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ON THE SOLUTION OF THE STEADY-STATE DIFFUSION PROBLEM FOR FERROMAGNETIC PARTICLES IN A MAGNETIC FLUID[1]

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Abstract. A mathematical model for the diffusion process of ferromagnetic particles in a magnetic fluid is described. The unique solvability of the steady-state particle concentration problem is investigated and an analytical expression for its solution is found. In case that the fluid is under the action of a high-gradient magnetic field a Stefan-type diffusion problem can arise. An algorithm for solving the Stefan-type steady-state problem is developed.

Key words: diffusion process, ferromagnetic particles, magnetic field, magnetic fluid.

1 Introduction

Because of their ability for ponderomotive interaction with an external magnetic field, magnetic fluids have not only initiated the development of a new direction in fluid mechanics but became a new technological material which found a wide application in engineering [1, 3, 4, 9]. A magnetic fluid is a stable colloidal suspension of ferromagnetic particles in a carrier liquid (oil, water, bio-compatible liquid). In order to prevent the particles to stick together, they are covered by a special nonmagnetic surface-active substance. The size of particles is of the order of 10nm $= 10^{-8}$m, and they are in the Brownian motion

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state in the carrier liquid. Owing to the fact that the particles possess magnetic properties, not only Brownian motion but also magnetophoresis diffusion process takes place in a magnetic fluid [5, 9]. This becomes more important when the magnetic fluid is under the influence of a high-gradient magnetic field. Both processes always occur in a magnetic fluid and their study is of undoubted interest.

The main objective of this paper is the study of the steady-state magnetic particle diffusion problem admitting an explicit analytical solution. Moreover, the Stefan type diffusion problem which appears if the magnetic fluid is subjected to a sufficiently strong magnetic field will be an object of special interest.

2 Mathematical Model for the Diffusion of Magnetic Particles

For describing the diffusion process mathematically, the classical theory of diffusion of suspended particles in the field of a body force may be applied to the magnetic fluid. As shown in [6], the mass conservation law for Brownian particles reads as follows

$$\begin{cases} \rho\left(\dfrac{\partial C}{\partial t} + \mathbf{v}\cdot\nabla C\right) + \nabla\cdot\mathbf{i} = 0, \\ \mathbf{i} = -\rho D\nabla C + \rho b C\mathbf{f}, \end{cases} \tag{2.1}$$

where C denotes the volume concentration of the particles in the colloid; $\mathbf{i}$ the mass-flux density of the particles; D the diffusion coefficient; ρ the density of a particle material; $b = D/kT$ the particle mobility (Einstein's correlation); $\mathbf{v}$ the velocity of convective motion; $\mathbf{f}$ an external force acting on a particle; $k = 1.3806568 \times 10^{-23}\mathrm{JK}^{-1}$ the Boltzmann constant; T the particle temperature; and t the time variable. Equation (2.1) is valid for any suspension of spherical Brownian particles (not necessarily magnetic) of the same size. The term $\rho D\nabla C$ corresponds to the usual diffusion and the term $\rho bC\mathbf{f}$ to the diffusion process caused by the action of external fields.

The macroscopic interaction of a magnetic fluid with an external nonuniform magnetic field is determined by the force acting on each particle. If the magnetic moment of the particle $\mathbf{m}$ is directed along the vector of magnetic intensity $\mathbf{H}$, the force acting on the particle is defined by the expression $\mu_0 m\nabla H$ where $\mu_0 = 4\pi\times 10^{-7}\mathrm{Hm}^{-1}$ is the magnetic constant (magnetic permeability of vacuum). As the particle performs the Brownian motion, the force is determined by the time-averaged value of the projection of the moment $\mathbf{m}$ on the field direction that is equal to $mL(\xi H)$ where

$$L(\gamma) = \coth\gamma - \frac{1}{\gamma}$$

is the Langevin function and $\xi = \mu_0 m/kT$. Thus, the ferromagnetic particle is subjected to the magnetic force

$$\mathbf{f} = \mu_0 m L(\xi H)\nabla H. \tag{2.2}$$

The force $\mathbf{f}$ generates a diffusion motion of magnetic particles (magnetophoresis) with respect to the carrier liquid. The particle concentration increases in regions where the magnetic intensity is high. In contrast to this forced motion, the usual mechanism of diffusion of particles in regions with low concentration takes place. The competition of these two mechanisms brings about a redistribution of the particles in the magnetic fluid and, consequently, of a volume magnetic force acting upon the fluid. Concerning the gravity force, estimations show that its influence on diffusion of Brownian particles is negligible.

By virtue of (2.1) and (2.2), the mass conservation law for magnetic particles in the magnetic fluid becomes [2]

$$\frac{\partial C}{\partial t} + \mathbf{v} \cdot \nabla C = D\nabla \cdot \left(\nabla C - CL(\xi H)\nabla(\xi H)\right), \quad \xi = \frac{\mu_0 m}{kT}. \tag{2.3}$$

Equation (2.3) is supplemented by the impermeability condition of the boundaries $\mathbf{i} \cdot \mathbf{n} = 0$, i.e. the particle flux through the boundaries is equal to zero, or fully written

$$\frac{\partial C}{\partial n} - \xi L(\xi H)\frac{\partial H}{\partial n} C = 0. \tag{2.4}$$

As initial condition, it is reasonable to consider a uniform distribution of particles, which means

$$C = C_0 = const, \quad t = 0. \tag{2.5}$$

Equation (2.3) completed by the conditions (2.4) and (2.5) represents our mathematical model of the diffusion process of ferromagnetic particles in a magnetic fluid.

Integrating equation (2.3) over the magnetic-fluid volume V and taking into consideration (2.4) and (2.5), it is not difficult to show that the solution of the problem (2.3)–(2.5) satisfies the condition of equal mean particle concentration for all times t:

$$\frac{1}{V}\int\limits_V C(x,t)\,dx = C_0, \quad \text{for all } \ t \geq 0.$$

The magnetic characteristics of a magnetic fluid is determined by its magnetization M which depends on the magnetic-field intensity H and the particle concentration C. In ferrohydrodynamics, the magnetization law for a non-uniformly concentrated magnetic fluid is defined by the formula

$$M = M(H, C) = \frac{M_s}{C_0} L(\xi H)C, \tag{2.6}$$

where M_s is the magnetic-fluid saturation magnetization; C_0 the mean concentration corresponding the uniform particle distribution.

Because of the dependence (2.6), the diffusion problem (2.3)–(2.5) is, in general, of coupled type since it is interconnected with other problems of magnetic-fluid mechanics. For example, if $\nabla C \times \nabla H \neq 0$, i.e. the vectors ∇C and ∇H are not parallel, magneto-concentration convection arises [5] and this motion is described by the Navier-Stokes equations

$$\begin{cases} \rho_0 \left(\dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p + \eta \nabla^2 \mathbf{v} + \mu_0 M(H,C) \nabla H, \\ \nabla \cdot \mathbf{v} = 0 \end{cases}$$

extended by the magnetic terms of the stress tensor. The Maxwell equations for the magnetic field in a magnetic fluid are of the form

$$\nabla \times \mathbf{H} = 0, \quad \nabla \cdot (\mu \mathbf{H}) = 0; \quad \mu = 1 + \frac{M(H,C)}{H}.$$

Equilibrium shapes of a free magnetic-fluid surface are described by the generalized Young-Laplace equation

$$\sigma K = \frac{1}{2}\mu_0 \left(\frac{M(H,C)}{H} \mathbf{H} \cdot \mathbf{n} \right)^2 + \mu_0 \int_0^H M(H,C)\, dH + p_f - p_0 \,.$$

Here ρ_0 is the fluid density, p the pressure in the fluid, η the dynamic viscosity, K is the sum of principal curvatures of a free surface, σ is the surface tension coefficient, p_0 the pressure in a surrounding nonmagnetic medium, p_f the thermodynamic pressure in the fluid.

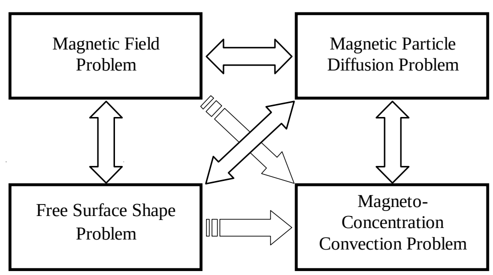

Figure 1. Structure of the coupled problem.

Thus, the mathematical model for the time-dependent diffusion process of ferromagnetic particles in a magnetic fluid with a free surface leads to a coupled problem consisting of at least four nonlinear subproblems – the structure of the magnetic field, the diffusion of magnetic particles, the equilibrium free-surface shapes, and the magneto-concentration convection. Since the solution of each of them depends on the solutions of the others, they should be solved altogether. At the steady state, ∇C is directed along ∇H and so magneto-concentration convection does not arise. The structure of the coupled problem is presented in Fig. 1 where the arrows indicate interrelations between the subproblems.

3 Steady-State Concentration Problem

For $t \to \infty$ we obtain the steady-state concentration problem which can be written in the form

$$\begin{cases} \nabla \cdot (\nabla C - C\nabla(\ln\varphi)) = 0 \quad - \quad \text{in the fluid domain } V, \\ \dfrac{\partial C}{\partial n} - \dfrac{\partial(\ln\varphi)}{\partial n} C = 0 \quad - \quad \text{at the boundary } \mathit{\Gamma}, \\ \int\limits_V C\, dV = C_0 V, \end{cases} \tag{3.1}$$

where

$$\varphi = \varphi(H) = \exp\Big(\int\limits_0^{\xi H} L(\gamma)\, d\gamma\Big) = \frac{\sinh(\xi H)}{\xi H}, \quad \xi = \frac{\mu_0 m}{kT}. \tag{3.2}$$

Note that φ is a monotone positive function with respect to H and a bi-unique positive mapping of the spatial coordinates.

In order to simplify the theoretical analysis of the problem, we introduce the a change of the dependent variable C by

$$C = C^* \varphi,$$

which transforms problem (3.1) into the homogeneous Neumann boundary-value problem for C^*:

$$\begin{cases} \nabla \cdot (\varphi \nabla C^*) = 0 \quad - \quad \text{in the domain } V, \\ \dfrac{\partial C^*}{\partial n} = 0 \quad - \quad \text{at the boundary } \mathit{\Gamma}, \\ \int\limits_V \varphi C^* dV = C_0 V. \end{cases} \tag{3.3}$$

Problem (3.3) admits an analytical solution. In order to find it we multiply differential equation (3.3) by the positive function C^* and modify its left-hand side as follows:

$$C^* \nabla \cdot (\varphi \nabla C^*) = -\varphi \left|\nabla C^*\right|^2 + \nabla \cdot (\varphi C^* \nabla C^*) = 0.$$

Integrating this expression over the fluid volume V and applying the Green's formula, we obtain that an integral over a non-negative function equals to zero:

$$\int\limits_V \varphi \left|\nabla C^*\right|^2 dV = \int\limits_V \nabla \cdot (\varphi C^* \nabla C^*)\, dV = \int\limits_{\mathit{\Gamma}} \varphi C^* \underbrace{\frac{\partial C^*}{\partial n}}_{=0} d\mathit{\Gamma} = 0$$

This is possible if and only if $\nabla C^* \equiv 0$, i.e. $C^* \equiv const$. Thus, only a constant can be a solution of the problem (3.3) and this constant is uniquely determined by the integral condition:

$$\int\limits_V \varphi C^* dV = C^* \int\limits_V \varphi\, dV = C_0 V \quad \Longrightarrow \quad C^* = C_0 V \Big/ \int\limits_V \varphi dV.$$

Consequently, the function

$$C = \varphi \frac{C_0 V}{\int\limits_V \varphi \, dV} \tag{3.4}$$

is the unique solution of steady-state problem (3.1), (3.2). Note that the concentration C is only a function of the magnetic intensity H, i.e. the isolines of the concentration C are also isolines of the intensity H in the magnetic-fluid volume V. The obvious advantage of the solution representation (3.4) is that it is valid for any closed domain $V \subset \mathbb{R}^d$ and any space dimension d.

4 Concentration Problem of Stefan Type

The Stefan type problem can arise if the magnetic fluid is under the influence of a strong magnetic field. The particles diffuse in the direction of the magnetic gradient ∇H, and if the gradient is sufficiently large, the particle concentration reaches a maximum possible value $C_{\max}$ in the vicinity of magnet poles. The maximal achievable particle concentration corresponds to the dense packing of particles. In this case, the magnetic-fluid volume V consists of two different domains, a domain V_1 in which $C < C_{\max}$ and a domain V_2 where $C = C_{\max}$. The interface γ between the domains is a priori unknown and needs to be determined, see Fig. 2.

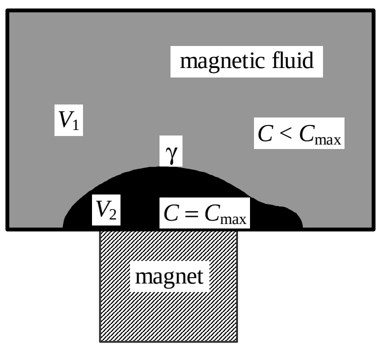

Figure 2. Illustration of the Stefan problem.

It should be noted that if problem (3.1)–(3.2) is solved in the whole domain V and magnetic field is sufficiently strong, the solution (3.4) becomes unrealistic near the magnet poles because the computed particle concentration can achieve values $C \gg 1 > C_{\max}$. Incidentally, for the majority of magnetic fluids, the value of $C_{\max}$ lies in the range $0.3 \lesssim C_{\max} \lesssim 0.4$ [9].

For solving the Stefan problem, it is necessary to determine the interface between the domains V_1 and V_2 which is the isoline corresponding a particle concentration $C = C_{\max}$. If the interface is found, the steady-state concentration problem has to be restricted on the domain V_1 where it has an exact solution analogous to (3.4).

A strategy for solving the Stefan problem could be the following:

1) We use the fact that the sought interface between the domains V_1 and V_2 is an isoline of both the concentration C and the intensity H. If magnetic field intensity H is given, we can determine the intensity isoline $\gamma(H_0)$ for any constant $H_{\min} \le H_0 \le H_{\max}$. The isoline $\gamma(H_0)$ divides the domain V into the subdomain $V_1(H_0)$ where $H < H_0$ and the subdomain $V_2(H_0)$ where $H > H_0$.

2) Assuming $C = C_{\max}$ in the domain V_2, we solve the steady-state concentration problem only in the domain V_1. According to (3.4), its solution is

$$C = \varphi \frac{C_{01} V_1}{\int\limits_{V_1} \varphi \, dV}, \quad C_{01} = \frac{1}{V_1} \int\limits_{V_1} C \, dV,$$

where C_{01} is an unknown mean concentration in the domain V_1. The value of C_{01} is determined from the integral condition:

$$C_0 V = \int\limits_{V} C \, dV = \int\limits_{V_1} C dV + \int\limits_{V_2} C \, dV = C_{01} V_1 + C_{\max} V_2.$$

It follows that

$$C_{01} = \frac{C_{\max} V_1 - (C_{\max} - C_0) V}{V_1}.$$

3) Now it is necessary to find the value of H_0 that the continuity condition

$$C_1 \Big|_{\gamma(H_0)} = C_2 \Big|_{\gamma(H_0)}$$

is fulfilled at the interface $\gamma(H_0)$ where C_1 and C_2 are the particle concentrations in the domains V_1 and V_2, respectively. It yields the equation with respect to H_0:

$$\varphi(H_0) \frac{C_{01} V_1(H_0)}{\int\limits_{V_1(H_0)} \varphi \, dV} = C_{\max}.$$

In other words, for the function

$$\psi(H_0) := \frac{\varphi(H_0) \left[C_{\max} V_1(H_0) - (C_{\max} - C_0) V \right]}{\int\limits_{V_1(H_0)} \varphi(H) \, dV} - C_{\max}$$

it is required to find H_0^* such that $\psi(H_0^*) = 0$. The equation $\psi(H_0) = 0$ can be solved by the dichotomy method or the secant method.

4) After the equation $\psi(H_0) = 0$ has been solved, we find the particle concentration:

$$C = \begin{cases} \varphi(H) \dfrac{C_{\max} V_1 - (C_{\max} - C_0) V}{\int\limits_{V_1} \varphi(H) \, dV} & - \quad \text{in the domain } V_1, \\ C_{\max} \quad - \quad \text{in the domain } V_2. \end{cases} \tag{4.1}$$

If the equation $\psi(H_0) = 0$ does not have a solution in the range $H_{\min} \le H_0 \le H_{\max}$, it implies that $V_1 = V$, $V_2 = \emptyset$ in formulas (4.1).

Finally, it should be noted that the developed algorithm has been successfully applied to the well known problem on stability of a static magnetic-fluid seal under the action of an external pressure drop [7, 8]. Earlier in [7], the problem was solved without considering the diffusion process of magnetic particles. The new computations have shown that the particle concentration in the sealing layer under high-gradient magnetic field of an annular magnet becomes appreciably nonuniform up to the formation of a domain with dense particle packing resulting in the Stefan problem for the particle concentration. This implies that the diffusion of particles can greatly influence the critical pressure drop that is holding by the seal.

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