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SPLINES IN CONVEX SETS UNDER CONSTRAINTS OF TWO-SIDED INEQUALITY TYPE IN A HYPERPLANE

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Abstract. The problem of minimization of a smoothing functional under inequality constraints is considered in a hyperplane. The conditions of the existence of a solution are obtained and some properties of this solution are investigated. It is proved that the solution is a spline. The method for its construction is suggested.

Key words: smoothing problem, spline, simplex method.

1 Smoothing Histosplines

Let a mesh $\triangle_n : a = t_0 < t_1 < \ldots < t_n = b$ be given for the interval $[a, b]$, and let $F = \{f_1, \ldots, f_n\}$ be a corresponding histogram, i.e. f_i is the frequency for the interval $[t_{i-1}, t_i]$, where $i = 1, \ldots, n$. The mesh sizes are denoted by $h_i = t_i - t_{i-1}$, $i = 1, \ldots, n$.

In many practical applications it is of interest to have a function g that satisfies the area matching histopolation conditions

$$\int_{t_{i-1}}^{t_i} g(t)dt = f_i h_i,\ i = 1, \ldots, n.$$

We will take into account that the information on frequencies $f_i,\ i = 1, \ldots, n$, is obtained in practice as a result of measuring, experiment or preliminary calculations and it may be inexact. Hence for given numbers $\varepsilon_i \geq 0,\ i = 1, \ldots, n,$ we consider more general histopolation conditions

$$\Big| \int_{t_{i-1}}^{t_i} g(t)dt - f_i h_i \Big| \leq \varepsilon_i, \quad i = 1, \ldots, n. \tag{1.1}$$

Let us formulate the following problem.

Problem 1

$$\int_a^b \left(g^{(q)}(t)\right)^2 dt \longrightarrow \min_{g\in D_1(\varepsilon)},$$

$$D_1(\varepsilon) = \left\{g \in W_2^q[a,b] : \Big| \int_{t_{i-1}}^{t_i} g(t)dt - f_i h_i \Big| \le \varepsilon_i, \quad i = 1,\ldots,n \right\},$$

where $W_2^q[a,b]$ is the Sobolev space. In the case of exact information (i.e. $\varepsilon_i = 0$ for all i) we have a histopolation problem the solution of which is a spline s (called *a histospline*) from the space $S(\triangle_n)$ of integral splines of degree $2q$ and defect 1 over the mesh $\triangle_n$ (see, e.g. [5]):

$$S_{2q,1}(\triangle_n) = \Big\{ s \in W_2^q[a,b] : \int_{t_{i-1}}^{t_i} g(t)\ dt = 0, \quad i = 1,\ldots,n,$$

$$\Longrightarrow \int_a^b g^{(q)}(t)s^{(q)}(t)\ dt = 0 \quad \text{for all } g \in W_2^q[a,b] \Big\}.$$

In the case of inexact information (i.e. $\varepsilon_i > 0$ for some i) it is a problem of smoothing histopolation. If $n \le q$, then any polynomial of degree $q-1$, which satisfies the condition of histopolation (1.1), gives the solution of Problem 1. If $n > q$ and no algebraic polynomial of degree $q-1$ satisfies the inequalities (1.1), then Problem 1 has a unique solution (e.g. [5]). This solution is a spline from the space $S_{2q,1}(\triangle_n)$, which minimizes the smoothing functional under restrictions. This spline is called *a smoothing histospline*. Such problem is investigated in [2].

The main purpose of the present paper is to consider Problem 1 with one additional restriction. We formulate the following problem.

Problem 2

$$\int_a^b \left(g^{(q)}(t)\right)^2 dt \longrightarrow \min_{g\in D_2(\varepsilon)},$$

$$D_2(\varepsilon) = \left\{g \in \mathbf{W}_2^q[a,b] : \int_a^b g(t)\,dt = 1, \Big| \int_{t_{i-1}}^{t_i} g(t)\,dt - f_i h_i \Big| \le \varepsilon_i,\ i = 1,\ldots,n \right\}.$$

The condition $\int_a^b g(t)dt = 1$ appears under approximation of a given normalized histogram F with frequencies f_i, $i = 1,\ldots,n$. We investigate this problem in a more general case in a Hilbert space (see Problem 3) and obtain the existence and the characteristics of its solution. We reduce Problem 3 to the problem of "almost" linear programming problem with some nonlinear conditions (see Problem 5) and propose the method for finding its solution by one modification of the simplex algorithm.

2 The Generalization of the Problem of Smoothing Histopolation

Let X, Y be Hilbert spaces and assume that a linear operator $T : X \to Y$ and linear functionals $k_i : X \to I\!R$, $i = 1, \dots, n$, are continuous. For given vectors $\boldsymbol{r} = (r_1, \dots, r_n)$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_i \geq 0,\ i = 1, \dots, n$, we consider the following conditional minimization problem.

Problem 3

$$\| Tx \|_Y \longrightarrow \min_{x \in D_3(\varepsilon)},$$

$$D_3(\varepsilon) = \Big\{ x : \big| k_i x - r_i \big| \leq \varepsilon_i,\ i = 1, \dots, n,\ \sum_{i=1}^{n} k_i x = \sum_{i=1}^{n} r_i \Big\}.$$

In the case $\varepsilon_i = 0,\ i = 1,\ \dots\ , n,$ a solution of this problem is called an interpolating spline for a vector $\boldsymbol{r}$ and it belongs to the space

$$S(T, A) = \Big\{ s \in X : \ < Ts, Tx >= 0 \ \text{ for all } \ x \in \operatorname{Ker} A \Big\},$$

corresponding to the operators T and $A = (k_1, \dots, k_n)$. In the case of inexact information ($\varepsilon_i > 0$) Problem 3 without the last condition defines splines in a convex set (in the special case *smoothing splines*) [5, 6]. Such splines belong to the space $S(T, A)$, also.

Let us suppose that $\operatorname{Im} A = I\!R^n,\ \operatorname{Im} T = Y$ and the sum $\operatorname{Ker} T + \operatorname{Ker} A$ is closed. Under these assumptions for each vector $\boldsymbol{r} \in I\!R^n$ there exists an interpolating spline $s \in S(T, A)$. Let us denote

$$Z_{\boldsymbol{r}} = \Big\{ \boldsymbol{z} \in I\!R^n : \sum_{i=1}^{n} z_i = \sum_{i=1}^{n} r_i \Big\}, \quad X_{\boldsymbol{r}} = \Big\{ x \in X : Ax \in Z_{\boldsymbol{r}} \Big\},$$

$$P_{\boldsymbol{r},\boldsymbol{\varepsilon}} = \prod_{i=1}^{n} \Big[r_i - \varepsilon_i, r_i + \varepsilon_i \Big], \quad C_{\boldsymbol{r},\boldsymbol{\varepsilon}} = \Big\{ x \in X_{\boldsymbol{r}} : Ax \in P_{\boldsymbol{r},\boldsymbol{\varepsilon}} \Big\}.$$

We rewrite Problem 3 in the form

$$||Tx||_Y \longrightarrow \min_{x \in C_{r,\varepsilon}}$$

and state the following results.

Theorem 1. *A solution of Problem 3 exists. An element $\sigma \in C_{\boldsymbol{r},\boldsymbol{\varepsilon}}$ is a solution of this problem if and only if there exists an element $\boldsymbol{\lambda} \in I\!R^n$ such that*

$$T^*T(\sigma) = A^*\boldsymbol{\lambda} \ \ \textit{and} \ \ < \boldsymbol{\lambda}, \boldsymbol{\omega} - A\sigma > \geq 0 \quad \textit{for all} \ \ \boldsymbol{\omega} \in P_{\boldsymbol{r},\boldsymbol{\varepsilon}} \cap Z_{\boldsymbol{r}}.$$

Proof. This theorem is a particular case of Theorem 2 from [6] (p. 10). To prove it we check that under our assumptions the sum $C_{\boldsymbol{r},\boldsymbol{\varepsilon}} + \mathrm{Ker} T$ is closed, thus in this case all conditions of Theorem 2 are true.

Taking into account that $T^{-1}(T(C_{\boldsymbol{r},\boldsymbol{\varepsilon}})) = C_{\boldsymbol{r},\boldsymbol{\varepsilon}} + \mathrm{Ker}\, T$ it is sufficient to prove that the set $T(C_{\boldsymbol{r},\boldsymbol{\varepsilon}})$ is closed. We rewrite $C_{\boldsymbol{r},\boldsymbol{\varepsilon}}$ as the sum

$$C_{\boldsymbol{r},\boldsymbol{\varepsilon}} = S_{\boldsymbol{r},\boldsymbol{\varepsilon}} + \mathrm{Ker}\, A.$$

Here $S_{\boldsymbol{r},\boldsymbol{\varepsilon}}$ is the class of splines from the space $S(T, A)$ described by taking splines $s_1, \ldots, s_n \in S(T, A)$, which satisfy the conditions

$$k_j s_i = \delta_{ij}, \quad j = 1, \ldots, n, \quad i = 1, \ldots, n,$$

where δ_{ij} is the Kronecker symbol. Under these notations

$$S_{\boldsymbol{r},\boldsymbol{\varepsilon}} = \Big\{ \sum_{i=1}^{n} w_i s_i \in S(T, A) \colon \boldsymbol{w} \in P_{\boldsymbol{r},\boldsymbol{\varepsilon}} \cap Z_{\boldsymbol{r}} \Big\}.$$

Here the spline $\sum\limits_{i=1}^{n} w_i s_i$ is interpolating for $\boldsymbol{w} = (w_1, \ldots, w_n) \in P_{\boldsymbol{r},\boldsymbol{\varepsilon}} \cap Z_{\boldsymbol{r}}$. Then

$$T(C_{\boldsymbol{r},\boldsymbol{\varepsilon}}) = T(S_{\boldsymbol{r},\boldsymbol{\varepsilon}} + \mathrm{Ker}\, A) = T(S_{\boldsymbol{r},\boldsymbol{\varepsilon}}) + T(\mathrm{Ker}\, A) \text{ and } T(S_{\boldsymbol{r},\boldsymbol{\varepsilon}}) \perp T(\mathrm{Ker}\, A).$$

It is known that $T(\mathrm{Ker}\, A)$ is closed if and only if $\mathrm{Ker}\, A + \mathrm{Ker}\, T$ is closed. Taking into account also that the set

$$T(S_{\boldsymbol{r},\boldsymbol{\varepsilon}}) = \Big\{ \sum_{i=1}^{n} w_i T(s_i) \colon \boldsymbol{w} \in P_{\boldsymbol{r},\boldsymbol{\varepsilon}} \cap Z_{\boldsymbol{r}} \Big\}$$

is closed, we prove that $T(C_{\boldsymbol{r},\boldsymbol{\varepsilon}})$ is closed, so the sum $C_{\boldsymbol{r},\boldsymbol{\varepsilon}} + \mathrm{Ker}\, T$ is closed too. $\square$

Corollary 1. A solution of Problem 3 belongs to the space of splines $S(T, A)$.

In the remaining part of the paper, we suppose that $\varepsilon_i > 0$, $i = 1, \ldots, n$.

Theorem 2. *An element $\sigma \in C_{\boldsymbol{r},\boldsymbol{\varepsilon}}$ is a solution of Problem 3 if and only if there exist elements $\boldsymbol{\lambda} \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ such that*

$$\begin{aligned} & T^*T\sigma = A^*\boldsymbol{\lambda}, \\ & \lambda_i = \gamma \;\; \text{if} \;\; |k_i\sigma - r_i| < \varepsilon_i, \\ & \lambda_i \geq \gamma \;\; \text{if} \;\; k_i\sigma - r_i = -\varepsilon_i, \\ & \lambda_i \leq \gamma \;\; \text{if} \;\; k_i\sigma - r_i = \varepsilon_i, \;\; \text{for} \;\; i = 1, \ldots, n. \end{aligned} \tag{2.1}$$

Proof. Under the assumption that there exist elements $\boldsymbol{\lambda} \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ such that $T^*T\sigma = A^*\boldsymbol{\lambda}$ and the conditions (2.1) are true we prove that

$$< \boldsymbol{\lambda}, \boldsymbol{\omega} - A\sigma > \geq 0 \;\text{ for all }\; \boldsymbol{\omega} \in P_{\boldsymbol{r},\boldsymbol{\varepsilon}} \cap Z_{\boldsymbol{r}}.$$

Taking into account the conditions on λ_i for $i \in I_-, I_+, I_0$, where

$$I_+ = \Big\{ i : \ k_i\sigma = r_i + \varepsilon_i \Big\}, \ \ I_- = \Big\{ i : \ k_i\sigma = r_i - \varepsilon_i \Big\}, \ \ I_0 = \Big\{ i : \ |k_i\sigma - r_i| < \varepsilon_i \Big\},$$

we obtain

$$\sum_{i=1}^{n} \lambda_i(\omega_i - k_i\sigma) = \sum_{I\in I_+} \lambda_i(\omega_i - k_i\sigma) + \sum_{i\in I_-} \lambda_i(\omega_i - k_i\sigma) + \sum_{i\in I_0} \lambda_i(\omega_i - k_i\sigma)$$

$$\geq \sum_{I\in I_+} \gamma(\omega_i - k_i\sigma) + \sum_{i\in I_-} \gamma(\omega_i - k_i\sigma) + \sum_{i\in I_0} \gamma(\omega_i - k_i\sigma) = \gamma \sum_{i=1}^{n} (\omega_i - k_i\sigma) = 0.$$

Therefore we get that

$$\sum_{i=1}^{n} \lambda_i(\omega_i - k_i\sigma) \geq 0, \quad \forall \boldsymbol{\omega} \in P_{\boldsymbol{r},\boldsymbol{\varepsilon}} \cap Z_{\boldsymbol{r}}$$

and by Theorem 1 σ is a solution of Problem 3. Now if σ is a solution of Problem 3, then according to Theorem 1 there exists element $\boldsymbol{\lambda} \in I\!R^n$ such, that

$$T^*T\sigma = A^*\boldsymbol{\lambda} \text{ and } <\boldsymbol{\lambda}, \boldsymbol{\omega} - A\sigma> \geq 0 \text{ for all } \boldsymbol{\omega} \in P_{\boldsymbol{r},\boldsymbol{\varepsilon}} \cap Z_{\boldsymbol{r}}.$$

Let us fix an index $i_0 \in I_+$ and take $i_1 \in I_- \cup I_0$. By choosing $\omega_i = k_i\sigma$ for $i \neq i_0$, $i \neq i_1$, $\omega_{i_0} = k_{i_0}\sigma - \delta$, $\omega_{i_1} = k_{i_1}\sigma + \delta$ for some $0 < \delta < \min\{\varepsilon_{i_0}, \varepsilon_{i_1}\}$ we can show that

$$<\boldsymbol{\lambda}, \boldsymbol{\omega} - A\sigma> = \sum_{i=1}^{n} \lambda_i(\omega_i - k_i\sigma) = -\lambda_{i_0}\delta + \lambda_{i_1}\delta \geq 0,$$

therefore $\lambda_{i_1} \geq \lambda_{i_0}$ for all $i_1 \in I_- \cup I_0$ and $i_0 \in I_+$, i.e. for every index $i \in I_+$ it holds $\lambda_i \leq \min\{\lambda_j : j \in I_- \cup I_0\}$.

Now we can fix $i_0 \in I_-$ and take $i_1 \in I_+ \cup I_0$. By choosing $\omega_i = k_i\sigma$ for $i \neq i_0$, $i \neq i_1$, $\omega_{i_0} = k_{i_0}\sigma + \delta$, $\omega_{i_1} = k_{i_1}\sigma - \delta$ for some $0 < \delta < \min\{\varepsilon_{i_0}, \varepsilon_{i_1}\}$ we show that

$$<\boldsymbol{\lambda}, \boldsymbol{\omega} - A\sigma> = \sum_{i=1}^{n} \lambda_i(\omega_i - k_i\sigma) = \lambda_{i_0}\delta - \lambda_{i_1}\delta \geq 0,$$

therefore $\lambda_{i_1} \leq \lambda_{i_0}$ for all $i_1 \in I_+ \cup I_0$ and $i_0 \in I_-$, i.e. for every index $i \in I_-$ it holds $\lambda_i \geq \max\{\lambda_j : j \in I_+ \cup I_0\}$.

If we fix an index $i_0 \in I_0$ and take $i_1 \in I_0$, then by choosing $\omega_i = k_i\sigma$ for $i \neq i_0$, $i \neq i_1$, $\omega_{i_0} = k_{i_0}\sigma - \delta$, $\omega_{i_1} = k_{i_1}\sigma + \delta$ for some $0 < \delta < \min\{\varepsilon_{i_0}, \varepsilon_{i_1}\}$ we can show that

$$<\boldsymbol{\lambda}, \boldsymbol{\omega} - A\sigma> = \sum_{i=1}^{n} \lambda_i(\omega_i - k_i\sigma) = -\lambda_{i_0}\delta + \lambda_{i_1}\delta \geq 0,$$

therefore $\lambda_{i_0} \leq \lambda_{i_1}$. But by choosing $\omega_i = k_i\sigma$ for $i \neq i_0$, $i \neq i_1$, $\omega_{i_0} = k_{i_0}\sigma + \delta$, $\omega_{i_1} = k_{i_1}\sigma - \delta$, $\delta > 0$, we can show that

$$< \boldsymbol{\lambda}, \boldsymbol{\omega} - A\sigma >= \sum_{i=1}^{n} \lambda_i(\omega_i - k_i\sigma) = \lambda_{i_0}\delta - \lambda_{i_1}\delta \geq 0,$$

so $\lambda_{i_1} \leq \lambda_{i_0}$. Therefore for all i_1 and i_0 from I_0 we have $\lambda_{i_1} = \lambda_{i_0}$. Let us denote $\gamma = \lambda_i$, $i \in I_0$. We have proved that

$$\begin{aligned}
&\lambda_i = \gamma \quad \text{if } \ i \in I_0, \ \text{ i.e. } \ |k_i\sigma - r_i| < \varepsilon_i, \\
&\lambda_i \geq \gamma \quad \text{if } \ i \in I_-, \ \text{ i.e. } \ k_i\sigma - r_i = -\varepsilon_i, \\
&\lambda_i \leq \gamma \quad \text{if } \ i \in I_+, \ \text{ i.e. } \ k_i\sigma - r_i = \varepsilon_i, \ \text{ for } \ i = 1, \ldots, n.
\end{aligned}$$

□

3 The Equivalent Problem of Quadratic Programming

Taking into account that the solution of Problem 3 is a spline, we can restrict the class of functions X by the space $S(T, A)$ and rewrite the smoothing functional $\|T\sigma\|_Y$ as a function of n new non-negative variables

$$z_i = k_i\sigma - r_i + \varepsilon_i, \qquad i = 1, \ldots, n. \tag{3.1}$$

If we denote by $s_i \in S(T, A)$ the spline which satisfies the conditions $k_j s_i = \delta_{ij}$, $j = 1, \ldots, n$, $i = 1, \ldots, n$, where δ_{ij} is the Kronecker symbol, then $s_1, \ldots, s_n$ is a basis of the space $S(T, A)$. Let us express the spline σ (a solution of Problem 3) with respect to $\boldsymbol{z}$:

$$\sigma = \sum_{i=1}^{n} (r_i - \varepsilon_i + z_i)s_i.$$

Taking into account that σ is a solution of Problem 3 by Theorem 2 there exists an element $\boldsymbol{\lambda}(\sigma) \in I\!R^n$ such that $T^*T\sigma = A^*\boldsymbol{\lambda}(\sigma)$. Using the following two equalities

$$\begin{aligned}
&< T^*T\sigma, s_j >=< A^*\boldsymbol{\lambda}(\sigma), s_j >=< \boldsymbol{\lambda}(\sigma), As_j >= \lambda_j(\sigma), \\
&< T^*T\sigma, s_j >=< \sigma, T^*Ts_j >=< \sigma, A^*\boldsymbol{\lambda}(s_j) > \\
&\qquad\qquad\qquad\qquad =< A\sigma, \boldsymbol{\lambda}(s_j) >= \sum_{i=1}^{n} k_i\sigma\lambda_i(s_j)
\end{aligned}$$

we prove that

$$\lambda_j(\sigma) = \sum_{i=1}^{n} (r_i - \varepsilon_i + z_i)\lambda_{ji}, \quad j = 1, \ldots, n, \tag{3.2}$$

where $(\lambda_{ij})_{j=1,\ldots,n}$ are the coefficients of the basis spline s_i, i.e. $\lambda_{ij} = \lambda_j(s_i)$.

By introducing the matrix $\mathbf{D} = (\lambda_{ji})_{i,j=1,\dots,n}$ and vectors $\boldsymbol{z} = (z_i)_{i=1,\dots,n}$, and $\boldsymbol{c} = (c_i)_{i=1,\dots,n}$, where

$$c_i = \sum_{j=1}^{n}(r_j - \varepsilon_j)(\lambda_{ji} + \lambda_{ij}), \quad h = \sum_{i=1}^{n}\sum_{j=1}^{n}(r_j - \varepsilon_j)(r_i - \varepsilon_i)\lambda_{ji}$$

we rewrite

$$\| T\sigma \|^2 =< T\sigma, T\sigma >=< T^*T\sigma, \sigma >=< \boldsymbol{\lambda}(\sigma), A\sigma >= \sum_{j=1}^{n}\lambda_j(\sigma)k_j\sigma$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}(r_j - \varepsilon_j + z_j)(r_i - \varepsilon_i + z_i)\lambda_{ji} = \sum_{i=1}^{n} z_i \sum_{j=1}^{n}(r_j - \varepsilon_j)(\lambda_{ji} + \lambda_{ij})$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{n} z_i z_j \lambda_{ji} + \sum_{i=1}^{n}\sum_{j=1}^{n}(r_j - \varepsilon_j)(r_i - \varepsilon_i)\lambda_{ji} = \boldsymbol{z}\mathbf{D}\boldsymbol{z}^T + \boldsymbol{c}\boldsymbol{z}^T + h$$

and reduce Problem 3 to the matrix form.

Problem 4

$$\boldsymbol{z}\mathbf{D}\boldsymbol{z}^T + \boldsymbol{c}\boldsymbol{z}^T \longrightarrow \min_{\boldsymbol{z}\geq\mathbf{0}, \quad \boldsymbol{z}\leq 2\boldsymbol{\varepsilon}, \quad (\boldsymbol{z}-\boldsymbol{\varepsilon})\boldsymbol{e}^T=0,}$$

where $\boldsymbol{e}$ is the vector with n unit components.

Lemma 1. *The matrix* $\mathbf{D}$ *is symmetric and positive semidefinite.*

Proof. Using Theorem 1 and doing simple transformations of expressions for $\lambda_{ij} = \lambda_j(s_i)$ and $\lambda_{ji} = \lambda_i(s_j)$ we prove that

$$\lambda_{ji} = \lambda_i(s_j)k_i(s_i) = \sum_{l=1}^{n}\lambda_l(s_j)k_l(s_i) =< Ts_j, Ts_i >,$$

and, similarly, that $\lambda_{ij} =< Ts_i, Ts_j >$, thus we prove the equality $\lambda_{ji} = \lambda_{ij}$.

The inequality $\boldsymbol{z}\mathbf{D}\boldsymbol{z}^T \geq 0$ for any vector $\boldsymbol{z} \in \mathbb{R}^n$ is proved by using the identity $\boldsymbol{z}\mathbf{D}\boldsymbol{z}^T = \|Ts(\boldsymbol{z})\|^2$, where $s(\boldsymbol{z})$ is a spline interpolating for $\boldsymbol{z}$. This identity is obtained by direct calculations

$$\boldsymbol{z}\mathbf{D}\boldsymbol{z}^T = \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ji}z_i z_j = \sum_{j=1}^{n} z_j \sum_{i=1}^{n} z_i \lambda_j(s_i) = \sum_{j=1}^{n} z_j \lambda_j(s(\boldsymbol{z}))$$
$$= \sum_{j=1}^{n} k_j(s(\boldsymbol{z}))\lambda_j(s(\boldsymbol{z})) =< Ts(\boldsymbol{z}), Ts(\boldsymbol{z}) >= ||Ts(\boldsymbol{z})||^2.$$

□

Thus Problem 3 is reduced to Problem 4 of quadratic programming with symmetric and positive semidefinite matrix under linear restrictions.

4 The Equivalent Problem of "Almost" Linear Programming under Some Nonlinear Conditions

We use Wolfe's method (e.g. [4]) to reduce Problem 4 to the problem of "almost" linear programming with some nonlinear conditions. The reasoning in this reduction is similar to that of [1, 2] and we consider only the most important steps. We start with the Lagrange function

$$F(\boldsymbol{z}, \lambda^0, \boldsymbol{\lambda}^1) = \boldsymbol{z}\mathbf{D}\boldsymbol{z}^T + \boldsymbol{c}\boldsymbol{z}^T + \boldsymbol{\lambda}^1(\boldsymbol{z} - 2\boldsymbol{\varepsilon})^T + \lambda^0(\boldsymbol{z} - \boldsymbol{\varepsilon})\boldsymbol{e}^T,$$

where $\lambda^0 \in I\!R$, $\boldsymbol{\lambda}^1 = (\lambda_i^1)_{i=1,\dots,n} \in I\!R^n$ are the Lagrange multipliers. Taking into account the necessary and sufficient conditions for $\boldsymbol{z}$ to be a solution of Problem 4 (see, e.g. [4])

$$\begin{aligned}
&\nabla_{\boldsymbol{z}} F(\boldsymbol{z}, \lambda^0, \boldsymbol{\lambda}^1) \ge 0, \quad \nabla_{\boldsymbol{z}} F(\boldsymbol{z}, \lambda^0, \boldsymbol{\lambda}^1)\boldsymbol{z}^T = 0, \quad \boldsymbol{z} \ge \mathbf{0};\\
&\nabla_{\boldsymbol{\lambda}^1} F(\boldsymbol{z}, \lambda^0, \boldsymbol{\lambda}^1) \le 0, \quad \nabla_{\boldsymbol{\lambda}^1} F(\boldsymbol{z}, \lambda^0, \boldsymbol{\lambda}^1)\boldsymbol{\lambda}^{1T} = 0, \quad \boldsymbol{\lambda}^1 \ge \mathbf{0};\\
&\nabla_{\lambda^0} F(\boldsymbol{z}, \lambda^0, \boldsymbol{\lambda}^1) = 0,
\end{aligned}$$

and by introducing slack non-negative variables

$$\boldsymbol{\mu}^T = 2(\mathbf{D}\boldsymbol{z}^T) + \boldsymbol{c}^T + (\boldsymbol{\lambda}^1)^T + \lambda^0\boldsymbol{e}^T \quad \text{and} \quad \bar{\boldsymbol{z}} = 2\boldsymbol{\varepsilon} - \boldsymbol{z},$$

we can rewrite Problem 4 as a linear programming minimization problem of $\boldsymbol{u}\boldsymbol{e}^T$ for an auxiliary non-negative vector $\boldsymbol{u} \in I\!R^n$ under some nonlinear restrictions.

Problem 5

$$\begin{cases}
\boldsymbol{u}\boldsymbol{e}^T \longrightarrow \min\\
2\mathbf{D}\boldsymbol{z}^T + \boldsymbol{c}^T + (\boldsymbol{\lambda}^1)^T + \lambda^0\boldsymbol{e}^T - \boldsymbol{\mu}^T + \mathbf{E}\boldsymbol{u}^T = 0,\\
\boldsymbol{z} + \bar{\boldsymbol{z}} = 2\boldsymbol{\varepsilon}, \ (\boldsymbol{z} - \boldsymbol{\varepsilon})\boldsymbol{e}^T = 0,\\
\boldsymbol{\mu}\boldsymbol{z}^T = 0, \ \boldsymbol{\lambda}^1\bar{\boldsymbol{z}}^T = 0,\\
\boldsymbol{z} \ge \mathbf{0}, \ \bar{\boldsymbol{z}} \ge \mathbf{0}, \ \boldsymbol{\lambda}^1 \ge \mathbf{0}, \boldsymbol{\mu} \ge \mathbf{0}, \ \boldsymbol{u} \ge \mathbf{0},
\end{cases}$$

where $\mathbf{E}$ is the diagonal matrix with components $0, 1$ and -1. The existence of a non-negative solution of Problem 3 implies that zero is the solution of Problem 5.

Theorem 3. *Let Problem 3 has the unique solution. Then Problem 3 is equivalent to Problem 5 in the following sense:*

- *Problem 5 has the unique solution too;*
- *The solution of Problem 3 determines the solution of Problem 5 and the solution of Problem 5 determines the solution of Problem 3 by* (3.1).

Proof. By formulating Problems 3-5 in a natural order we see that the solution of Problem 3 determines the solution of Problem 5 by (3.1) and the system of restrictions of Problem 5 with $\boldsymbol{u} = \mathbf{0}$. Thus we have established a connection between the solutions of Problem 5 and Problem 3.

Let us consider the solution of Problem 5. Under the assumption that Problem 3 is solvable for this solution we have $\boldsymbol{u} = \mathbf{0}$. We denote by σ an interpolating spline for the vector $\boldsymbol{z} + \boldsymbol{r} - \boldsymbol{\varepsilon}$ (see (3.1)). It is easy to show, that $\sigma \in C_{\boldsymbol{r},\boldsymbol{\varepsilon}}$. To prove that this spline gives a solution of Problem 3 we check the necessary and sufficient conditions (2.1) from Theorem 2 for σ to be a solution of Problem 3.

According to (3.2) and Lemma 1 for the vector $\boldsymbol{\lambda}$ of coefficients of spline σ we have $\boldsymbol{\lambda} = \mathbf{D}\boldsymbol{z}^T + \frac{1}{2}\boldsymbol{c}^T$. From restrictions of Problem 5 with $\boldsymbol{u} = \mathbf{0}$ we obtain

$$2\boldsymbol{\lambda} = 2\mathbf{D}\boldsymbol{z}^T + \boldsymbol{c}^T = -(\boldsymbol{\lambda}^1)^T - \lambda^0 \boldsymbol{e}^T + \boldsymbol{\mu}^T.$$

Now it is easy to verify the conditions (2.1) with $\gamma = -\frac{1}{2}\lambda^0$:

1. If $k_i\sigma - r_i = -\varepsilon_i$, i.e. $z_i = 0,\ \bar{z}_i = 2\varepsilon_i$, then $\lambda_i^1 = 0,\ \mu_i \geq 0$ and so $\lambda_i = \frac{1}{2}(-\lambda^0 + \mu_i) \geq -\frac{1}{2}\lambda^0$;
2. If $k_i\sigma - r_i = \varepsilon_i$, i.e. $z_i = 2\varepsilon_i,\ \bar{z}_i = 0$, then $\mu_i = 0,\ \lambda_i^1 \geq 0$ and so $\lambda_i = \frac{1}{2}(-\lambda_i^1 - \lambda^0) \leq -\frac{1}{2}\lambda^0$;
3. If $|k_i\sigma - r_i| < \varepsilon_i$, i.e. $z_i \neq 0,\ \bar{z}_i \neq 0$, then $\mu_i = 0,\ \lambda_i^1 = 0$ and so $\lambda_i = -\frac{1}{2}\lambda^0$.

By Theorem 2 a solution of Problem 5 gives the solution of Problem 3. □

5 The Modification of the Simplex Method

Problem 5 differs from problems of linear programming in two simple nonlinear conditions

$$\boldsymbol{\mu}\boldsymbol{z}^T = 0, \quad \boldsymbol{\lambda}(\bar{\boldsymbol{z}})^\top = 0.$$

For the solution of this new problem a modification of the simplex method based on the Wolfe and Daugavet works ([3, 4]) is suggested. We give a short description of this algorithm.

Initial plan. We choose $\boldsymbol{z} = \bar{\boldsymbol{z}} = \boldsymbol{\varepsilon},\ \boldsymbol{\lambda} = \mathbf{0},\ \boldsymbol{\mu} = \mathbf{0}$ and take an initial value of u_i as

$$u_i = \Big|2(\mathbf{D}\boldsymbol{z}^T)_i + c_i\Big|,\ i = 1, \ldots, n.$$

The signs of u_i, $i = 1, \ldots, n$ (i.e. the diagonal elements of matrix $\mathbf{E}$) are chosen in such a way that the equations

$$2\mathbf{D}\boldsymbol{z}^T + \boldsymbol{c}^T + \mathbf{E}\boldsymbol{u}^T = 0$$

are satisfied.

Iterations. Every step of the method is a transformation of the simplex table, taking into account the lexicographic ordering (it allows us to avoid iterative loops) and the additional conditions $\boldsymbol{\mu}\boldsymbol{z}^\top = 0$, $\boldsymbol{\lambda}^1\bar{\boldsymbol{z}}^T = 0$. We can show that the additional nonlinear conditions do not prevent us from doing it.

It is proved that if some simplex iteration can not be done without violation of these nonlinear conditions then the last basic solution gives $\boldsymbol{u}\boldsymbol{e}^T = 0$, i.e. we have the solution of Problem 5.

Solution. This method gives us the values of the components of the vector $\boldsymbol{r} - \boldsymbol{\varepsilon} + \ \boldsymbol{z}$. The corresponding interpolating spline is the solution of Problem 3. It can be constructed by known spline interpolation methods.

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