Mathematical Modelling and Analysis
Volume 13 Number 2, 2008, pages 241–250
 ISSN 1392-6292 print, ISSN 1648-3510 online

ON THE UNIQUE SOLVABILITY OF A NON-LOCAL BOUNDARY VALUE PROBLEM FOR LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS[1]

A. RONTÓ[1], V. PYLYPENKO[2] and N. DILNA[3]

[1] *Institute of Mathematics, AS of Czech Republic*
Žižkova St. 22, 61062 Brno, Czech Republic
[2] *National Technical University of Ukraine,Kiev Polytechnic Institute*
Peremohy Ave. 37, 03056 Kiev, Ukraine
[3] *Institute of Mathematics, NAS of Ukraine*
Mathematical Institute of the Slovak Academy of Sciences
Tereshchenkivska St. 3, 01601 Kiev, Ukraine,
Štefánikova 49, Bratislava, Slovak Republic

E-mail: a.ronto@gmail.com; vita.pilipenko@gmail.com;
E-mail: dilna@imath.kiev.ua

Received October 15, 2007; revised November 28, 2007; published online May 1, 2008

Abstract. General conditions are obtained for the unique solvability of a non-local boundary value problem for systems of linear functional differential equations.

Key words: Boundary value problem, functional differential equation, non-local condition, unique solvability, differential inequality.

1 Introduction

The paper deals with the question on the existence and uniqueness of a solution of a non-local boundary-value problem for linear functional differential equations of the general form. More precisely, we consider the system of functional differential equations

$$u_k'(t) = (l_k u)(t) + q_k(t), \quad t \in [a, b], \ \ k = 1, 2, \ldots, n, \tag{1.1}$$

[1] The research was supported in part by AS CR, Institutional Research Plan No. AV0Z10190503, GA CR, Grant No. 201/06/0254, and DFFD, Grant No. 0107U003322, and National Scholarship Programme of the Slovak Republic.

subjected to the non-local boundary conditions

$$u_k(a) = h_k(u), \quad k = 1, 2, \dots, n, \tag{1.2}$$

where $-\infty < a < b < +\infty$, $n \in \mathbb{N}$,

$$l_k : D([a,b], \mathbb{R}^n) \to L_1([a,b], \mathbb{R}), \quad k = 1, 2, \dots, n,$$

are linear operators, $\{q_k \mid k = 1, 2, \dots, n\} \subset L_1([a,b], \mathbb{R})$ are given functions, and $h_k : D([a,b], \mathbb{R}^n) \to \mathbb{R}$, $k = 1, 2, \dots, n$, are continuous linear functionals. Here, $D([a,b], \mathbb{R}^n)$ and $L_1([a,b], \mathbb{R})$ are, respectively, the Banach spaces of absolutely continuous and Lebesgue integrable vectors functions on the interval $[a,b]$. By a solution of problem (1.1), (1.2), as usual (see, e. g., [1]), we mean a vector function $u = (u_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$ whose components are absolutely continuous, satisfy system (1.1) almost everywhere on the interval $[a,b]$, and possesses property (1.2).

It should be noted that equations (1.1) may contain terms with derivatives and, thus, the statements presented in what follows are applicable, in particular, to neutral type linear functional differential equations.

The aim of this note is to prove the unique solvability of problem (1.1), (1.2) on the assumption that the linear operators l_k, $k = 1, 2, \dots, n$, appearing in (1.1) can be estimated by certain other linear operators generating problems with conditions (1.2) for which the statement on the integration of differential inequality holds. The precise formulation of the property mentioned is given by the following definition.

Definition 1. A linear operator $l = (l_k)_{k=1}^n : D([a,b], \mathbb{R}^n) \to L_1([a,b], \mathbb{R}^n)$ is said to belong to the set $\mathcal{S}_{a,h}([a,b], \mathbb{R}^n)$ if the boundary value problem (1.1), (1.2) has a unique solution $u = (u_k)_{k=1}^n$ for any $\{q_k \mid k = 1, 2, \dots, n\} \subset L_1([a,b], \mathbb{R})$ and, moreover, the solution of (1.1), (1.2) possesses the property

$$\min_{t \in [a,b]} u_k(t) \geq 0, \qquad k = 1, 2, \dots, n, \tag{1.3}$$

whenever the functions q_k, $k = 1, 2, \dots, n$, appearing in (1.1) are non-negative almost everywhere on $[a,b]$.

Note that $\mathcal{S}_{a,h}([a,b], \mathbb{R})$ contains the set $\tilde{V}_{ab}^{+}(h)$ defined in the paper [6], where certain efficient conditions sufficient for the inclusion $l \in \tilde{V}_{ab}^{+}(h)$ are established in the case where the linear operator l admits a continuous extension to the space of continuous functions.

2 Notation

The following notation is used throughout the paper.

1. $\mathbb{R} := (-\infty, \infty)$, $\quad \mathbb{N} := \{1, 2, 3, \dots\}$.
2. $\|x\| := \max_{1 \leq k \leq n} |x_k|$ for $x = (x_k)_{k=1}^n \in \mathbb{R}^n$.

3. $L_1([a,b],\mathbb{R}^n)$ is the Banach space of all the Lebesgue integrable vector-functions $u : [a,b] \to \mathbb{R}^n$ with the standard norm

$$L_1([a,b],\mathbb{R}^n) \ni u \longmapsto \int_a^b \|u(\xi)\|\, d\xi.$$

4. $\operatorname{mes} A$ is the Lebesgue measure of a set $A \subset \mathbb{R}$.

5. $D([a,b],\mathbb{R}^n)$ is the Banach space of the absolutely continuous functions $[a,b] \to \mathbb{R}^n$ equipped with the norm

$$D([a,b],\mathbb{R}^n) \ni u \longmapsto \|u(a)\| + \int_a^b \|u'(\xi)\|\, d\xi.$$

6. If $h = (h_k)_{k=1}^n : D([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ are certain operators, then the symbol $D_h([a,b],\mathbb{R}^n)$ denotes the set of all $u = (u_k)_{k=1}^n$ from $D([a,b],\mathbb{R}^n)$ for which $u_k(a) = h_k(u)$, $k = 1, 2, \dots, n$.

7. The set $D_{h,1}([a,b],\mathbb{R}^n)$ is defined by the formula

$$D_{h,1}([a,b],\mathbb{R}^n) := \Big\{u = (u_k)_{k=1}^n \in D_h([a,b],\mathbb{R}^n) \mid \min_{\xi\in[a,b]} u_k(\xi) \geq 0 \text{ for all } k = 1, 2, \dots, n\Big\}. \quad (2.1)$$

8. The set $D_{h,2}([a,b],\mathbb{R}^n)$ is introduced by the formula

$$D_{h,2}([a,b],\mathbb{R}^n) := \Big\{u = (u_k)_{k=1}^n \in D_h([a,b],\mathbb{R}^n) \mid \min_{\xi\in[a,b]} u_k(\xi) \geq 0 \text{ and } \operatorname*{vrai\,min}_{\xi\in[a,b]} u_k'(\xi) \geq 0 \text{ for all } k = 1, 2, \dots, n\Big\}. \quad (2.2)$$

3 Main Theorem

The following theorem provides general conditions sufficient for the existence and uniqueness of a solution of problem (1.1), (1.2).

Theorem 1. *Let there exist linear operators $p_i = (p_{ik})_{k=1}^n : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i = 0, 1$, satisfying the inclusions*

$$p_1 \in \mathcal{S}_{a,h}([a,b],\mathbb{R}^n), \quad p_0 + p_1 \in \mathcal{S}_{a,h}([a,b],\mathbb{R}^n), \quad (3.1)$$

and such that the inequalities

$$|(l_k u)(t) - (p_{1k} u)(t)| \leq (p_{0k} u)(t), \quad t \in [a,b], \ \ k = 1, 2, \dots, n, \quad (3.2)$$

hold for an arbitrary non-negative absolutely continuous vector function $u : [a,b] \to \mathbb{R}^n$ with property (1.2)*. Then the boundary value problem* (1.1)*,* (1.2) *has a unique solution for arbitrary $\{q_k \mid k = 1, 2, \dots, n\} \subset L_1([a,b],\mathbb{R})$.*

Theorem 1 generalises [7, Theorem 3.3]. Note that assumption (3.1) in Theorem 1 can not be replaced neither by the condition

$$(1-\varepsilon)\, p_1 \in \mathcal{S}_{a,h}([a,b],\mathbb{R}^n), \quad p_0 + p_1 \in \mathcal{S}_{a,h}([a,b],\mathbb{R}^n), \tag{3.3}$$

nor by the condition

$$p_1 \in \mathcal{S}_{a,h}([a,b],\mathbb{R}^n), \quad (1-\varepsilon)\,(p_0 + p_1) \in \mathcal{S}_{a,h}([a,b],\mathbb{R}^n), \tag{3.4}$$

where ε is an arbitrarily small positive number. Indeed, let us fix some $\varepsilon \in [0,1)$ and consider the homogeneous Cauchy problem

$$u_1(a) = 0, \quad u_2(a) = 0, \tag{3.5}$$

for the system

$$u_1'(t) = \frac{1}{2\,(b-a)}\left(u_1(b) - u_2(b)\right), \tag{3.6}$$

$$u_2'(t) = -\frac{1}{2\,(b-a)}\left(u_1(b) - u_2(b)\right), \quad t \in [a,b]. \tag{3.7}$$

It is clear that (3.5), (3.6), (3.7) is a particular case of problem (1.1), (1.2), where $n = 2$, $q_1 = q_2 = 0$,

$$(l_i u)(t) = \frac{(-1)^{i+1}}{2(b-a)}\left(u_1(b) - u_2(b)\right), \quad t \in [a,b], \;\; i = 1, 2,$$

and $h_1 = h_2 = 0$. Problem (3.5), (3.6), (3.7) has the family of solutions

$$u_i(t) = \lambda\,(-1)^i\,(t-a), \quad t \in [a,b], \;\; i = 1, 2,$$

where $\lambda \in \mathbb{R}$ is arbitrary. However, condition (3.4) in this case is satisfied for all $\varepsilon \in (0,1)$ with

$$p_1 := 0, \quad p_0 u := \frac{1}{2\,(b-a)}\begin{pmatrix} u_1(b) + u_2(b) \\ u_1(b) + u_2(b) \end{pmatrix},$$

because initial value problem (3.5) for the system

$$u_1'(t) = \frac{1-\varepsilon}{2\,(b-a)}\left(u_1(b) + u_2(b)\right) + q_1(t),$$

$$u_2'(t) = \frac{1-\varepsilon}{2\,(b-a)}\left(u_1(b) + u_2(b)\right) + q_2(t), \quad t \in [a,b],$$

as it is easy to see, has a unique solution for any $q_i \in L_1([a,b],\mathbb{R})$, $i = 1, 2$, and this solution is non-negative for non-negative q_i, $i = 1, 2$.

In a similar way, one can specify an example showing the optimality of condition (3.3).

4 Corollaries

Theorem 1 allows one to formulate several corollaries.

Definition 2. A linear operator $l = (l_k)_{k=1}^n : D([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R}^n)$ is said to have positive restriction on $D_h([a, b], \mathbb{R}^n)$ if the inequality

$$\operatorname*{vrai\,min}_{t\in[a,b]} (l_k u)(t) \ge 0, \qquad k = 1, 2, \dots, n, \tag{4.1}$$

is true for any $u = (u_k)_{k=1}^n$ from $D_{h,1}([a, b], \mathbb{R}^n)$. An operator $l = (l_k)_{k=1}^n : D([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R}^n)$ is said to be positive if (4.1) holds for any non-negative $u = (u_k)_{k=1}^n$ from $D([a, b], \mathbb{R}^n)$.

Note that an operator $l : D([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R}^n)$ having positive restriction on $D_h([a, b], \mathbb{R}^n)$ need not be positive. This is the case, in particular, for the operator $l = (l_k)_{k=1}^n$,

$$(l_k u)(t) = (p_k u)(t) + h_k(u) - u_k(a), \qquad t \in [a, b],\ k = 1, 2, \dots, n,$$

where $p = (p_k)_{k=1}^n : D([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R}^n)$ is positive and

$$\operatorname{mes} \{t \in [a, b] \mid (p_{k_*} v)(t) < v_{k_*}(a) - h_{k_*}(v)\} \ne 0$$

for some $v = (v_k)_{k=1}^n \in D([a, b], \mathbb{R}^n)$ and $k_* \in \{1, 2, \dots, n\}$. The latter property is present, e. g., if

$$v_k(t) = (t - a)(b - a)^{-1}, \quad h_k(u) = -u_k(b),$$
$$(p_k u)(t) = \sum_{j=1}^{n} \alpha_{kj}(t) u_j(\omega_{kj}(t)), \quad k = 1, 2, \dots, n,\ \ t \in [a, b],$$

for all $u = (u_k)_{k=1}^n$ from $D([a, b], \mathbb{R}^n)$ where the integrable functions $\alpha_{ki} : [a, b] \to \mathbb{R}$ and the measurable functions $\omega_i : [a, b] \to [a, b]$, $k = 1, 2, \dots, n$, $i = 1, 2, \dots, n$, satisfy the condition

$$\min_{k=1,2,\dots,n} \operatorname{mes} \Big\{ t \in [a, b] \mid \sum_{j=1}^{n} \alpha_{kj}(t)(\omega_{kj}(t) - a) < b - a \Big\} \ne 0.$$

The following statement generalises the results from [2, Theorem 2.2].

Theorem 2. *Let there exist certain linear operators* $l_i : D([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R}^n)$, $i = 0, 1$, *which have positive restriction on* $D_h([a, b], \mathbb{R}^n)$, *satisfy the inclusions*

$$l_0 \in \mathcal{S}_{a,h}([a, b], \mathbb{R}^n), \quad -\frac{1}{2} l_1 \in \mathcal{S}_{a,h}([a, b], \mathbb{R}^n), \tag{4.2}$$

and are such that the inequalities

$$|(l_k u)(t) + (l_{1k} u)(t)| \le (l_{0k} u)(t), \quad t \in [a, b], \ \ k = 1, 2, \dots, n, \tag{4.3}$$

hold for an arbitrary non-negative absolutely continuous function $u : [a, b] \to \mathbb{R}^n$ *with property* (1.2). *Then the boundary value problem* (1.1), (1.2) *has a unique solution for arbitrary* $\{q_k \mid k = 1, 2, \dots, n\} \subset L_1([a, b], \mathbb{R})$.

Proof. It follows from assumption (4.3) and the positivity of the operator $l_1|_{D_h([a,b],\mathbb{R}^n)}$ that, for any u from $D_{h,1}([a,b],\mathbb{R}^n)$, the relations

$$|(l_k u)(t) + \tfrac{1}{2}(l_{1k}u)(t)| = |(l_k u)(t) + (l_{1k}u)(t) - \tfrac{1}{2}(l_{1k}u)(t)|$$
$$\leq (l_{0k}u)(t) + \frac{1}{2}|(l_{1k}u)(t)| = (l_{0k}u)(t) + \frac{1}{2}(l_{1k}u)(t),\ t \in [a,b],\ k = 1,2,\ldots,n$$

are true. This means that l admits estimate (3.2) with the operators p_0 and p_1 defined by the equalities

$$p_0 := l_0 + \frac{1}{2}l_1, \quad p_1 := -\frac{1}{2}l_1. \tag{4.4}$$

It now remains to note that assumption (4.2) ensures the validity of inclusion (3.1) for operators (4.4). Thus, under conditions (4.2) and (4.3), the operators $p_i : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i = 0,1$, defined by formulae (4.4) satisfy conditions (3.1) and (3.2) of Theorem 1. Applying Theorem 1, we arrive at the required assertion. □

Remark 1. Arguing similarly, one can show that the assertion of Theorem 2 is preserved if condition (4.2) is replaced by the assumption that

$$l_0 + (1-2\theta)\,l_1 \in \mathcal{S}_{a,h}([a,b],\mathbb{R}^n), \quad -\theta l_1 \in \mathcal{S}_{a,h}([a,b],\mathbb{R}^n) \tag{4.5}$$

for a certain $\theta \in (0,1)$.

Condition (4.3) is satisfied, in particular, if l can be represented in the form

$$l = l_0 - l_1, \tag{4.6}$$

where l_0 and l_1 are certain linear operators which have positive restriction on $D_h([a,b],\mathbb{R}^n)$. In the case where the operator l admits decomposition (4.6), the following statement is also true.

Theorem 3. *Let us assume that the operator l admits representation* (4.6) *where $l_i : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i = 0,1$, are linear operators which have positive restriction on $D_h([a,b],\mathbb{R}^n)$ and such that the inclusions*

$$l_0 \in \mathcal{S}_{a,h}([a,b],\mathbb{R}^n), \quad \frac{1}{2}(l_0 - l_1) \in \mathcal{S}_{a,h}([a,b],\mathbb{R}^n) \tag{4.7}$$

are satisfied. Then problem (1.1), (1.2) *has a unique solution for arbitrary* $\{q_k \mid k = 1,2,\ldots,n\} \subset L_1([a,b],\mathbb{R})$.

Proof. The positivity of the operators $l_i|_{D_h([a,b],\mathbb{R}^n)}$, $i = 0,1$, and equality (4.6) imply that, for any u from $D_{h,1}([a,b],\mathbb{R}^n)$, the inequalities

$$-(l_{1k}u)(t) \leq (l_k u)(t) \leq (l_{0k}u)(t), \qquad k = 1,2,\ldots,n, \tag{4.8}$$

are true for almost every t from $[a,b]$. Putting

$$p_i := \frac{1}{2}(l_0 + (-1)^i l_1), \quad i = 0,1, \tag{4.9}$$

and taking into account the obvious identities

$$p_0 + (-1)^i p_1 = l_i, \quad i = 0, 1,$$

we conclude that estimates (4.8) guarantee the fulfilment of conditions (3.1) and (3.2) of Theorem 1 for the operators $p_i : D([a,b], \mathbb{R}^n) \to L_1([a,b], \mathbb{R}^n)$, $i = 0, 1$, given by the formulae (4.9). Application of Theorem 1 completes the proof. □

Remark 2. Arguing similarly to Section 3, one can show that conditions (4.2), (4.5), and (4.7) are optimal in a certain sense.

5 Proof of the Main Theorem

The proof of Theorem 1 is based on rather general order-theoretical considerations. Let us consider the abstract operator equation

$$Fx = z, \tag{5.1}$$

where $F : E_1 \to E_2$ is a mapping, $\langle E_1, \|\cdot\|_{E_1} \rangle$ is a normed space, $\langle E_2, \|\cdot\|_{E_2} \rangle$ is a Banach space over the field $\mathbb{R}$, $K_i \subset E_i$, $i = 1, 2$, are closed cones, and z is an arbitrary element from E_2.

The following statement is due to M. A. Krasnoselskii, E. A. Lifshits, Yu. V. Pokornyi, and V. Ya. Stetsenko [4, Theorem 7] (see also [5, Theorem 49.4]).

Theorem 4. *Let the cone K_2 be normal and reproducing. Furthermore, let $B_k : E_1 \to E_2$, $k = 1, 2$, be additive and homogeneous operators such that B_1^{-1} and $(B_1 + B_2)^{-1}$ exist and possess the properties*

$$B_1^{-1}(K_2) \subset K_1, \quad (B_1 + B_2)^{-1}(K_2) \subset K_1, \tag{5.2}$$

and, furthermore, the relation

$$\{Fx - Fy - B_1(x - y),\ B_2(x - y) - Fx + Fy\} \subset K_2 \tag{5.3}$$

is satisfied for any pair $(x, y) \in E_1^2$ such that $x - y \in K_1$. Then equation (5.1) has a unique solution $u \in E_1$ for an arbitrary element $z \in E_2$.

Let us recall that a cone $K \subset E$ in a Banach space $\langle E, \|\cdot\|_E \rangle$ is normal if and only if the relation

$$\inf \{\gamma \in (0, +\infty) \mid \|x\|_E \le \gamma \|y\|_E \ \forall \{x, y\} \subset E :\ y - x \in K\} < +\infty$$

is true. By definition, the cone K is reproducing in E if and only if an arbitrary element x from E can be represented in the form $x = u - v$, where u and v belong to K (see, e. g., [3, 5]).

Lemma 1. 1. *$D_h([a,b], \mathbb{R}^n)$ is a normed space with the norm*

$$D_h([a,b], \mathbb{R}^n) \ni u \longmapsto \int_a^b \|u'(\xi)\|\, d\xi + \|u(a)\|.$$

2. *The set* $D_{h,1}([a,b],\mathbb{R}^n)$ *is a cone in the space* $D_h([a,b],\mathbb{R}^n)$*.*

3. *The set* $D_{0,2}([a,b],\mathbb{R}^n)$ *is a normal and reproducing cone in the space* $D_0([a,b],\mathbb{R}^n)$*.*

Proof. The assertions of Lemma 1 follow immediately from the definitions of the sets $D_h([a,b],\mathbb{R}^n)$ and $D_{h,1}([a,b],\mathbb{R}^n)$ (see notation 6 and formulae (2.1) and (2.2) in Section 2). □

The next lemma establishes the relation between the property described by Definition 1 and the positive invertability of a certain operator.

Lemma 2. *If* $l = (l_k)_{k=1}^n : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$ *is a linear operator such that*

$$l \in \mathcal{S}_{a,h}([a,b],\mathbb{R}^n) \tag{5.4}$$

then the operator $V_l : D_h([a,b],\mathbb{R}^n) \to D_0([a,b],\mathbb{R}^n)$ *given by the formula*

$$D_h([a,b],\mathbb{R}^n) \ni u \longmapsto V_l u := u - \int_a^{\cdot} (lu)(\xi)\, d\xi - h(u) \tag{5.5}$$

is invertible and, moreover, its inverse V_l^{-1} *satisfies the inclusion*

$$V_l^{-1}(D_{0,2}([a,b],\mathbb{R}^n)) \subset D_{h,1}([a,b],\mathbb{R}^n).$$

Proof. Let the mapping l belong to the set $\mathcal{S}_{a,h}([a,b],\mathbb{R}^n)$. Given an arbitrary function $y = (y_k)_{k=1}^n \in D_0([a,b],\mathbb{R}^n)$, consider the equation

$$V_l u = y. \tag{5.6}$$

In view of notation 6, Sec. 2, we have

$$y_k(a) = 0, \qquad k = 1,2,\dots,n. \tag{5.7}$$

By virtue of assumption (5.4), there exists a unique absolutely continuous $u = (u_k)_{k=1}^n$ such that

$$u_k'(t) = (l_k u)(t) + y_k'(t), \quad t \in [a,b], \quad k = 1,2,\dots,n, \tag{5.8}$$

$$u_k(a) = h_k(u), \quad k = 1,2,\dots,n. \tag{5.9}$$

Moreover, if the functions y_k, $k = 1,2,\dots,n$, are non-negative and non-decreasing, the components of u possess property (1.3). Integrating both parts of (5.8) and taking (5.7) and (5.9) into account, we find that $u = (u_k)_{k=1}^n$ is the unique solution of equation (5.6). □

The following statement is an obvious consequence of formula (5.5).

Lemma 3. *For arbitrary linear operators* $p_i : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$*,* $i = 1,2$*, the identity*

$$V_{p_1} + V_{p_2} = 2V_{\frac{1}{2}(p_1+p_2)}$$

is true.

Let us now turn to the proof of Theorem 1. Consider the initial value problem (1.1), (1.2). An absolutely continuous vector function $u = (u_k)_{k=1}^n : [a, b] \to \mathbb{R}^n$ is a solution of (1.1), (1.2) if, and only if it satisfies the equation

$$u(t) = \int_a^t (lu)(s)ds + \int_a^t q(s)ds + h(u), \quad t \in [a, b]. \tag{5.10}$$

Let us put $E_1 = D_h([a, b], \mathbb{R}^n)$, $E_2 = D_0([a, b], \mathbb{R}^n)$ and define the mapping $F : E_1 \to E_2$ by setting

$$(Fu)(t) := u(t) - \int_a^t (lu)(s)ds - h(u), \quad t \in [a, b], \tag{5.11}$$

for any u from $D_h([a, b], \mathbb{R}^n)$. Then equation (5.10) takes form (5.1) with

$$z(t) := \int_a^t q(s)ds, \qquad t \in [a, b].$$

Assumption (3.2) means that the estimate

$$-(p_{0k}u)(t) + (p_{1k}u)(t) \le (l_k u)(t) \le (p_{0k}u)(t) + (p_{1k}u)(t), \quad t \in [a, b],$$

is true for any u from $D_{h,1}([a, b], \mathbb{R}^n)$ and all $k = 1, 2, \dots, n$. Therefore, for all such u, the relation

$$\begin{aligned} u_k'(t) - (p_{0k}u)(t) - (p_{1k}u)(t) &\le u_k'(t) - (l_k u)(t) \\ &\le u_k'(t) + (p_{0k}u)(t) - (p_{1k}u)(t), \quad k = 1, 2, \dots, n, \end{aligned} \tag{5.12}$$

holds for almost every $t \in [a, b]$. Integrating (5.12) and taking property (1.2) into account, we obtain that the inequality

$$\begin{aligned} u_k(t) - h_k(u) - \int_a^t \left[(p_{0k}u)(\xi) + (p_{1k}u)(\xi)\right] d\xi &\le u_k(t) - h_k(u) - \int_a^t (l_k u)(\xi)d\xi \\ \le u_k(t) - h_k(u) + \int_a^t \left[(p_{0k}u)(\xi) - (p_{1k}u)(\xi)\right] d\xi, &\qquad t \in [a, b], \end{aligned} \tag{5.13}$$

holds for any $(u_k)_{k=1}^n$ from $D_{h,1}([a, b], \mathbb{R}^n)$, and all $k = 1, 2, \dots, n$. Let us define the linear mappings $B_{ik} : D_h([a, b], \mathbb{R}^n) \to D_0([a, b], \mathbb{R})$, $i = 1, 2$, $k = 1, 2, \dots, n$, by putting

$$B_{ik}u := u_k(\cdot) - h_k(u) + (-1)^i \int_a^{\cdot} \left[(p_{0k}u)(\xi) - (-1)^i (p_{1k}u)(\xi)\right] d\xi \tag{5.14}$$

for an arbitrary u from $D_h([a, b], \mathbb{R}^n)$ and construct the corresponding mappings $B_i : D_h([a, b], \mathbb{R}^n) \to D_0([a, b], \mathbb{R}^n)$, $i = 1, 2$, according to the formula

$$D_h([a, b], \mathbb{R}^n) \ni u \longmapsto B_i u := \begin{pmatrix} B_{i1}u \\ B_{i2}u \\ \vdots \\ B_{in}u \end{pmatrix}, \qquad i = 1, 2.$$

Then estimates (5.12) and (5.13), formula (5.5), and the definition of the sets $D_{h,1}([a,b],\mathbb{R}^n)$ and $D_{0,2}([a,b],\mathbb{R}^n)$ imply that

$$\{B_2u - V_lu, V_lu - B_1u\} \subset D_{0,2}([a,b],\mathbb{R}^n) \text{ for an arbitrary } u \text{ from } D_{h,1}([a,b],\mathbb{R}^n).$$

The last property means that mapping (5.11) satisfies condition (5.3) with K_1 and K_2 defined by the formulae

$$K_1 = D_{h,1}([a,b],\mathbb{R}^n), \quad K_2 = D_{0,2}([a,b],\mathbb{R}^n). \tag{5.15}$$

By virtue of Lemma 1, the set K_1 forms a cone in the normed space $D_h([a,b],\mathbb{R}^n)$, whereas the set K_2 is a normal and reproducing cone in the Banach space $D_0([a,b],\mathbb{R}^n)$.

It follows from Lemma 3 that the identity

$$\frac{1}{2}\left(V_{p_1-p_0} + V_{p_1+p_0}\right) = V_{p_1}$$

is true. However, according to equalities (5.14), we have $B_i = V_{p_1-(-1)^i p_0}$, $i = 1, 2$. Therefore, by virtue of assumption (3.1) and Lemma 2, we conclude that the inverse operators B_1^{-1} and $(B_1 + B_2)^{-1}$ exist and possess properties (5.2) with respect to cones (5.15). Applying Theorem 4, we establish the unique solvability of the initial value problem (1.1), (1.2) for arbitrary $q_k \in L_1([a,b],\mathbb{R})$, $k = 1, 2, \ldots, n$. Theorem 1 is proved.

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