2}\, t$. In these new variables the original problem is transformed into the boundary value problem

$$\begin{aligned} \widetilde{L}\, \widetilde{u}(\xi,\tau) &= \widetilde{f}(\xi,\tau), \quad (\xi,\tau) \in \widetilde{G}, && (3.3a)\\ \widetilde{u}(\xi,\tau) &= \widetilde{\varphi}(\xi,\tau), \quad (\xi,\tau) \in \widetilde{S}. && (3.3b) \end{aligned}$$

Here $\widetilde{v}(\xi,\tau) = v\big(x(\xi), t(\tau)\big)$, where $v(x,t)$ is any one of the functions $u(x,t)$, $\dots$, $\varphi(x,t)$, and

$$\widetilde{G}^0 = \{(\xi,\tau) :\ \xi = \xi(x),\ \tau = \tau(t),\ (x,t) \in G^0\},$$

where G^0 is either of the sets G, S.

In the domain $\widetilde{G}$ with the boundary condition (3.3b) on $\widetilde{S}$, the differential equation (3.3a) is regular with respect to the parameters ε_i. Hence the Schauder estimates give

$$\left| \frac{\partial^{k+k_0}}{\partial \xi^k \partial \tau^{k_0}}\, \widetilde{u}(\xi,\tau) \right| \le M, \quad (\xi,\tau) \in \overline{\widetilde{G}}.$$

Returning to the variables x and t, we obtain the bound

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, u(x,t) \right| \le M\, {\varepsilon_1}^{-k}\, {\varepsilon_2}^{-2k_0}, \quad (x,t) \in \overline{G},\ k + 2k_0 \le K, \tag{3.4}$$

where $K = l + 2$, $l = l_{(3.2)}$. We have now shown the following result:

Theorem 1. *The solution of the boundary value problem* (2.2), (2.1) *satisfies the estimate* (3.1)*. If the hypothesis* (3.2) *is satisfied, then the estimate* (3.4) *is valid.*

3.2. Let us refine the estimate (3.4). Here we need estimates of regular and singular parts of the solution. A type of components to the singular part of the solution and parts of the domain boundary in whose neighbourhood these components arise depend on the relations between the scalar parameters ε_1 and ε_2. When deriving the estimates for the boundary layer and the initial layer functions, we use solutions of auxiliary problems defined on extensions of the set $\overline{G}$. Such approach allows us to construct estimates for each singular component separately and also to avoid "artificial" compatibility conditions that ensure the smoothness of the solution components required for justification convergence the constructed schemes. We assume that

$$\begin{aligned} &a,\ c,\ p,\ f \in C^{l+4,(l+4)/2}(\overline{G}), \qquad (3.5a)\\ &\varphi \in C^{l+2+\alpha}(S_0) \bigcap C^{l+2+\alpha,(l+2+\alpha)/2}(\overline{S}^{\,L}) \bigcap C(S), \quad l \ge 0, \quad \alpha > 0, \end{aligned}$$

and that on the set S^c compatibility conditions are fulfilled that guarantee

$$u \in C^{l+2+\alpha,(l+2+\alpha)/2}(\overline{G}). \qquad (3.5b)$$

The problem solution will now be decomposed in the following way:

$$u(x,t) = U(x,t) + W(x,t) + V(x,t) + Q(x,t), \quad (x,t) \in \overline{G}, \qquad (3.6)$$

where $U(x,t)$ is the regular part of the solution while $W(x,t)$, $V(x,t)$ and $Q(x,t)$ are components of the singular part of the solution, i.e., the initial, boundary and initial–boundary layers.

Extend the data of problem (2.2) beyond the boundary S to a larger domain $\overline{G}^{\,e0}$: the function $\varphi^e(x,t)$ is smooth on $\overline{G}^{\,e0}$ with $\varphi^e(x,t) = \varphi(x,t)$, $(x,t) \in S$, and the functions $f^e(x,t)$ (which extends f) and $\varphi^e(x,t)$ are assumed to be equal to zero outside a sufficiently small neighbourhood of the set $\overline{G}$. Define the function $U^e(x,t)$, $(x,t) \in \overline{G}^{\,e0}$, to be the solution of the problem

$$L^e_{(2.2)}\, U^e(x,t) = f^e(x,t),\ (x,t) \in G^{e0}, \ \ U^e(x,t) = \varphi^e(x,t),\ (x,t) \in S^{e0}. \ (3.7)$$

Now take $U(x,t)$ to be the restriction of $U^e(x,t)$ to $\overline{G}$.

Next, choose the domains G^{e1} and G^{e2} as extensions of G beyond the sets S^L and S_0 respectively, where $\overline{G}^{\,e1}, \overline{G}^{\,e2} \subset \overline{G}^{\,e0}$. The sets S_0 and S^L are parts of the boundaries of the extended domains G^{e1} and G^{e2} respectively. Let $W^e(x,t)$ be the solution of the problem

$$\begin{aligned} L^e_{(2.2)}\, W^e(x,t) &= 0, && (x,t) \in G^{e1},\\ W^e(x,t) &= \varphi^e(x,t) - U^e(x,t), && (x,t) \in S^{e1}. \end{aligned} \qquad (3.8)$$

Then choose $W(x,t)$ to be the restriction to $\overline{G}$ of the function $W^e(x,t)$. The function $V(x,t)$ is the restriction to $\overline{G}$ of the function $V^e(x,t)$, $(x,t) \in \overline{G}^{\,e2}$, where $V^e(x,t)$ is the solution of the problem

$$\begin{aligned} L^e_{(2.2)}\, V^e(x,t) &= 0, && (x,t) \in G^{e2},\\ V^e(x,t) &= \varphi^e(x,t) - U^e(x,t), && (x,t) \in S^{e2}. \end{aligned} \qquad (3.9)$$

The function $Q(x,t)$ is the solution of the problem

$$\begin{aligned} L_{(2.2)}Q(x,t) &= 0, && (x,t)\in G,\\ Q(x,t) &= \varphi(x,t) - [U^e(x,t)+W^e(x,t)+V^e(x,t)], && (x,t)\in S. \end{aligned}$$

This completes the construction of the decomposition (3.6).

A further decomposition is needed: write the solution of (3.7) as

$$U^e(x,t) = U_0^e(x,t) + \varepsilon_1^2\, U_1^e(x,t) + v_U^e(x,t), \quad (x,t)\in \overline{G}^{\,e0}.$$

Here $U_0^e(x,t)$ and $U_1^e(x,t)$ are solutions of the problems

$$\begin{aligned} L^e_{(3.10)}\, U_0^e(x,t) &= f^e(x,t), && (x,t)\in \overline{G}^{\,e0}\setminus S_0^{e0},\\ U_0^e(x,t) &= \varphi^e(x,t), && (x,t)\in S_0^{e0},\\ L^e_{(3.10)}\, U_1^e(x,t) &= -a^e(x,t)\,\tfrac{\partial^2}{\partial x^2}\, U_0^e(x,t), && (x,t)\in \overline{G}^{\,e0}\setminus S_0^{e0},\\ U_1^e(x,t) &= 0, && (x,t)\in S_0^{e0}, \end{aligned} \tag{3.10}$$

and $L^e_{(3.10)}$ is an extension of the operator $L \equiv -\varepsilon_2^2\, p(x,t)\,\dfrac{\partial}{\partial t} - c(x,t)$.

The problems (3.7) and (3.10) can be differentiated with respect to t. Estimating the functions U_0^e, U_1^e and v_U^e in turn on $\overline{G}^{\,e0}$, one obtains the bound

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, U(x,t)\right| \le M\,[1+\varepsilon_1^{4-k}], \quad (x,t)\in\overline{G}, \quad k+2k_0\le K, \tag{3.11a}$$

where $K = l+2$, $l = l_{(3.5)}$.

Decompose the solutions of (3.8) and (3.9) as the sums

$$\begin{aligned} W^e(x,t) &= W_0^e(x,t) + v_W^e(x,t), && (x,t)\in\overline{G}^{\,e1},\\ V^e(x,t) &= V_0^e(x,t) \;+ v_V^e(x,t), && (x,t)\in\overline{G}^{\,e2}, \end{aligned} \tag{3.12}$$

where the functions $W_0^e(x,t)$, $(x,t)\in\overline{G}^{\,e1}$ and $V_0^e(x,t)$, $(x,t)\in\overline{G}^{\,e2}$ are solutions of the problems

$$\begin{aligned} L^e_{(3.10)}\, W_0^e(x,t) &= 0, && (x,t)\in\overline{G}^{\,e1}\setminus S^{e1L},\\ W_0^e(x,t) &= \varphi^e(x,t) - U^e(x,t), && (x,t)\in S_0^{e1L};\\ L^e_{(3.13)}\, V_0^e(x,t) &= 0, && (x,t)\in\overline{G}^{\,e2}\setminus S_0^{e2},\\ V_0^e(x,t) &= \varphi^e(x,t) - U^e(x,t), && (x,t)\in S_0^{e2}, \end{aligned} \tag{3.13}$$

and $L^e_{(3.13)}$ is an extension of the operator $L\equiv \varepsilon_1^2\, a(x,t)\,\dfrac{\partial^2}{\partial x^2} - c(x,t)$. Estimating the components in (3.12), we find the bounds

$$\begin{aligned} &\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, W(x,t)\right| \le M\,\varepsilon_2^{-2k_0}\,[1+\varepsilon_2^{4-2k_0}]\,\exp\big(-m_2\,\varepsilon_2^{-2}\,t\big),\\ &\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, V(x,t)\right| \le M\,\varepsilon_1^{-k_1}\,[1+\varepsilon_2^{4-2k_0}]\,\exp\big(-m_1\,\varepsilon_1^{-1}\,r(x,\mathit{\Gamma})\big),\\ &(x,t)\in\overline{G}, \quad k+2k_0\le K. \end{aligned} \tag{3.11b}$$

For the component $Q(x,t)$ we obtain the bound

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} Q(x,t)\right| \le M\,\varepsilon_1^{-k_1}\ \varepsilon_2^{-2k_0}\,[1+\varepsilon_2^{4-2k_0}] \\ \times \min\Big[\exp\big(-m_1\,\varepsilon_1^{-1}\,r(x,\varGamma)\big),\ \exp(-m_2\,\varepsilon_2^{-2}\,t)\Big], \\ (x,t)\in\overline{G},\quad k+2k_0\le K. \tag{3.11c}$$

In the bounds (3.11b) and (3.11c) we have $K = l+2$ where $l = l_{(3.5)}$ while m_1 and m_2 are arbitrary numbers satisfying the condition $m_i = m_{i(3.11)} < m^0_{i(3.11)}$, $i = 1,2$, where

$$m^0_{1(3.11)} = \big(1+M_{(3.11)}\big)^{-1}\,c_0^{1/2}\,(a^0)^{-1/2}, \\ m^0_{2(3.11)} = \big(1+M_{(3.11)}\big)^{-2}\,c_0\,(p^0)^{-1}, \tag{3.11d}$$

and $M_{(3.11)}$ is an arbitrary number satisfying the condition $M_{(3.11)} \ge \varepsilon_1+\varepsilon_2$, where ε_1, $\varepsilon_2 \in (0,1]$.

Theorem 2. *Assume that conditions* (3.5) *are fulfilled for the data of boundary value problem* (2.2), (2.1) *and its solution. Then functions $U(x,t)$, $W(x,t)$, $V(x,t)$ and $Q(x,t)$ of* (3.6) *satisfy the bounds* (3.11).

Remark 1. An examination of the main terms in the asymptotic representations of the singular components of the problem solution reveals that boundary layers appear when $\varepsilon_1 = o(1)$ while initial layers appear when $\varepsilon_2 = o(1)$. If both conditions $\varepsilon_1 = o(1)$ and $\varepsilon_2 = o(1)$ are fulfilled, then the solution also contains initial-boundary layers, where the initial-boundary layers are parabolic in nature, while the initial and boundary layers are regular. Under the condition that $\varepsilon_1 = o(1)$ and $\varepsilon_2 \approx 1$, only the parabolic boundary layer appears while under the condition $\varepsilon_1 \approx 1$ and $\varepsilon_2 = o(1)$, we have only the parabolic initial layer.

4 Grid Approximations of the Initial–Boundary Value Problem (2.2), (2.1)

4.1. We now construct a finite difference scheme that uses a classical approximation of the boundary value problem (2.2), (2.1) on rectangular grids. On the set $\overline{G}$ we introduce the grid

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega} \times \overline{\omega}_0, \tag{4.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are, in general, nonuniform meshes on the segments $[0,d]$ and $[0,T]$ respectively. For x^i, $x^{i+1} \in \overline{\omega}$ and t^j, $t^{j+1} \in \overline{\omega}_0$, set $h^i = x^{i+1}-x^i$, $h = \max_i h^i$, $h_t^j = t^{j+1}-t^j$ and $h_t = \max_j h_t^j$. Let $(N+1)$ and (N_0+1) be the number of nodes in the meshes $\overline{\omega}$ and $\overline{\omega}_0$. Assume that $h \le MN^{-1}$ and $h_t \le MN_0^{-1}$. For the problem (2.2), (2.1) we consider the difference scheme

$$\varLambda_{(4.2)}\, z(x,t) = f(x,t),\quad (x,t)\in G_h,\quad z(x,t) = \varphi(x,t),\quad (x,t)\in S_h. \tag{4.2}$$

Here $G_h = G \bigcap \overline{G}_h$, $S_h = S \bigcap \overline{G}_h$,

$$\Lambda_{(4.2)} \equiv \varepsilon_1^2 \, a(x,t) \, \delta_{\overline{x}\widehat{x}} - \varepsilon_2^2 \, p(x,t) \, \delta_{\overline{t}} - c(x,t), \quad (x,t) \in G_h,$$

$\delta_{\overline{x}\widehat{x}} \, z(x,t)$ and $\delta_{\overline{t}} \, z(x,t)$ are the second-order and the first-order (backward) difference derivatives:

$$\begin{aligned} \delta_{\overline{x}\widehat{x}} \, z(x,t) &= 2(h^i + h^{i-1})^{-1} [\delta_x \, z(x,t) - \delta_{\overline{x}} \, z(x,t)], \\ \delta_x \, z(x,t) &= \left(h^i\right)^{-1} \left[z(x^{i+1},t) - z(x,t)\right], \\ \delta_{\overline{x}} \, z(x,t) &= \left(h^{i-1}\right)^{-1} \left[z(x,t) - z(x^{i-1},t)\right], \quad x = x^i. \end{aligned}$$

The difference operator $\Lambda_{(4.2)}$ is $\overline{\varepsilon}$-uniformly monotone [11].

By using comparison theorems, one can verify that the solution of the problem (4.2), (4.1) is $\overline{\varepsilon}$-uniformly bounded:

$$|\, z(x,t) \,| \le M, \quad (x,t) \in \overline{G}_h.$$

Taking into account the estimates of Theorem 1 for $K = 4$, one can show that

$$|u(x,t) - z(x,t)| \le M \Big[\frac{N^{-1}}{\varepsilon_1 + N^{-1}} + \frac{N_0^{-1}}{\varepsilon_2^2 + N_0^{-1}} \Big], \quad (x,t) \in \overline{G}_h. \tag{4.3}$$

Thus the scheme (4.2), (4.1) converges under the condition

$$N^{-1} = o(\varepsilon_1), \quad N_0^{-1} = o(\varepsilon_2^2) \quad \text{as } N,\, N_0 \to \infty. \tag{4.4}$$

If the mesh

$$\overline{G}_h \tag{4.5}$$

is equidistant, then one can prove the sharper estimate

$$|u(x,t) - z(x,t)| \le M \Big[\frac{N^{-2}}{(\varepsilon_1 + N^{-1})^2} + \frac{N_0^{-1}}{\varepsilon_2^2 + N_0^{-1}} \Big], \quad (x,t) \in \overline{G}_h. \tag{4.6}$$

It follows that the finite difference scheme (4.2), (4.1) converges under the condition (4.4).

Theorem 3. *Let the data of the boundary value problem* (2.2), (2.1) *satisfy the conditions* (2.3) *and assume for the problem solution that the estimates of Theorem 1 are fulfilled for $K = 4$. Then under the condition* (4.4)*, the solutions of the finite difference scheme* (4.2) *on the meshes* (4.1) *and* (4.5) *converge to the solution of the boundary value problem with the bounds* (4.3) *and* (4.6) *respectively.*

4.2. We now construct a grid that condenses in the boundary and initial layer regions and on which the solution of the finite difference scheme converges $\overline{\varepsilon}$-uniformly. Set

$$\overline{G}_h = \overline{D}_h^s \times \overline{\omega}_0^s = \overline{\omega}^s \times \overline{\omega}_0^s, \tag{4.7a}$$

where $\overline{\omega}^s = \overline{\omega}^s(\sigma_1)$ and $\overline{\omega}_0^s = \overline{\omega}_0^s(\sigma_2)$ are piecewise-equidistant meshes on $[0,d]$ and $[0,T]$ respectively; here σ_1 and σ_2 are parameters depending on N, N_0 and $\overline{\varepsilon}$. The mesh sizes in $\overline{\omega}^s$ (see, e.g., [15, 14, 13]) are $h^{(1)} = 4\sigma_1 N^{-1}$ on the intervals $[0,\sigma_1]$ and $[d-\sigma_1, d]$, and $h^{(2)} = 2(d-2\sigma_1)N^{-1}$ on $[\sigma_1, d-\sigma_1]$. The mesh sizes in $\overline{\omega}_0^s$ are $h_0^{(1)} = 2\sigma_2 N_0^{-1}$ on the interval $[0,\sigma_2]$ and $h_0^{(2)} = 2(T-\sigma_2)N_0^{-1}$ on $[\sigma_2, T]$. The values σ_1 and σ_2 are specified by

$$\begin{aligned} \sigma_1 &= \sigma_{1(4.7)}(\overline{\varepsilon}, N_1) = \min\left\{ 4^{-1}\, d,\ M_1\, \varepsilon_1 \ln N_1 \right\}, \\ \sigma_2 &= \sigma_{2(4.7)}(\overline{\varepsilon}, N_0) = \min\left\{ 2^{-1}\, T,\ M_2\, \varepsilon_2^2 \ln N_0 \right\}, \end{aligned} \tag{4.7b}$$

where M_1, M_2 are arbitrary constants. Thus the grid $\overline{G}_{h(4.7)}$ is defined by the parameters N, N_0, $\overline{\varepsilon}$ and by the constants M_1 and M_2, i.e., $\overline{G}_{h(4.7)} = \overline{G}_{h(4.7)}(N, N_0, \overline{\varepsilon};\, M_1, M_2) = \overline{G}_{h(4.7)}(M_1, M_2)$.

From the estimates of Theorem 2 (for $K = 4$) one can deduce $\overline{\varepsilon}$-uniform convergence of the solution of the finite difference scheme (4.2), (4.7) to the solution of the boundary value problem (2.2), (2.1). The convergence rate of the solution of the finite difference scheme is estimated using a technique from [13, 14, 15]. When

$$M_1 > 6\,(m^0_{1(3.11)})^{-1} \text{ and } M_2 > 9\,(m^0_{2(3.11)})^{-1}, \tag{4.8}$$

we obtain the $\overline{\varepsilon}$-uniform convergence estimate

$$|u(x,t) - z(x,t)| \le M\,[N^{-2} \ln^2 N + N_0^{-1} \ln N_0], \quad (x,t) \in \overline{G}_h, \tag{4.9a}$$

for the solution of the finite difference scheme. Also, one has the $\overline{\varepsilon}$-dependent estimate

$$|u(x,t) - z(x,t)| \le M\Big[\frac{N^{-2}}{(\varepsilon_1 + \ln^{-1} N)^2} + \frac{N_0^{-1}}{\varepsilon_2^2 + \ln^{-1} N_0}\Big], \ (x,t) \in \overline{G}_h. \tag{4.9b}$$

So for any fixed value of the parameter $\overline{\varepsilon}$, these schemes converge at the rate $\mathcal{O}\big(N^{-2} + N_0^{-1}\big)$. Thus, we have

Theorem 4. *For the components of the solution of the boundary value problem* (2.2), (2.1) *in the representation* (3.6)*, assume that the estimates of Theorem 2 hold for $K = 4$. Then the solution of the finite difference scheme* (4.2), (4.7) *is $\overline{\varepsilon}$-uniformly convergent to the solution of* (2.2), (2.1)*. If in addition the condition* (4.8) *is satisfied then the estimates* (4.9) *are valid.*

Remark 2. In the case when for an approximation of the boundary value problem (2.2), (2.1) the scheme (4.2) is used on a mesh that condenses only in x or only in t, the scheme converges only under the condition $N_0^{-1} = o(\varepsilon_2^2)$ or $N^{-1} = o(\varepsilon_1)$ respectively. The scheme (4.2) on such meshes is not $\overline{\varepsilon}$-uniformly convergent.

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