Mathematical Modelling and Analysis
Volume 13 Number 2, 2008, pages 223–232
 ISSN 1392-6292 print, ISSN 1648-3510 online

GENERALIZED LINEAR METHODS AND GAP TAUBERIAN REMAINDER THEOREMS

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Received October 6, 2007; revised December 28, 2007; published online May 1, 2008

Abstract. Some gap Tauberian remainder theorems are proved for generalized linear methods $\mathcal{A} = (A_{nk})$, where A_{nk} are bounded linear operators from Banach space X into X. In these theorems the investigated sequences have infinitely many constant pieces.

Key words: generalized linear methods of summability, gap Tauberian remainder theorems.

1 Introduction

Already in 1704 N. Fatzius (see [7]) used linear transformation to accelerate the convergence to compute π. The first convergence acceleration methods derived and used were linear methods. In recent years the main results in convergence acceleration are proved for nonlinear methods (see [2]). Though nonlinear sequence transformations usually give better results, they use substantially bigger volume of computation. In some occasions the precision of intermediate computing is not so important and we can successfully use linear methods (see [20]).

Let X be a Banach space. A sequence $x = (\xi_k)$ $(\xi_k \in X)$ is called λ-bounded if

$$\exists \lim \xi_k = \xi \quad \wedge \quad \lambda_k (\xi_k - \xi) = O(1),$$

where $0 < \lambda_k \nearrow$. Let m_X^λ be the set of all $\lambda-$bounded sequences. Let $\mathcal{L}(X,X)$ be the space of all bounded linear operators from X into X. A sequence $x = (\xi_k)$ is called summable (see [5, 9, 12, 26] and [15]) by a generalized method $\mathcal{A} = (A_{nk})$ if $y = (\eta_n)$ with

$$\eta_n = \sum_{k=0}^{\infty} A_{nk} \xi_k$$

and $A_{nk} \in \mathcal{L}(X,X)$ is convergent. The transformation $\mathcal{A}$ is called preserving $\lambda-$boundedness (see [23]) if

$$\mathcal{A}m_X^\lambda \subset m_X^\lambda.$$

Let $\mu = (\mu_k)$ with $0 < \mu_k \nearrow$. The transformation $\mathcal{A}$ is called accelerating $\lambda-$boundedness if

$$\mathcal{A}m_X^\lambda \subset m_X^\mu$$

with $\lim \mu_k / \lambda_k = \infty$.

There is a problem to determine certain sets T of sequences with the following property

$$\left(\mathcal{A}x \in m_X^\lambda\right) \wedge (x \in T) \Rightarrow \left(x \in m_X^\lambda\right).$$

In the "usual" Tauberian remainder theorems (see [20, 21, 22, 23, 24], [16] and [10, 11]) the Tauberian condition $x \in T$ is connected with the certain regularity of the oscillation of a sequence x. Besides the "usual" Tauberian remainder theorems, there are important Tauberian remainder theorems in which the sequence (ξ_n) has infinitely many constant pieces. In the case $\lambda_n = O(1)$ these theorems are named gap Tauberian theorems (see [1, 27]) and in the case $\lambda_n \neq O(1)$ gap Tauberian remainder theorems (see [18, 19]). The first Tauberian theorem for gap series was proved (see [4]) by G. H. Hardy and J. E. Littlewood. In [1] and [27] (see also [3, 8, 13, 14, 25]) are presented several classical gap Tauberian theorems for certain classic methods of summability.

In this paper we present several more gap Tauberian remainder theorems.

2 Generalized Riesz Method and Gap Tauberian Remainder Theorems

Let us denote by $(\mathcal{R}, P_n)$ or shortly by $\mathcal{R}$ the generalized Riesz method (see [12]), defined by

$$R_{nk} = \begin{cases} R_n P_k & (k = 0, 1, \ldots, n), \\ \theta & (k > n), \end{cases}$$

where $\mathcal{R} = (R_{nk})$ and P_k, $R_n \in \mathcal{L}(X,X)$, while R_n is determined by

$$R_n \sum_{k=0}^{n} P_k \zeta = \zeta \quad (\zeta \in X, n \in \mathbf{N}_0). \tag{2.1}$$

Let (B_n) be a sequence of operators $B_n \in \mathcal{L}(X,X)$ satisfying the conditions

$$(n+1)\|B_{n+1} - B_n\| = O(\|B_n\|) \tag{2.2}$$

and

$$B_0 = \theta, \quad B_n \neq \theta \quad (n \in \mathbf{N}). \tag{2.3}$$

Let us denote by $(\mathcal{Z}, B_n)$ or shortly by $\mathcal{Z}$ the generalized Zygmund method (see [24]) as a generalized Riesz method $(\mathcal{R}, P_n)$ with $P_n = B_{n+1} - B_n$. Let (k_ν) be a sequence of positive integers with

$$k_{\nu+1} - k_\nu > \vartheta k_\nu \qquad (\vartheta > 0). \tag{2.4}$$

Let $Q(n, k_\nu) = \sum_{k=k_\nu}^{n} P_k$.

Proposition 1. (see also [6], [17]–[19]). *Let*

$$R_n^{-1}, \quad Q^{-1}(n,k_\nu) \in \mathcal{L}(X,X), \tag{2.5}$$

$$n = [(1+\vartheta)k_\nu], \tag{2.6}$$

$$\left\|Q^{-1}(n,k_\nu) R_n^{-1}\right\| = O(1), \tag{2.7}$$

$$\left\|Q^{-1}(n,k_\nu) R_{k_\nu-1}^{-1}\right\| = O(1), \tag{2.8}$$

If the conditions (2.4),

$$\xi_k - \xi_{k-1} = \theta \qquad (k \neq k_\nu), \tag{2.9}$$

$$\lambda_{k_{\nu+1}} = O(\lambda_{k_\nu}) \tag{2.10}$$

$$\mathcal{R}x \in m_X^\lambda \tag{2.11}$$

are satisfied, then

$$x \in m_X^\lambda. \tag{2.12}$$

Proof. Using

$$\eta_n = R_n \sum_{k=0}^{n} P_k \xi_k,$$

(2.1), (2.6) and (2.9) we get

$$\begin{aligned} R_n^{-1}\eta_n &= \sum_{k=0}^{n} P_k\xi_k = \sum_{k=0}^{k_\nu-1} P_k\xi_k + \sum_{k=k_\nu}^{n} P_k\xi_k \\ &= \sum_{k=0}^{k_\nu-1} P_k\xi_k + \Big(\sum_{k=k_\nu}^{n} P_k\Big)\xi_{k_\nu} = \sum_{k=0}^{k_\nu-1} P_k\xi_k + Q(n,k_\nu)\,\xi_{k_\nu}. \end{aligned}$$

Therefore we have

$$\begin{aligned} \xi_{k_\nu} &= Q^{-1}(n,k_\nu)\Big(R_n^{-1}\eta_n - \sum_{k=0}^{k_\nu-1} P_k\xi_k\Big) \\ &= Q^{-1}(n,k_\nu)\,R_n^{-1}\eta_n - Q^{-1}(n,k_\nu)\,R_{k_\nu-1}^{-1}\eta_{k_\nu-1} \end{aligned}$$

and

$$\begin{aligned} \lambda_{k_\nu}(\xi_{k_\nu} - \eta) &= \lambda_{k_\nu} Q^{-1}(n,k_\nu)\left(R_n^{-1}\eta_n - R_{k_\nu-1}^{-1}\eta_{k_\nu-1} - R_n^{-1}\eta + R_{k_\nu-1}^{-1}\eta\right) \\ &= Q^{-1}(n,k_\nu)\,\lambda_{k_\nu}\left(R_n^{-1}(\eta_n-\eta) - R_{k_\nu-1}^{-1}(\eta_{k_\nu-1}-\eta)\right) \\ &= Q^{-1}(n,k_\nu)\,\frac{\lambda_{k_\nu}}{\lambda_n} R_n^{-1}\gamma_n - Q^{-1}(n,k_\nu)\,\frac{\lambda_{k_\nu}}{\lambda_{k_\nu-1}} R_{k_\nu-1}^{-1}\gamma_{k_\nu-1}, \end{aligned}$$

where $\gamma_k = \lambda_k(\eta_k - \eta)$ and $\lim \eta_k = \eta$. Since $0 < \lambda_k \nearrow \infty$, the conditions (2.6) and (2.10) are satisfied we get

$$\frac{\lambda_{k_\nu}}{\lambda_n} = O(1), \qquad \frac{\lambda_{k_\nu}}{\lambda_{k_\nu-1}} = O(1).$$

From (2.11) we get $\gamma_k = O(1)$. Therefore using (2.7) and (2.8) it follows that

$$\lambda_{k_\nu}(\xi_{k_\nu} - \eta) = O(1). \tag{2.13}$$

Using (2.9), (2.10) and (2.13) we get that (2.12) is fulfilled. □

Corollary 1. (see also [18] and [19]). If $I \in \mathcal{L}(X, X)$ is the identity operator,

$$P_k = p_k I \qquad (k \in \mathbf{N}_0, \ \ p_k \in \mathbf{R}) \tag{2.14}$$

and the conditions (2.4), (2.6), (2.9), (2.10), (2.11),

$$\sum_{\nu=0}^{k} p_\nu \neq 0 \qquad (k \in \mathbf{N}_0), \qquad \sum_{k=k_\nu}^{n} p_k \neq 0, \tag{2.15}$$

$$\Big|\sum_{k=0}^{n} p_k\Big| = O(1)\Big|\sum_{k=k_\nu}^{n} p_k\Big|, \tag{2.16}$$

$$\Big|\sum_{k=0}^{k_\nu-1} p_k\Big| = O(1)\Big|\sum_{k=k_\nu}^{n} p_k\Big| \tag{2.17}$$

are satisfied, then the assertion (2.12) is valid.

Proof. Taking (2.14) we get

$$R_n^{-1} = \Big(\sum_{k=0}^{n} p_k\Big) I \in \mathcal{L}(X, X) \ \ \text{and} \ \ \|R_n^{-1}\| = \Big|\sum_{k=0}^{n} p_k\Big|.$$

As $Q(n.k_\nu) = \left(\sum_{k=k_\nu}^{n} p_k\right) I$ and the condition (2.15) is satisfied, then

$$Q^{-1}(n, k_\nu) = \Big(\sum_{k=k_\nu}^{n} p_k\Big)^{-1} I \in \mathcal{L}(X, X),$$

$$\|Q^{-1}(n, k_\nu)\| = \Big|\sum_{k=k_\nu}^{n} p_k\Big|^{-1}.$$

Using (2.16) and (2.17) we get that the conditions (2.7) and (2.8) are satisfied. So the conditions of Proposition 1 are satisfied and the assertion of Corollary 1 follows from the assertion of Proposition 1. □

Corollary 2. If the sequences $k = (k_\nu)$, $x = (\xi_\nu)$, $\lambda = (\lambda_\nu)$ and (B_ν) satisfy the conditions (2.2), (2.3), (2.4), (2.6), (2.9), (2.10),

$$(B_{n+1} - B_{k_\nu})^{-1} \in \mathcal{L}(X, X), \tag{2.18}$$

$$\left\|(B_{n+1} - B_{k_\nu})^{-1} B_{n+1}\right\| = O(1), \tag{2.19}$$

$$\left\|(B_{n+1} - B_{k_\nu})^{-1} B_{k_\nu}\right\| = O(1) \tag{2.20}$$

and the requirement $\mathcal{Z}x \in m_X^\lambda$ is satisfied, then (2.12) is valid.

Proof. Let us use Proposition 1, taking $P_k = B_{k+1} - B_k$. We get

$$Q(n, k_\nu) = B_{n+1} - B_{k_\nu}, \quad Q^{-1}(n, k_\nu) = (B_{n+1} - B_{k_\nu})^{-1}.$$

As $R_n^{-1} = B_{n+1}$ and $B_n \in \mathcal{L}(X, X)$, then $R_n^{-1} \in \mathcal{L}(X, X)$. The conditions $R_n^{-1} \in \mathcal{L}(X, X)$ and (2.18) imply that the condition (2.5) is satisfied. From (2.19) we get (2.7) and from (2.20) we get (2.8). As the conditions (2.4), (2.6), (2.9), (2.10) and (2.11) are satisfied, then the assertion of Corollary 2 follows from the assertion of Proposition 1. □

3 Generalized Zygmund Method and Gap Tauberian Remainder Theorem

Let us define the generalized method $\mathcal{Z}^{(1)}$ as $(\mathcal{Z}, B_n)$. Using the recurrent relation $\mathcal{Z}^{(m)} = \mathcal{Z}^{(1)}\mathcal{Z}^{(m-1)}$ we define the method $\mathcal{Z}^{(m)}$ for $m \in \mathbf{N}\backslash\{1\}$. If $B_k = kI$, then

$$\mathcal{Z}^{(m)} = \mathcal{H}^{(m)} \qquad (m \in \mathbf{N}),$$

where $\mathcal{H}^{(m)}$ is the generalized Hölder method of order m. Let $\eta_\nu^{(0)} = \xi_\nu$ for $\nu \in \mathbf{N}_0$. So we have

$$\eta_n^{(m)} = B_{n+1}^{-1} \sum_{\nu=0}^{n} (B_{\nu+1} - B_\nu)\, \eta_\nu^{(m-1)} \quad (m \in \mathbf{N}). \tag{3.1}$$

Lemma 1. *If the sequences $k = (k_\nu)$, $x = (\xi_i)$ and (B_ν) satisfy the conditions (2.2), (2.3), (2.4) and (2.9), then for $k_\nu \le k_\nu + p < k_{\nu+1}$ we have*

$$\eta_{k_\nu+p}^{(1)} = B_{k_\nu+p+1}^{-1}\Big[\sum_{i=1}^{\nu} (B_{k_i} - B_{k_{i-1}})\, \xi_{k_{i-1}} + (B_{k_\nu+p+1} - B_{k_\nu})\, \xi_{k_\nu}\Big], \tag{3.2}$$

$$\Delta\eta_{k_\nu+p}^{(1)} = \left(B_{k_\nu+p}^{-1} - B_{k_\nu+p+1}^{-1}\right) \sum_{i=1}^{\nu} B_{k_i}\left(\xi_{k_i} - \xi_{k_{i-1}}\right). \tag{3.3}$$

Proof. Let $k_0 = 0$ and $p \ne 0$. Using (3.1), (2.4) and (2.9) we get

$$\begin{aligned} \eta_{k_\nu+p}^{(1)} &= B_{k_\nu+p+1}^{-1}\Big(\sum_{j=0}^{\nu-1}\sum_{i=k_j}^{k_{j+1}-1} (B_{i+1} - B_i)\, \xi_{k_j} + \sum_{i=k_\nu}^{k_\nu+p} (B_{i+1} - B_i)\, \xi_{k_\nu}\Big) \\ &= B_{k_\nu+p+1}^{-1}\Big(\sum_{i=1}^{\nu} (B_{k_i} - B_{k_{i-1}})\, \xi_{k_{i-1}} + (B_{k_\nu+p+1} - B_{k_\nu})\, \xi_{k_\nu}\Big). \end{aligned}$$

So the assertion (3.2) is valid. Using (3.2) we for $k_\nu < k_\nu + p < k_{\nu+1}$ conclude

$$\begin{aligned} \Delta\eta_{k_\nu+p}^{(1)} &= \eta_{k_\nu+p}^{(1)} - \eta_{k_\nu+p-1}^{(1)} \\ &= \left(B_{k_\nu+p+1}^{-1} - B_{k_\nu+p}^{-1}\right) \sum_{i=1}^{\nu} (B_{k_i} - B_{k_{i-1}})\xi_{k_{i-1}} + (B_{k_\nu+p} - B_{k_\nu+p+1}) B_{k_\nu} \xi_{k_\nu}. \end{aligned}$$

That is why using Abel's identity for $p \neq 0$ we get that the assertion (3.3) is true. Analogously can be proved that the assertions (3.2) and (3.3) are valid for $p = 0$. $\square$

Lemma 2. *If* $k_\nu \leq k_\nu + p < k_{\nu+1}$, $p \in \boldsymbol{N}_0$ *and the sequences* $k = (k_\nu)$, $x = (\xi_\nu)$, $\lambda = (\lambda_\nu)$ *and* (B_ν) *satisfy the conditions* (2.2), (2.3), (2.4) *and* (2.9),

$$\nu \lambda_{k_\nu+p} \left\| B_{k_\nu+p} \right\| \left\| B_{k_\nu+p}^{-1} - B_{k_\nu+p+1}^{-1} \right\| = O\left(1\right), \tag{3.4}$$

$$\left\| B_{k_\nu} \right\| \left\| \xi_{k_\nu} - \xi_{k_{\nu-1}} \right\| = O\left(1\right), \tag{3.5}$$

then we have that

$$\lambda_k \left\| B_k \right\| \left\| \Delta \eta_k^{(1)} \right\| = O\left(1\right). \tag{3.6}$$

Proof. If $k_\nu \leq k_\nu + p < k_{\nu+1}$, then using (3.3) we get

$$\begin{aligned} &\lambda_{k_\nu+p} \left\| B_{k_\nu+p} \right\| \left\| \Delta \eta_{k_\nu+p}^{(1)} \right\| \\ &\qquad \leq \lambda_{k_\nu+p} \left\| B_{k_\nu+p} \right\| \left\| B_{k_\nu+p}^{-1} - B_{k_\nu+p+1}^{-1} \right\| \sum_{i=1}^{\nu} \left\| B_{k_i} \right\| \left\| \xi_{k_i} - \xi_{k_{i-1}} \right\|. \end{aligned}$$

Applying (3.4) and (3.5) we get

$$\lambda_{k_\nu+p} \left\| B_{k_\nu+p} \right\| \left\| \Delta \eta_{k_\nu+p}^{(1)} \right\| = O\left(1\right).$$

That means the assertion (3.6) is valid. $\square$

Lemma 3. *If* $m \in \boldsymbol{N}$ *and the sequences* $k = (k_\nu)$, $x = (\xi_\nu)$, $\lambda = (\lambda_\nu)$ *and* (B_ν) *satisfy the conditions* (2.2), (2.3),

$$\lambda_n \left\| B_n \right\| \left\| \Delta \eta_n^{(m)} \right\| = O\left(1\right), \tag{3.7}$$

$$\lambda_n \left\| B_n \right\| \left\| B_n^{-1} - B_{n+1}^{-1} \right\| \sum_{k=1}^{n} 1/\lambda_k = O\left(1\right), \tag{3.8}$$

then we get that

$$\lambda_n \left\| B_n \right\| \left\| \Delta \eta_n^{(m+1)} \right\| = O\left(1\right). \tag{3.9}$$

Proof. As

$$\eta_n^{(m+1)} = B_{n+1}^{-1} \sum_{k=0}^{n} \left(B_{k+1} - B_k\right) \eta_k^{(m)},$$

then

$$\begin{aligned}\Delta\eta_n^{(m+1)} &= \eta_n^{(m+1)} - \eta_{n-1}^{(m+1)} \\ &= B_{n+1}^{-1}\sum_{k=0}^{n}\left(B_{k+1}-B_k\right)\eta_k^{(m)} - B_n^{-1}\sum_{k=0}^{n-1}\left(B_{k+1}-B_k\right)\eta_k^{(m)} \\ &= \left(B_{n+1}^{-1}-B_n^{-1}\right)\sum_{k=1}^{n}\left(B_k-B_{k-1}\right)\eta_{k-1}^{(m)} + B_{n+1}^{-1}\left(B_{n+1}-B_n\right)\eta_n^{(m)}.\end{aligned}$$

Using Abel's identity we get

$$\Delta\eta_n^{(m+1)} = \left(B_n^{-1} - B_{n+1}^{-1}\right)\sum_{k=1}^{n} B_k\Delta\eta_k^{(m)}.$$

So we have

$$\lambda_n\left\|B_n\right\|\left\|\Delta\eta_n^{(m+1)}\right\| \le \lambda_n\left\|B_n\right\|\left\|B_n^{-1}-B_{n+1}^{-1}\right\|\sum_{k=1}^{n}\left\|B_k\right\|\left\|\Delta\eta_k^{(m)}\right\|.$$

Therefore using the conditions (3.7) and (3.8) we get

$$\lambda_n\left\|B_n\right\|\left\|\Delta\eta_n^{(m+1)}\right\| = O\left(1\right)\lambda_n\left\|B_n\right\|\left\|B_n^{-1}-B_{n+1}^{-1}\right\|\sum_{k=1}^{n}1/\lambda_k = O\left(1\right)$$

and the assertion (3.9) is valid. □

Lemma 4. *If* $m \in \boldsymbol{N}$ *and the conditions* (2.2), (2.3), (3.7) *and*

$$\lambda_n\left\|B_{n+1}^{-1}\right\|\sum_{k=1}^{n}1/\lambda_k = O\left(1\right) \tag{3.10}$$

are satisfied, then

$$\lambda_n\left(\eta_n^{(m+1)} - \eta_n^{(m)}\right) = O\left(1\right). \tag{3.11}$$

Proof. As

$$\eta_n^{(m+1)} - \eta_n^{(m)} = B_{n+1}^{-1}\sum_{k=1}^{n-1}\left(B_{k+1}-B_k\right)\left(\eta_k^{(m)} - \eta_n^{(m)}\right),$$

then using Abel's identity we get

$$\eta_n^{(m+1)} - \eta_n^{(m)} = -B_{n+1}^{-1}\sum_{k=1}^{n}B_k\Delta\eta_k^{(m)}$$

This connection and the conditions (3.7) and (3.10) imply

$$\lambda_n\left\|\eta_n^{(m+1)} - \eta_n^{(m)}\right\| = O\left(1\right)\lambda_n\left\|B_{n+1}^{-1}\right\|\sum_{k=1}^{n}\lambda_k\left\|B_k\right\|\left\|\Delta\eta_k^{(m)}\right\|/\lambda_k$$

and

$$\lambda_n \left\| \eta_n^{(m+1)} - \eta_n^{(m)} \right\| = O(1)\, \lambda_n \left\| B_{n+1}^{-1} \right\| \sum_{k=1}^{n} 1/\lambda_k = O(1)\,.$$

So the assertion (3.11) is valid. □

Proposition 2. *If* $m \in \boldsymbol{N} \setminus \{1\}$ *and the sequences* $k = (k_\nu)\,,\ x = (\xi_\nu)\,,\ \lambda = (\lambda_\nu)$ *and* (B_ν) *satisfy the conditions* (2.2)*,* (2.3)*,* (2.4)*,* (2.9)*,* (2.10)*,* (3.4)*,* (3.5)*,* (3.8)*,* (3.10) *and* $\mathcal{Z}^{(m)} x \in m_X^\lambda$*, then* $\mathcal{Z}^{(1)} x \in m_X^\lambda$*.*

Proof. The condition $\mathcal{Z}^{(m)} x \in m_X^\lambda$ means that

$$\lambda_n \left(\eta_n^{(m)} - \eta \right) = O(1)\,,$$

where $\mathcal{Z}^{(m)} x = \left(\eta_n^{(m)} \right)$ and $\lim \eta_n^{(m)} = \eta$. We have

$$\left\| \eta_n^{(m-1)} - \eta \right\| \le \left\| \eta_n^{(m-1)} - \eta_n^{(m)} \right\| + \left\| \eta_n^{(m)} - \eta \right\|. \tag{3.12}$$

Lemmas 1 and 2 imply, that (3.6) is valid. Therefore using Lemma 3 we get that the assertion (3.9) is valid for every $m \in \mathbf{N}$. Using Lemma 4 we get that the assertion (3.11) is valid for every $m \in \mathbf{N}$. So using (3.12) we get that $\mathcal{Z}^{(m-1)} x \in m_X^\lambda$. Step by step we prove that the assertion $\mathcal{Z}^{(1)} x \in m_X^\lambda$ is valid. □

Remark 1. Taking in Proposition 2 $B_k = kI$ $(k \in \mathbf{N}_0)$ we get the conditions under which

$$\mathcal{H}^{(m)} x \in m_X^\lambda \quad \Rightarrow \quad H^{(1)} x \in m_X^\lambda,$$

where $\mathcal{H}^{(m)}$ $(m \in N \setminus \{1\})$ is the generalized Hölder method of order m.

Corollary 3. If $m \in \mathbf{N} \setminus \{1\}$ and the sequences $k = (k_\nu)\,,\ x = (\xi_\nu)\,,\ \lambda = (\lambda_\nu)$ and (B_ν) satisfy the conditions (2.2), (2.3), (2.4), (2.9), (2.10), (2.18), (2.19), (2.20), (3.4), (3.5), (3.8), (3.10) and $\mathcal{Z}^{(m)} x \in m_X^\lambda$, then the assertion (2.12) is valid.

Proof. If $m \in \mathbf{N} \setminus \{1\}$, then using Proposition 2 and Corollary 2 we get that the assertion (2.12) is valid. □

4 Conclusions

- We obtain several new results for the linear methods of summability, where the elements of the matrix are the linear operators from Banach space X into X.
- We prove a gap Tauberian remainder theorem for the generalized Riesz method of summability.
- We prove a gap Tauberian remainder theorem for the generalized Zygmund method of summability.

- We draw several conclusions from these gap Tauberian remainder theorems.
- Among other conclusions we get a gap Tauberian remainder theorem for the generalized Hölder method of summability.

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