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ON SOLUTIONS OF NEUMANN BOUNDARY VALUE PROBLEM FOR THE LIÉNARD TYPE EQUATION [1]

S. ATSLEGA

Daugavpils University

Parades Str. 1, LV-5400 Daugavpils, Latvia

E-mail: oglana@tvnet.lv

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Abstract. We provide conditions on the functions $f(x)$ and $g(x)$, which ensure the existence of solutions to the Neumann boundary value problem for the equation $x'' + f(x)x'^2 + g(x) = 0$.

Key words: Neumann boundary value problem, Liénard equation, critical points, homoclinic solutions, conservative equation.

1 Introduction

Intensive literature is devoted to investigation of the Liénard equation

$$x'' + f(x)x' + g(x) = 0 \tag{1.1}$$

due to its importance in applications. This type equations were intensely studied as they can be used to model oscillating circuits. Of applications of the Liénard equation see the book [[7],Ch. XII, §3] and references therein.

Existence (also nonexistence) of periodic solutions is the main subject of investigations. This depends of course on properties of functions f and g. Burton points out that phase portraits for (1.1) are well known if the function $f(x)$ is supposed to be positive and $g(x)$ is assumed to be odd, that is, $xg(x) > 0$ ([2], [3]). Let $F(x) = \int_0^x f(s)\,ds$. The existence of periodic solutions of (1.1) was studied in [6] provided that $F(x)$ can change sign and the amplitudes of $F(x)$ are decreasing. The function $g(x)$ was supposed to be of odd type.

On the other hand, it is known that conservative equation

$$x'' + g(x) = 0 \tag{1.2}$$

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always has periodic solutions if function $g(x)$ has simple zeros where $g'(x) > 0$. The equivalent system

$$\begin{cases} x' = y, \\ y' = -g(x) \end{cases} \tag{1.3}$$

then has critical points of the type "center" and "small"-amplitude periodic solutions appear. In the case of $xg(x) > 0$ the only critical point is $(0;0)$ and a set (continuum) of closed curves exist in a neighborhood of the critical point.

If function $g(x)$ satisfies the condition

$$xg(x) > 0 \ \text{ for } \ x \in (-\infty, p_1) \cup (p_{i+1}, +\infty),$$

where $p_1 < p_{i+1}$ and there exist $(i-1)$ simple $(g' \neq 0)$ zeros in (p_1, p_{i+1}), then equation (1.2) may have "large"-amplitude periodic solutions. The respective closed orbits go around several critical points. For details one may consult [4].

From the point of view of the boundary value problems (BVP) periodic solutions may give rise to solutions which satisfy some boundary conditions. We have considered the Neumann BVP for conservative equation (1.2) in t [1]. In this paper we obtain a similar result for equation (1.1) subjected to the Neumann boundary conditions

$$x'(0) = 0, \quad x'(1) = 0. \tag{1.4}$$

Recently an article by Sabatini was published [5], where the equation

$$x'' + f(x)x'^2 + g(x) = 0 \tag{1.5}$$

was studied. Among other things the transformation was presented which turns equation (1.5) to the conservative form

$$u'' + h(u) = 0. \tag{1.6}$$

In this paper we use this transformation to get the results on the existence and multiplicity of solutions to the Neumann BVP using previously obtained related results for conservative equation. In Section 2 we recall the results for conservative equation (1.2). In Section 3 reduction of equation (1.5) to (1.6) is considered. Results for equation (1.5) are established in Section 4. Examples illustrating the results are given in Section 5.

2 Analysis of Equation $x'' + g(x) = 0$

Let $g(x)$ be a continuously differentiable function like in Fig. 1. Zeros of $g(x)$ are $p_1 < p_2 < p_3 < p_4 < p_5$. The equivalent system has three saddle points at $(p_1, 0)$, $(p_3, 0)$, $(p_5, 0)$ and centers at $(p_2, 0)$ and $(p_4, 0)$.

The typical phase portrait is given in Fig. 2. There are two sets of "small" amplitude periodic solutions located in neighborhoods of $(p_2, 0)$ and $(p_4, 0)$.

No other nontrivial periodic solutions exist if three local maxima of the primitive $G(x)$ are such that either $G(p_1) > G(p_3) > G(p_5)$ or $G(p_1) < G(p_3) < G(p_5)$ or $G(p_5) < G(p_1) < G(p_3)$ or $G(p_1) < G(p_5) < G(p_3)$.

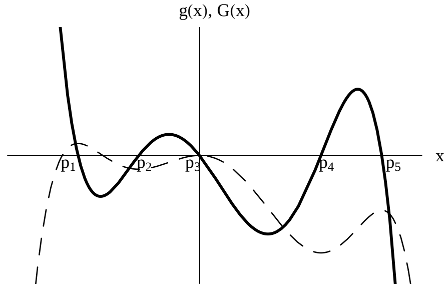

Figure 1. Functions $g(x)$ and $G(x)$ (the primitive).

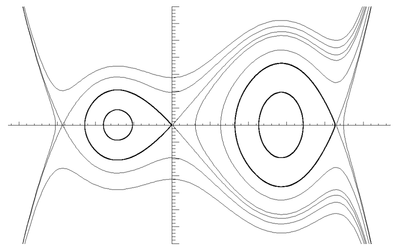

Figure 2. The phase plane.

Theorem 1. *Let the conditions*

$$m_1^2\pi^2 < |g_x(p_2)| < (m_1+1)^2\pi^2, \quad m_2^2\pi^2 < |g_x(p_4)| < (m_2+1)^2\pi^2$$

hold. Then the Neumann BVP (1.2), (1.4) *has at least* $2m_1+2m_2$ *nonconstant "small"-amplitude periodic solutions.*

Situation is quite different if $G(p_3)$ is less than $G(p_1)$ and $G(p_5)$. Then appear "large" amplitude periodic solutions like given in Fig. 4.

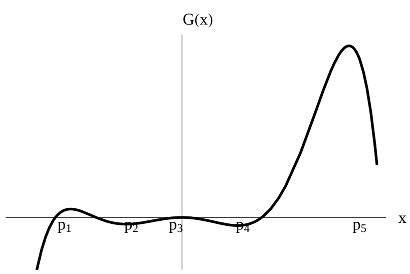

Figure 3. The primitive $G(x)$.

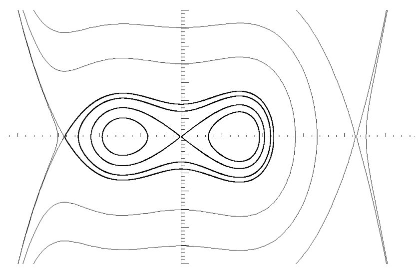

Figure 4. The phase plane for the case $G(p_3) < G(p_1) < G(p_5)$.

Theorem 2. *If* $G(p_3)$ *is less (strictly) than* $G(p_1)$ *and* $G(p_5)$, *then equation* (1.2) *has "large"-amplitude periodic solutions, that is, solutions with trajectories going around the critical points* $(p_2;0)$ *and* $(p_4;0)$.

Consider the function

$$T(x_0) = \frac{1}{\sqrt{2}} \int_{x_0}^{x_1(x_0)} \frac{ds}{\sqrt{G(s)-G(x_0)}},$$

which is defined in the interval $(x_0^*; x_0^{**})$, where $x_1(x_0)$ is the first zero to the right of x_0 of the function $G(s)-G(x_0)$ and x_0^*, x_0^{**} are the minimal values of the respective homoclinic solutions.

Theorem 3. *Suppose that an integer* k *is such that*

$$nT_{min} < 1 < (n+1)T_{min}.$$

Then the Neumann BVP (1.2), (1.4) *has at least* $4n$ *nonconstant "large"-amplitude periodic solutions.*

Theorems 1, 2 and 3 were proved in [1].

3 Reduction of $x'' + f(x)x'^2 + g(x) = 0$ to $u'' + h(u) = 0$

Let $F(x) = \int_0^x f(s)\,ds$ and $G(x) = \int_0^x g(s)\,ds$. The function $\Phi(x)$ was introduced in [5] by the formula

$$\Phi(x) = \int_0^x e^{F(s)}\,ds. \tag{3.1}$$

It is evident that $\Phi(x)$ satisfies the condition $x\Phi(x) > 0$ for $x \neq 0$. The growth rate of $\Phi(x)$ depends on properties of the primitive $F(x)$. It is important that $\Phi(x)$ is strictly monotone function for any F since $\Phi'(x) = e^{F(x)} > 0\ \forall x \in \mathbb{R}$. Then the relation

$$\Phi(x) = u \tag{3.2}$$

defines $u = u(x)$ and the inverse function $x = x(u)$ exists. We will use these functions defined for various F throughout in our considerations. Our further study employs the following basic result from [5].

Lemma 1 [[5], Lemma 1]. *The function* $x(t)$ *is a solution to* (1.5) *if and only if* $u(t) = \Phi(x(t))$ *is a solution to*

$$u'' + g(x(u))e^{F(x(u))} = 0. \tag{3.3}$$

Denote $h(u) = g(x(u))e^{F(x(u))}$. Therefore, $H(u) = \int_0^u g(x(s))e^{F(x(s))}\,ds$. The existence of periodic solutions and the existence of solutions to the Neumann BVP depends entirely on properties of the primitive H.

Let us state some easy assertions about equation (1.5) and the equivalent system

$$\begin{cases} x' &= \quad y, \\ y' &= -f(x)y^2 - g(x). \end{cases} \tag{3.4}$$

Proposition 1. *Critical points and their character are the same for systems* (1.3) *and* (3.4).

Proof. Critical points of both systems are the points $(x_i, 0)$, where x_i are zeros of $g(x)$. Points $(p_1, 0)$, $(p_3, 0)$, $(p_5, 0)$ are saddle points and $(p_2, 0)$ and $(p_4, 0)$ are the centers. Consider linearized at a point $(p_i, 0)$ system (1.3)

$$\begin{cases} \xi' &= \quad \eta, \\ \eta' &= -g_x(p_i)\xi, \end{cases} \tag{3.5}$$

where p_i is a zero of $g(x)$. Consider also linearized at a point $(p_i, 0)$ system (3.4)

$$\begin{cases} \alpha' &= \beta, \\ \beta' &= -[f_x(x)y + g_x(x)]\alpha + [2f(x)y]\beta \\ &= -g_x(x)\alpha, \quad \text{at} \quad \mathrm{(x,y) = (p_i, 0)}, \end{cases}$$

i.e. we have the system

$$\begin{cases} \alpha' &= \beta, \\ \beta' &= -g_x(p_i)\alpha. \end{cases} \tag{3.6}$$

Systems (3.5) and (3.6) up to notation are the same. □

Consider a system

$$\begin{cases} u' &= v, \\ v' &= -g(x(u))e^{F(x(u))}, \end{cases} \tag{3.7}$$

which is equivalent to equation (3.3).

Proposition 2. *Critical points* $(x, 0)$ *and* $(u(x), 0)$ *of systems* (1.3) *and* (3.7) *are in 1-to-1 correspondence and their characters are the same.*

Proof. The first follows from the fact that if p is a zero of $g(x)$ then $u(p)$ is a zero of $g(x(u))$. The second follows from Proposition 1. □

Proposition 3. *Periodic solutions* $x(t)$ *of equation* (1.5) *turn to periodic solutions* $u(t) = \Phi(x(t))$ *by transformation* (3.2).

Proposition 4. *Homoclinic solutions of* (1.5) *turn to homoclinic solutions of equation* (3.3) *by transformation* (3.2).

Proposition 5. *Let* p *be a zero of* $g(x)$. *Then* $g_x(p) = h_u(u(p))$.

Proof. Let us differentiate $h(u)$. Notice that from (3.1) and (3.2)

$$\frac{du}{dx} = e^F, \quad \frac{dx}{du} = e^{-F}.$$

Then

$$h_u = \frac{dg(x(u))}{du\, e^F(x(u))} + \frac{g(x(u))de^F(x(u))}{du} = \frac{dg(x)}{dx}\frac{dx}{du\; e^F(x(u))} + \frac{g(x(u))F(x(u))dx}{du\; e^F(x(u))}$$

$$= g_x(x)e^{-F(x(u))}e^F(x(u)) + g(x)F(x(u))e^{-F(x(u))}e^F(x(u))$$

$$= g_x(x) + g(x)F(x(u)).$$

Then

$$h_u(u(p)) = g_x(p) + g(p)F(p) = g_x(p).$$

□

4 The Neumann BVP

Consider equation (3.3). This equation is conservative and Theorem 2 applies. The function $H(u) = \int_0^u g(x(s))e^{F(x(s))}\,ds$ has the same structure as $G(x)$ thus it has exactly 3 points of maxima and 2 minimum points. Moreover, $H(u)$ has three local maxima at the points $u(p_1)$, $u(p_3)$ and $u(p_5)$, where u is as in (3.2), and two local minima at the points $u(p_2)$ and $u(p_4)$.

Theorem 4. *Let the conditions*

$$m_1^2\pi^2 < |g_x(p_2)| < (m_1+1)^2\pi^2, \quad m_2^2\pi^2 < |g_x(p_4)| < (m_2+1)^2\pi^2$$

hold. Then the Neumann BVP (1.5), (1.4) *has at least* $2m_1+2m_2$ *nonconstant "small"-amplitude periodic solutions.*

Proof. By application of Theorem 1 to equation (1.6) and using the equalities $g_x(p) = h_u(u(p))$ at the points p_2 and p_4 we get the required result. □

Theorem 5. *Let the inequalities hold:*

$$H(u(p_3)) < H(u(p_1)), \quad H(u(p_3)) < H(u(p_5)).$$

Then equation (1.6) *has "large"-amplitude periodic solutions which enclose exactly two critical points* $u(p_2)$ *and* $u(p_4)$*. If there is a "large"-amplitude solution with the half of minimal period* T *such that* $nT < 1 < (n+1)T$*, then there exist also at least* $4n$ *solutions to the Neumann BVP.*

Proof. Consider equation (1.6). Let us parametrize the set of "large"–amplitude solutions by $u(0)$. Assume that $u_* = u(0)$ for a solution with the period T. We change $u(0)$ in the opposite direction towards the values corresponding to homoclinic solutions (like in the proof of Theorem 2 in [1]). Since then period of solutions tends to $+\infty$, we got at least $2n$ solutions. In addition we note that all trajectories are symmetric with respect to the u-axis, therefore there exist $2n$ solutions which are monotonically decreasing since their trajectories (halves of the closed ones) are in the lower half-plane $\{(u,u') : \ u' < 0\}$. Thus there exist at least $4n$ solutions. The theorem is proved. □

5 Example

In this section we give two examples which illustrate theoretical results presented above.

Example 1. Consider the second-order nonlinear boundary value problem

$$\begin{cases} x'' - x^5 + 3x^4 + 13x^3 - 27x^2 - 36x = 0, \\ x'(0) = x'(1) = 0. \end{cases} \tag{5.1}$$

Function $g(x) = -x^5 + 3x^4 + 13x^3 - 27x^2 - 36x$ has five simple zeros. The equivalent two–dimensional system has three critical points of the type "saddle". There are also two critical points of the type "center". Respectively the function

$$G(x) = -\frac{1}{6}x^6 + \frac{3}{5}x^5 + \frac{13}{4}x^4 - 9x^3 - 18x^2$$

has three local minima and consequently two local maxima as it is shown in Fig. 5.

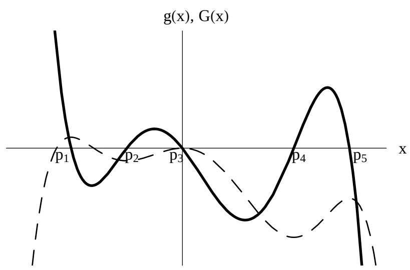

Figure 5. Functions $g(x)$ and $G(x)$ (the primitive).

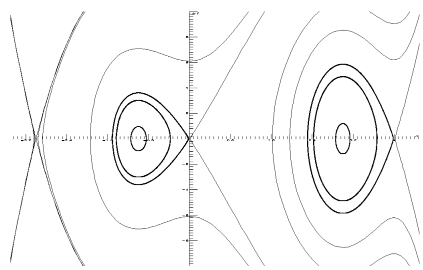

Figure 6. The phase plane.

We notice that if $g_x(p_2) = g_x(-1) = -40$, then it follows that inequalities $2^2\pi^2 < 40 < 3^2\pi^2$ hold, and therefore boundary value problem (5.1) has at least 4 nonconstant solutions. Similarly, if $g_x(p_4) = g_x(3) = -72$, then the condition $2^2\pi^2 < 72 < 3^2\pi^2$ holds. So boundary value problem (5.1) has at least 4 nonconstant solutions (see Fig. 7). Hence the BVP(5.1) has at least 8

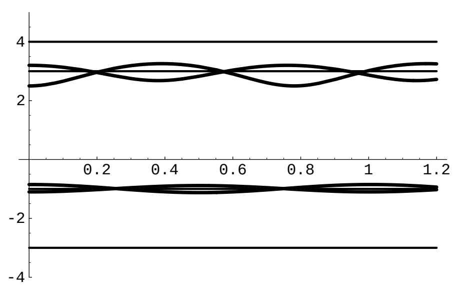

Figure 7. Solutions in neighborhoods of the p_2 and p_4.

nonconstant solutions.

Example 2. Consider the second-order nonlinear boundary value problem

$$\begin{cases} x'' + 0.5xx'^2 - x^5 + 3x^4 + 13x^3 - 27x^2 - 36x = 0, \\ x'(0) = x'(1) = 0 \end{cases} \tag{5.2}$$

with $f(x) = 0.5x$ and $g(x) = -x^5 + 3x^4 + 13x^3 - 27x^2 - 36x$. Then we get

$$F(x) = \int_0^x \frac{s}{2}\,ds = \frac{x^2}{4}, \quad G(x) = \int_0^x g(s)\,ds = -\frac{1}{6}x^6 + \frac{3}{5}x^5 + \frac{13}{4}x^4 - 9x^3 - 18x^2.$$

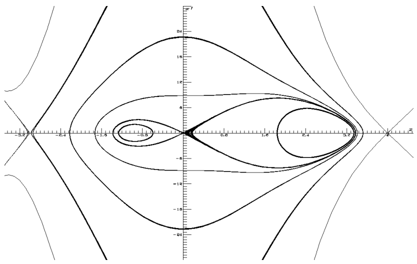

Figure 8. Phase plane.

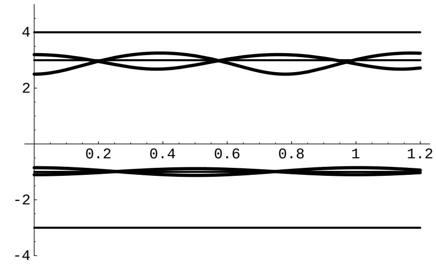

Figure 9. Solutions of (5.2) in neighbourhoods of the p_2 and p_4.

The phase plane is presented in Fig. 8. The transformation turns equation (5.2) to the conservative form

$$u'' + h(u) = 0, \quad \text{where } \ h(u) = g\big(x(u)\big)e^{0.25(x(u))^2}.$$

It follows from $g_x(p_2) = g_x(-1) = -40$, $g_x(p_4) = g_x(3) = -72$ that the conditions

$$2^2\pi^2 < 40 < 3^2\pi^2, \quad 2^2\pi^2 < 72 < 3^2\pi^2$$

hold. Then by Theorem 4 BVP (5.2) has at least 8 solutions (see Fig. 9).

A respective two-dimensional system has exactly 5 critical points:

$$u_1 = u(-3) = -8.12623, \quad u_2 = u(-1) = -1.08997, \quad u_3 = u(0) = 0,$$
$$u_4 = u(3) = 8.12623, \quad u_5 = u(4) = 32.9053.$$

After simple computations we get that

$$H(u_1) = 1406.39, \quad H(u_3) = 0, \quad H(u_5) = 11201.7.$$

Suppose that $H(u)$ is such that

$$H(u(p_3)) < H(u(p_1)), \quad H(u(p_3)) < H(u(p_5)).$$

Then by Theorem 5 equation (5.2) has "large"-amplitude periodic solutions which enclose exactly two critical points $u(p_2)$ and $u(p_4)$. Numerical experiment shows that there exists T such that $T < 1 < 2T$. So the boundary value problem (5.2) has at least 4 "large" solutions due to Theorem 5 (see Fig. 10).

Hence the BVP (5.2) has at least 12 solutions.

6 Conclusions

Conservative equations $x''+g(x) = 0$ have families of "large" amplitude periodic solutions if profile of the primitive $G(x)$ is such that there are local maxima which are bigger than the intermediate maxima. These periodic solutions may satisfy the Neumann boundary conditions if the rotation speed along closed

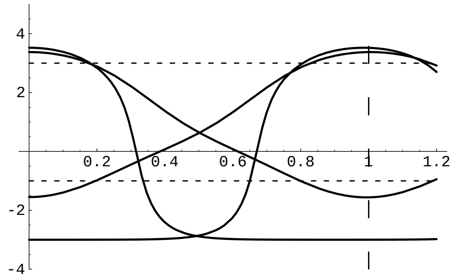

Figure 10. Solutions of problem (5.2) in neighbourhoods of the p_2 and p_4.

phase trajectories is relatively high. The Liénard type equation $x'' + f(x)x'^2 + g(x) = 0$ can be reduced to a conservative equation $u''+h(u) = 0$. The primitive $H(u) = \int_0^u h(s)\,ds$ can be expressed in terms of functions $f(x)$ and $g(x)$. It is possible therefore that for some appropriate f and g the behavior of this primitive ensures the existence of families of "large" amplitude solutions. If the rotation speed along the trajectories is "high" (or the time period is "small" enough) then the Liénard type equation has multiple solutions of the Neumann boundary value problem.

References

[1] S. Atslega. Multiplicity results for the Neumann boundary value problem. *Mathematical Modelling and Analysis*, **12**(2):179–186, 2007.

[2] T.A. Burton. The nonlinear wave equation as a Liénard equation. *Funkcialaj Ekvacioj*, **34**:529–545, 1991.

[3] T. Hara and T. Yoneyama. On the global center of generalized Liénard equation and its application to stability problems II. *Funkcialaj Ekvacioj*, **3**(2):221–225, 1988.

[4] S. Ogorodnikova and F. Sadyrbaev. Estimations of the number of solutions to some nonlinear second order boundary value problems. *Mathematics. Differential Equations (Univ. of Latvia, Institute of Math. and Comp. Sci.*, **5**:24–32 (available electronically at http://www.lumii.lv/math.htm), 2005.

[5] M. Sabatini. On the period function of $x'' + f(x)x'^2 + g(x) = 0$. *Diff. Equations*, **196**:151–168, 2004.

[6] S. Sakata. On the existence of periodic solutions of a Lienard system. *Funkcialaj Ekvacioj*, **37**(1):155–174, 1994.

[7] G. Sansone. *Ordinary Differential Equations, II.* Izdatelstvo Inostrannoi Literatury, Moscow, 1954. (in Russian)