Inverse Problems for a Generalized Subdiffusion Equation with Final Overdetermination

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Abstract. We consider two inverse problems for a generalized subdiffusion equation that use the final overdetermination condition. Firstly, we study a problem of reconstruction of a specific space-dependent component in a source term. We prove existence, uniqueness and stability of the solution to this problem. Based on these results, we consider an inverse problem of identification of a space-dependent coefficient of a linear reaction term. We prove the uniqueness and local existence and stability of the solution to this problem.

Keywords: inverse problem, subdiffusion, final overdetermination, fractional diffusion.

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1 Introduction

Anomalous diffusion processes are described by different models [6]. Among them stands out the time (or space-time) fractional diffusion equation that is the most common way to represent a subdiffusion. For some situations such approach does not work [19]. Therefore, more general models that unify wider range of subdiffusion processes are introduced [19, 25].

In this paper we use an operator that is more general than the fractional time derivative:

$$D_t^{\{k\}}v = \frac{d}{dt}k * v, \tag{1.1}$$

where $*$ denotes the time convolution, i.e. $(v_1 * v_2)(t) = \int_0^t v_1(t - \tau)v_2(\tau)d\tau$. Taken $k = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, (1.1) transforms into a well-known Riemann-Liouville frac-

tional derivative. The operator corresponding to the Caputo fractional derivative denoted as $\mathbb{D}^C_{(k)}v$ was introduced by Luchko and Yamamoto in [23] and also in [15].

The toolkit for treating such a type of derivative have been developed by Prüss et al. [5]. They have created a setting to introduce the operator inverse to $D_t^{\{k\}}$ through the concept of Completely Positive kernels [5]: a kernel $M \in L_{1,loc}(\mathbb{R}_+)$ is called completely positive if there are $k_0 \geq 0$ and nonnegative and nonincreasing $k_1 \in L_{1,loc}(\mathbb{R}_+)$ such that $M * (k_0\delta + k_1) = 1$ holds. The applications of this concept can be found in [1, 33, 34]. Another approach to this issue has been developed by Kochubei [19].

Often parameters of models are unknown. Then additional observations are performed and inverse problems solved to reconstruct unknown quantities [12, 13, 16, 17, 20, 21]. In the present paper we consider two inverse problems (IPs) that use final observation data: IP1 is to identify a space-dependent factor f of a source term $g(t,x)f(x)$; IP2 is to reconstruct a coefficient $r(x)$ of a linear reaction term.

IP1 for fractional and perturbed fractional diffusion equations is studied in several papers. Theoretical and numerical results are obtained in the particular case $g = g(t)$ [7,17,18,26] and in the case $g = g(t,x)$ [30,32]. In latter papers the existence and uniqueness of solutions are proved for almost all scalar diffusion coefficients. IP1 for a semilinear fractional diffusion equation is considered in [15]. Uniqueness of the solution is proved.

In this paper we consider IP1 for a more general diffusion equation that includes the operator (1.1) instead of the fractional derivative. We prove the uniqueness of the solution to IP1 by applying a modified version of the positivity principle from [15]. That falls into category of maximum principle results [13, 20, 22]. Similar approaches to the inverse problems are well-known in the domain of parabolic equations [2,12]. Next we prove the existence and stability of the solution of IP1 by means of the Fredholm alternative. The uniqueness of solution of IP2 follows from the IP1-results. Finally, we prove local existence and stability of the solution to IP2 by means of the contraction argument.

2 Formulation of direct and inverse problems

Let us consider the generalized subdiffusion equation

$$U_t(t,x) = (M * LU)_t(t,x) + Q(t,x), \qquad (2.1)$$

where U physical state, t is the time, $x \in \mathbb{R}^n$ is a space variable, Q is a source term, the operator $L = L(x)$ is such that

$$L(x) = L_1(x) + r(x)I, \text{ where } L_1(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n a_j(x)\frac{\partial}{\partial x_j}$$

and I is the unity operator. The kernel M is a memory function related to a non-locality of the diffusion process.

There are two ways to derive the equation (2.1) from physical laws. One method consists in modelling continuous time random walk processes in micro-level and taking a continuous limit in a macro-level [4] and another one uses conservative laws and specific constitutive relations with memory [27].

Real world applications of the equation (2.1) include diffusion in fractal and porous media, e.g. propagation of pollution, heat flow in media with memory, dynamics of protein in cells, transport in dielectrics and semiconductors, usage of optical tweezers, Hamiltonian chaos etc. [3, 4, 6, 27, 31].

Let us assume that there is a function k such that $k * M = M * k = 1$. Then if we apply $k*$ to (2.1), we obtain an equation that contains the explicit differential operator L and is called the normal form of (2.1): $k * U_t(t, x) = LU(t, x) + H(t, x)$, where $H(x, t) := k*Q(t, x)$. The term $k*U_t$ can be rewritten in the form $D_t^{\{k\}}(U - U(0, \cdot))$ that does not contain the 1st order derivative of U. Therefore, we get the equation

$$D_t^{\{k\}}(U - U(0, \cdot)) = LU(t, x) + H(t, x). \tag{2.2}$$

Conversely, in case of sufficiently regular U, the equation (2.1) follows from (2.2) by means of the application of the operator $\frac{\partial}{\partial t}M*$.

The equation (2.1) and its analogue (2.2) incorporate the following possibilities:

1. The kernel $M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, $0 < \beta < 1$, represents a power-type memory. Then (2.1) becomes the celebrated time fractional diffusion equation $U_t = \varkappa D^{1-\beta}LU + Q$, where $D^{1-\beta}v = \left(\frac{t^{\beta-1}}{\Gamma(\beta)} * v\right)_t$ is the Riemann-Liouville fractional derivative of the order $1 - \beta$ [4, 17, 20, 26]. For such M, it holds $k = \frac{t^{-\beta}}{\Gamma(1-\beta)}$ and $[k * (v - v(0)]_t = k * v_t = \partial_t^\beta v$ is the Caputo fractional derivative.

2. The kernel M or its associate k is a linear combination of power functions [25, 31]:

$$M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} + \sum_{j=1}^{l} p_j \frac{t^{\beta_j-1}}{\Gamma(\beta_j)}, \quad 0 < \beta < \beta_j < 1, \quad p_j \geq 0,$$

$$k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)} + \sum_{j=1}^{l} q_j \frac{t^{-\beta_j}}{\Gamma(1-\beta_j)}, \quad 0 < \beta_j < \beta < 1, \quad q_j \geq 0.$$

3. The kernel M has the form $M(t) = \int_0^1 p(s)\frac{t^{s-1}}{\Gamma(s)}ds$ where $p \geq 0$ is a nonvanishing integrable function (cf. [3, 25, 31]). Such a kernel stands for the distributed order fractional derivative that is used for modeling diffusion with a logarithmic growth of the mean square displacement [19].

4. Tempered fractional calculus [29], that is another way to generalize a fractional calculus, falls into the case

$$M(t) = \frac{1}{\Gamma(\beta)}e^{-\lambda t}t^{\beta-1} + \frac{\lambda}{\Gamma(\beta)}\int_0^t e^{-\lambda\tau}\tau^{\beta-1}d\tau, \quad \lambda > 0.$$

This type of kernel is used for modelling the transition from anomalous to normal diffusion.

Every presented example of M (or k) has a completely monotonic associate k (or M) that solves $k * M = 1$ (see Section 3).

Let $\Omega \in \mathbb{R}^n$ be an open bounded domain with the boundary $\partial\Omega$. In *direct problem* we have to find a function u that solves the initial-boundary value problem

$$
\begin{aligned}
&D_t^{\{k\}}(U - \Phi)(t, x) = LU(t, x) + H(t, x), \quad x \in \Omega, \, t \in (0, T), \\
&U(0, x) = \Phi(x), \quad x \in \Omega, \\
&\mathcal{B}(U - b)(t, x) = 0, \quad x \in \partial\Omega, \, t \in (0, T).
\end{aligned}
\tag{2.3}
$$

Here Φ and b are given functions and

$$
\mathcal{B}v(x) = v(x) \quad \text{or} \quad \mathcal{B}v(x) = \omega(x) \cdot \nabla v(x),
$$

with $\omega \cdot \nu > 0$ and $\nu(x)$ denoting the outer normal of $\partial\Omega$ at $x \in \Omega$. An important particular case is $\omega = \left(\sum_{j=1}^n a_{ij}\nu_j|_{i=1,\dots,n}\right)$. Then the condition $\mathcal{B}(U - b)|_{(t,x)\in(0,T)\times\partial\Omega} = 0$ corresponds to the flux specified at $\partial\Omega$.

Let us proceed to inverse problems. To this end we introduce the condition

$$
U(T, x) = \Psi(x), \quad x \in \Omega,
\tag{2.4}
$$

with a given observation function Ψ. Firstly, we formulate of an inverse source problem. Let

$$
H(t, x) = g(t, x)f(x) + h_0(t, x),
\tag{2.5}
$$

where the components gf and h_0 may correspond to different sources or sinks. The factor f is unknown and to be reconstructed by means of the data (2.4). Since the whole function U is also unknown, the first inverse problem consists in determination a pair of functions (f, U) that satisfies (2.3), (2.4) and (2.5).

In the second inverse problem, our aim is to identify the coefficient r of the linear reaction term rU. In the mathematical formulation, the problem consists in finding a pair (r, U) that satisfies (2.3) and (2.4). We can handle the case of zero initial condition $\Phi = 0$ (for details, see the end of Section 6).

Methods to be used in this paper require homogeneous boundary conditions. Therefore, we perform the change of the second unknown $u = U - b$ in our problems. It brings along shifts of data by addends containing b.

Firstly, from (2.3) we obtain the following problem for $u = U - b$:

$$
\begin{aligned}
&D_t^{\{k\}}(u - \varphi)(t, x) = Lu(t, x) + F(t, x), \quad x \in \Omega, \, t \in (0, T), \\
&u(0, x) = \varphi(x), \quad x \in \Omega, \\
&\mathcal{B}u(t, x) = 0, \quad x \in \partial\Omega, \, t \in (0, T),
\end{aligned}
\tag{2.6}
$$

where

$$
\varphi(x) = \Phi(x) - b(0, x),
\tag{2.7}
$$

$$
F(t, x) = H(t, x) + Lb(t, x) - D_t^{\{k\}}(b - b(0, \cdot))(t, x).
\tag{2.8}
$$

The overdetermination condition (2.4) in terms of u has the form

$$u(T, x) = \psi(x), \quad x \in \Omega, \tag{2.9}$$

where $\psi(x) = \Psi(x) - b(T, x)$. Plugging (2.5) into (2.8) we obtain

$$F(t, x) = g(t, x)f(x) + h(t, x), \tag{2.10}$$

where $h(t, x) = h_0(t, x) + Lb(t, x) - D_t^{\{k\}}(b - b(0, \cdot))(t, x)$.

In the reformulated first inverse problem (**IP1**), we seek for the pair of functions (f, u) that satisfies (2.6), (2.9) and (2.10).

Let us reformulate the second inverse problem, too. From the relations (2.3), (2.4) with $\Phi = 0$ by means of the change of variable $u = U - b$, we obtain the following problem for the pair (r, u):

$$
\begin{aligned}
&D_t^{\{k\}}u(t, x) = L_1 u(t, x) + r(x)(u + b)(t, x) + F_1(t, x) \quad x \in \Omega,\, t \in (0, T),\\
&u(0, x) = 0, \quad x \in \Omega, \qquad \mathcal{B}u(t, x) = 0, \quad x \in \partial\Omega,\, t \in (0, T), \qquad (2.11)\\
&u(T, x) = \psi(x), \quad x \in \Omega,
\end{aligned}
$$

where $b(0, x) = 0$, $x \in \Omega$, the function ψ is expressed by $\psi(x) = \Psi(x) - b(T, x)$ and $F_1(t, x) = H(t, x) + L_1 b(t, x) - D_t^{\{k\}} b(t, x)$.

Thus, the reformulated second inverse problem (**IP2**) is to find the pair of functions (r, u) that satisfies (2.11).

3 Basic assumptions

In this section we collect basic conditions on the domain, operator L and kernels k and M that will be assumed throughout the paper.

We assume that $\partial\Omega$ is uniformly of the class C^2 and $\omega \in (C^1(\partial\Omega))^n$. Moreover, we assume that $a_{ij}, a_j, r \in C(\overline{\Omega})$ and the principal part of L is uniformly elliptic, i.e. $\sum\limits_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2 \; \forall \xi \in \mathbb{R}^n$, $x \in \Omega$ for some $c > 0$.

Concerning the function k, we assume that

1. k belongs to $L_{1,loc}(0, \infty)$ and is a solution of the equation $M * k = 1$ with a kernel $M \in L_{1,loc}(0, \infty)$ that satisfies the conditions

$$
\begin{aligned}
&M \in C^1(0, \infty), \quad \lim_{t \to 0^+} M(t) = \infty, \quad M > 0, \quad M' \leq 0,\\
&- M' \text{ is nonincreasing and convex;}
\end{aligned} \tag{3.1}
$$

2. k has the following properties:

$$k \in C(0, \infty), \quad \lim_{t \to 0^+} k(t) = \infty, \quad k > 0, \quad k \text{ is nonincreasing,} \tag{3.2}$$

$$\exists t_k > 0 : k(t) \text{ is strictly decreasing in } (0, t_k). \tag{3.3}$$

The assumptions (3.1) ensure the existence of a sufficiently regular solution of the direct problem (see Lemma 3) and the assumptions (3.2), (3.3) are needed for the application of a positivity principle to this solution.

We mention that restricting generality a bit it is possible to reduce all conditions 1 and 2 to the single kernel M. Firstly, $M \in L_{1,loc}(0, \infty)$ and (3.1) imply the existence of a unique solution $k \in L_{1,loc}(0, \infty)$ of the equation $k * M = 1$ ([10], Ch. 5, Corollary 5.6). Secondly, all properties (3.2), (3.3) follow from conditions that are a bit stronger than (3.1). It is shown in the following lemma. Proof is in Appendix.

Lemma 1. *Let $M \in L_{1,loc}(0, \infty)$ satisfy (3.1) and $M' < 0$, $\log M$ - convex, $\log(-M')$ - convex. Then the solution of $M * k = 1$ satisfies (3.2), (3.3).*

The imposed assumptions on M and k hold for weakly singular completely monotonic kernels from

$$\mathcal{CM} = \{z \in L_{1,loc}(0, \infty) \bigcap C^\infty(0, \infty) : \lim_{t \to 0^+} z(t) = \infty, \ (-1)^i z^{(i)} > 0, \ i = 0, 1, \dots \}.$$

For M and k satisfying $M * k = 1$, it holds $M \in \mathcal{CM}$ if and only if $k \in \mathcal{CM}$ ([9], Theorem 3).

All examples of M and k given in Section 2 belong to $\mathcal{CM}$.

4 Preliminaries

4.1 Functional spaces

Let X be a Banach space. Since $k * M = 1$, we have

$$D_t^{\{k\}}(M * v) = \frac{d}{dt} k * M * v = \frac{d}{dt} 1 * v = v, \quad \forall v \in L_1((0, T); X), \qquad (4.1)$$

where $L_1((0, T); X)$ is the space of functions $u : (0, T) \to X$ that are integrable in the Bochner sense on $(0, T)$. This means that the operator $M*$ is a one-to-one mapping from $L_1((0, T); X)$ to $\{M * v : v \in L_1((0, T); X)\}$ and $D_t^{\{k\}}$ is the inverse of $M*$.

As usual, let $C([0, T]; X)$ stand for the Banach space of functions $u : [0, T] \to X$ that are continuous on $[0, T]$ with the norm $\|u\|_{C([0,T];X)} = \max_{t \in [0,T]} \|u(t)\|_X$ and $C_0([0, T]; X) = \{u \in C([0, T]; X) : u(0) = 0\}$. Based on the relation (4.1), we introduce the functional space

$$C_0^{\{k\}}([0, T]; X) := M * C([0, T]; X) = \{M * v : v \in C([0, T]; X)\}.$$

It is a Banach space with the norm

$$\|u\|_{C_0^{\{k\}}([0,T];X)} = \|D_t^{\{k\}} u\|_{C([0,T];X)}.$$

Since $M* \in \mathcal{L}(C([0, T]; X), C_0([0, T]; X))$, it holds

$$C_0^{\{k\}}([0, T]; X) \hookrightarrow C_0([0, T]; X).$$

We also define the space

$$C^{\{k\}}([0,T];X) := C_0^{\{k\}}([0,T];X) + X$$
$$= \{u : u(t) = u_1(t) + u_2, \ u_1 \in C_0^{\{k\}}([0,T];X), \ u_2 \in X\} \quad (4.2)$$

that is a Banach space with the norm

$$\|u\|_{C^{\{k\}}([0,T];X)} = \|u - u(0)\|_{C_0^{\{k\}}([0,T];X)} + \|u(0)\|_X.$$

Next we introduce the abstract Hölder spaces with corresponding norms

$$C_0^\alpha([0,T];X) = \Big\{ u \in C_0([0,T];X) \ :$$
$$\|u\|_{C_0^\alpha([0,T];X)} := \sup_{0<t_1<t_2<T} \frac{\|u(t_2) - u(t_1)\|_X}{(t_2 - t_1)^\alpha} < \infty \Big\},$$
$$C^\alpha([0,T];X) = C_0^\alpha([0,T];X) + X,$$
$$\|u\|_{C^\alpha([0,T];X)} = \|u - u(0)\|_{C_0^\alpha([0,T];X)} + \|u(0)\|_X,$$

where $0 < \alpha < 1$, and define the Banach spaces with norms

$$C_0^{\{k\},\alpha}([0,T];X) = M * C_0^\alpha([0,T];X), \quad (4.3)$$
$$\|u\|_{C_0^{\{k\},\alpha}([0,T];X)} = \|D_t^{\{k\}}u\|_{C_0^\alpha([0,T];X)},$$
$$C^{\{k\},\alpha}([0,T];X) = M * C^\alpha([0,T];X) + X,$$
$$\|u\|_{C^{\{k\},\alpha}([0,T];X)} = \|D_t^{\{k\}}(u - u(0))\|_{C^\alpha([0,T];X)} + \|u(0)\|_X.$$

Let us establish some connections between the spaces (4.2), (4.3) and the usual C, C^1- and Hölder spaces. For $C^{\{k\}}([0,T];X)$ the embeddings

$$C^1([0,T];X) \hookrightarrow C^{\{k\}}([0,T];X) \hookrightarrow C([0,T];X) \quad (4.4)$$

are valid. The right embedding follows from $M* \in \mathcal{L}(C([0,T];X))$[1] . To prove the left embedding, we choose some $u \in C^1([0,T];X)$. Then

$$\|u\|_{C^{\{k\}}([0,T];X)} = \|u - u(0)\|_{C_0^{\{k\}}([0,T];X)} + \|u(0)\|_X = \|k*u'\|_{C_0([0,T];X)} + \|u(0)\|_X$$

and since $k* \in \mathcal{L}(C([0,T];X), C_0([0,T];X))$, the left relation in (4.4) follows.

Analogous relations for the space $C_0^{\{k\},\alpha}([0,T];X)$ are

$$C_0^{1+\alpha}([0,T];X) \hookrightarrow C_0^{\{k\},\alpha}([0,T];X) \hookrightarrow C_0^\alpha([0,T];X) \quad (4.5)$$

where

$$C_0^{1+\alpha}([0,T];X) = \{u : u, u' \in C_0^\alpha([0,T];X)\}.$$

The right embedding in (4.5) is a consequence of the fact that $M* \in \mathcal{L}(C_0^\alpha([0,T];X))$ (see Lemma 4.2 in [14]) and the left embedding in (4.5) can be proved similarly to the left embedding in (4.4).

Under additional assumptions on M it is possible to show that the operator $M*$ increases the order of Hölder continuity of a function. Namely, the following lemma is valid. Its proof is deferred to Appendix.

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[1] The symbol $\mathcal{L}$ stands for the space of linear and bounded operators.

Lemma 2. *If* $M(t) \leq c_1 t^{\beta-1}$, $|M'(t)| \leq c_2 t^{\beta-2}$, $t \in (0, T)$ *for some* $c_1, c_2 \in$ $\mathbb{R}_+$, $0 < \beta \leq \alpha < 1$ *then* $M* \in \mathcal{L}(C_0^{\alpha-\beta}([0,T];X), C_0^{\alpha}([0,T];X))$.

Under conditions of Lemma 2, $C_0^{\{k\},\alpha-\beta}([0,T];X) \hookrightarrow C_0^{\alpha}([0,T];X)$. In the particular case $M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ (then $M*$ is the fractional integral of the order β), it holds the equality $C_0^{\{k\},\alpha-\beta}([0,T];X) = C_0^{\alpha}([0,T];X)$ [15].

By exchanging M and k in above relations, we obtain definitions and embeddings of spaces that contain $\{M\}$ instead of $\{k\}$ in the superscript.

4.2 Abstract Cauchy problem

Let $A : \mathcal{D}(A) \to X$ be a linear densely defined operator in a Banach space X. We say that A belongs to the class $\mathcal{S}(\eta, \theta)$ for $\eta \in \mathbb{R}$, $\theta \in (0, \pi)$ if

$$\rho(A) \supset \Sigma(\eta, \theta) = \{\lambda \in \mathbb{C} : \lambda \neq \eta, \, arg|\lambda - \eta| < \theta\} \quad \text{and}$$

$$\|(\mu - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{|\mu - \eta|} \quad \forall \mu \in \Sigma(\eta, \theta) \quad \text{for some constant } C > 0.$$

An operator $A \in \mathcal{S}(\eta, \theta)$ is closed. This implies that $X_A := \mathcal{D}(A)$ is a Banach space with the graph norm $\|w\|_{X_A} = \|w\|_X + \|Aw\|_X$.

Obviously, $\mathcal{S}(\eta, \theta_1) \subset \mathcal{S}(\eta, \theta_2)$ for $\theta_1 > \theta_2$. Operators of the class $\mathcal{S}(\eta, \theta)$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$, are the *sectorial* operators that generate analytic semigroups.

Now let us consider the Cauchy problem

$$D_t^{\{k\}}(u - \varphi)(t) = Au(t) + F(t), \quad t \in [0, T], \quad u(0) = \varphi, \tag{4.6}$$

with given $F : [0, T] \to X$ and $\varphi \in X$.

Lemma 3. *Let* $A \in \mathcal{S}(\eta, \frac{\pi}{2})$ *for some* $\eta \in \mathbb{R}$. *Then the following statements are valid.*
(i) *(uniqueness) Let* $u \in C^{\{k\}}([0,T];X) \bigcap C([0,T];X_A)$ *solve (4.6) and* $\varphi = 0$, $F = 0$. *Then* $u = 0$.
(ii) *Let* $F \in C_0^{\alpha}([0,T];X)$ *and* $\varphi = 0$. *Then (4.6) has a solution* u *in the space* $C_0^{\{k\},\alpha}([0,T];X) \bigcap C_0^{\alpha}([0,T];X_A)$. *This solution satisfies the estimate*

$$\|u\|_{C_0^{\{k\},\alpha}([0,T];X) \bigcap C_0^{\alpha}([0,T];X_A)} \leq C_1 \|F\|_{C_0^{\alpha}([0,T];X)}. \tag{4.7}$$

(iii) *Let* $F \in C^{\alpha}([0,T];X)$ *and* $\varphi \in X_A$. *Then (4.6) has a solution* u *in the space* $C^{\{k\}}([0,T];X) \bigcap C([0,T];X_A)$. *This solution satisfies the estimate*

$$\|u\|_{C^{\{k\}}([0,T];X) \bigcap C([0,T];X_A)} \leq C_2(\|F\|_{C^{\alpha}([0,T];X)} + \|\varphi\|_{X_A}). \tag{4.8}$$

The constants C_1 *and* C_2 *depend on* M *and* A.

Proof. The change of variable $v = D_t^{\{k\}}(u - \varphi) \Leftrightarrow u = M * v + \varphi$ reduces (4.6) of the integral equation

$$v(t) = A(M * v)(t) + F(t) + A\varphi, \quad t \in [0, T]. \tag{4.9}$$

Provided $F \in C([0,T];X)$, $\varphi \in X_A$, the function $u \in C^{\{k\}}([0,T];X) \bigcap C([0,T];X_A)$ solves (4.6) if and only if $v \in V := \{v \in C([0,T];X) : M * v \in C_0([0,T];X_A)\}$ solves (4.9). Similar one-to-one correspondence holds for $u \in C_0^{\{k\},\alpha}([0,T];X) \bigcap C_0^\alpha([0,T];X_A)$ and $v \in V^\alpha := \{v \in C_0^\alpha([0,T];X) : M * v \in C_0^\alpha([0,T];X_A)\}$ in the particular case $F \in C_0^\alpha([0,T];X)$, $\varphi = 0$.

Since M satisfies the conditions (3.1) and $A \in \mathcal{S}(\eta, \frac{\pi}{2})$, we can apply results of Ch. 3 of [28] to (4.9).

(i) Theorem 3.2 with Corollary 1.1 and Proposition 1.2 in [28] implies that there exists a family of operators $S : [0,\infty) \to \mathcal{L}(X)$ (called resolvent of (4.9)) so that a solution $v \in V$ (if it exists) is represented by the formula $v = \frac{d}{dt} S * F$. By assumptions of (i), (4.9) has a solution $v \in V$. Since $F = 0$, we have $v = 0$. Thus, $u = 0$.

(ii) Theorem 3.3 (i) [28] implies that for $F \in C_0^\alpha([0,T];X)$ there exists a solution $v \in V^\alpha$ of (4.9). This proves the existence of the solution $u \in C_0^{\{k\},\alpha}([0,T];X) \bigcap C_0^\alpha([0,T];X_A)$ of (4.6). The estimate (4.7) follows from the bounded inverse theorem.

(iii) It is sufficient to prove this assertion in case $F(t) \equiv \xi \in X$, because the problem with given pair of data (F, φ) can be splitted into two problems with the data $(F - F(0), 0)$ and $(F(0), \varphi)$, respectively. For the first problem, the assertion (ii) applies. Having proved (iii) for the second one, u is expressed as the sum of solutions of these two problems and satisfies (iii), too.

Thus, let us assume that $F(t) \equiv \xi \in X$. Due to Proposition 1.2 (ii) [28], (4.9) has the solution $v = S(\xi + A\varphi) \in V$. This implies the existence assertion of (iii). Due to the strong continuity of $S(t)$ [28], $\|S(t)\|_{\mathcal{L}(X)} \leq C_3$, $t \in [0,T]$, where C_3 is a constant. Thus, $\|v\|_{C([0,T],X)} \leq C_3 (\|\xi\|_X + \|A\varphi\|_X)$. Extracting the term $A(M*v)$ from (4.9) and estimating it we obtain $\|A(M*v)\|_{C_0([0,T],X)} \leq (C_3 + 1)(\|\xi\|_X + \|A\varphi\|_X)$. Consequently,

$$\|u\|_{C^{\{k\}}([0,T];X) \bigcap C([0,T];X_A)} = \|v\|_V + \|\varphi\|_{X_A} \leq C_4(\|\xi\|_X + \|\varphi\|_{X_A})$$

with a constant C_4. This implies (4.8). □

4.3 Statements on direct problem

In order to apply Lemma 3 to the direct problem (2.6), we must introduce appropriate Banach spaces of x-dependent functions and define realizations of the operator L in these spaces so that they belong to $\mathcal{S}\left(\eta, \frac{\pi}{2}\right)$.

Let us introduce the following spaces and operators:

1. $X_p = L_p(\Omega)$, $1 < p < \infty$,
    $A_p : X_{A_p} \to X_p$ with $X_{A_p} = \{z \in W_p^2(\Omega) : \mathcal{B}z|_{\partial\Omega} = 0\}$ and
    $A_p z = Lz$, $z \in X_{A_p}$.

2. $X_0 = \begin{cases} C_0(\overline{\Omega}) = \{z \in C(\overline{\Omega}) : z|_{\partial\Omega} = 0\} & \text{in case } \mathcal{B} = I, \\ C(\overline{\Omega}) & \text{in case } \mathcal{B} = \omega \cdot \nabla, \end{cases}$

    $A_0 : X_{A_0} \to X_0$ with $X_{A_0} = \{z \in \bigcap_{1<p<\infty} W_p^2(\Omega) : \mathcal{B}z|_{\partial\Omega} = 0, Lz \in X_0\}$

    and $A_0 z = Lz$, $z \in X_{A_0}$.

Corollary 1. Operators A_p, $p \in \{0\} \bigcup (1, \infty)$, are sectorial. Thus, Lemma 3 holds in cases $X = X_p$, $A = A_p$, $p \in \{0\} \bigcup (1, \infty)$ and applies to problem (2.6).

Proof. It follows from Theorems 3.1.2, 3.1.3 and Corollary 3.1.24 (ii) in [24]. □

Lemma 4. *Let $K \in L_1(0, T) \bigcap C^1(0, T)$, $\lim\limits_{t \to 0^+} K(t) = \infty$, $K > 0$, K be non-increasing and $\exists t_K > 0$: K is strictly decreasing in $(0, t_K)$. Moreover, let $F \in C([0, T] \times \overline{\Omega})$. Assume that u solves the problem*

$$D_t^{\{K\}}(u - \varphi)(t, x) = Lu(t, x) + F(t, x), \ t \in (0, T), \ x \in \Omega,$$
$$u(0, x) = \varphi, \ x \in \Omega$$

and satisfies the smoothness conditions $u \in C([0, T] \times \overline{\Omega})$, $u_{x_j} \in C((0, T] \times \overline{\Omega})$, $u \in C((0, T]; W_p^2(\Omega))$ for some $p > n$, $L_1 u \in C((0, T] \times \overline{\Omega})$, $D_t^{\{K\}}(u - \varphi) \in C((0, T] \times \overline{\Omega})$. Finally, let

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^\epsilon K(\tau) d\tau \sup_{0 \le s \le \epsilon} |u(t - s, x) - u(t, x)| = 0, \quad \forall t \in (0, T], \ x \in \overline{\Omega}. \ (4.10)$$

If $\varphi \ge 0$, $F \ge 0$ and $\mathcal{B}u|_{\partial \Omega} \ge 0$ then the following assertions are valid.
    (i) $u \ge 0$;
    (ii) *if $u(t_0, x_0) = 0$ in some point $(t_0, x_0) \in (0, T] \times \Omega_N$, where*

$$\Omega_N = \left\{ \begin{array}{ll} \Omega & \text{in case } \mathcal{B} = I \\ \overline{\Omega} & \text{in case } \mathcal{B} = \omega \cdot \nabla \end{array} \right. ,$$

    then $u(t, x_0) = 0$ for any $t \in [0, t_0]$.

    This lemma is a slight modification of a positivity principle that was proved in [15] for a semilinear equation in case of a more smooth solution $u \in C((0, T]; C^2(\overline{\Omega}))$ and strictly decreasing in $(0, T)$ kernel K.
    To prove Lemma 4, we need the following auxiliary result. It is proved in Appendix of the paper.

Lemma 5. *Let $w \in W_p^2(\Omega)$ for some $p > n$, $L_1 w \in C(\overline{\Omega})$ and $x^* = \operatorname*{argmin}\limits_{x \in \overline{\Omega}} w(x)$. In case $x^* \in \partial \Omega$ we also assume that $(\omega \cdot \nabla w)(x^*) \ge 0$. Then $L_1 w(x^*) \ge 0$.*

Proof of Lemma 4. Without a restriction of generality we assume that $r \le 0$. Otherwise it is possible to define $\tilde{u} = e^{-\sigma t} u$ as in [15] and to consider the corresponding problem for $\tilde{u}$. Such a problem also satisfies the assumptions of Lemma 4 and has the coefficient $\tilde{r} = r - \sigma \int_0^T e^{-\sigma s} K(s) ds$ in place of r. Since $\lim\limits_{t \to 0^+} K(t) = \infty$, for sufficiently large σ, $\tilde{r} \le 0$.
    Let us suppose that (i) does not hold. Then there exists $(t_1, x_1) \in (0, T] \times \overline{\Omega}$ such that $u(t_1, x_1) < 0$ and $(t_1, x_1) = \operatorname*{argmin}\limits_{x \in \overline{\Omega}, t \in [0, T]} u(t, x)$. It was shown in [15] (formula (37)) that the assumptions $D_t^{\{K\}}(u - \varphi) \in C((0, T] \times \overline{\Omega})$, (4.10),

$K > 0$ and K – nonincreasing together with the relations $u(t, x_1) \geq u(t_1, x_1)$ and $u(t_1, x_1) < 0$ imply $D_t^{\{K\}}(u - \varphi)(t_1, x_1) < 0$. On the other hand, Lemma 5 applies to the function $w = u(t_1, \cdot)$ at $x^* = x_1$. We obtain $L_1 u(t_1, x_1) \geq 0$. Also $r(x_1) u(t_1, x_1) \geq 0$ and $F \geq 0$. Thus, the left-hand side of the equation $D_t^{\{K\}}(u - \varphi)(t_1, x_1) = [Lu + F](t_1, x_1)$ is negative, but the right-hand side is nonnegative. We have reached a contradiction. The assertion (i) is valid.

Let us prove (ii). Let $u(t_0, x_0) = 0$ at $(t_0, x_0) \in (0, T] \times \Omega_N$. Define $\hat{t}_0 = \inf\{t : t \leq t_0, u(\tau, x_0) = 0 \text{ for } \tau \in [t, t_0]\}$. If (ii) is not valid, then $\hat{t}_0 > 0$ and $u(t, x_0) \geq \delta$, $t \in (t_2, t_3)$ for some $\delta > 0$ and $(t_2, t_3) \subset (0, \hat{t}_0)$ such that $\hat{t}_0 - t_2 < t_K$. Then, similarly to the proof in [15] p.138, from the assumptions $D_t^{\{K\}}(u - \varphi) \in C((0, T] \times \overline{\Omega})$, (4.10), $K > 0$, K – nonincreasing and relations $u \geq 0$, $u(t, x_0) \geq \delta > 0$, $t \in (t_2, t_3)$, we derive

$$D_t^{\{K\}}(u - \varphi)(\hat{t}_0, x_0) \leq \delta(K(\hat{t}_0 - t_2) - K(\hat{t}_0 - t_3)). \tag{4.11}$$

Since $0 < \hat{t}_0 - t_3 < \hat{t}_0 - t_2 < t_K$ and K is strictly decreasing in $(0, t_K)$, (4.11) implies $D_t^{\{K\}}(u - \varphi)(\hat{t}_0, x_0) < 0$. On the other hand, from $u(\hat{t}_0, x_0) = 0$ and $u(t, x) \geq 0$, $(t, x) \in (0, T] \times \Omega$, we conclude that $(\hat{t}_0, x_0) = \operatorname*{argmin}_{x \in \overline{\Omega}} u(\hat{t}_0, x)$. By Lemma 5, $L_1 u(\hat{t}_0, x_0) \geq 0$. Moreover, $(ru)(\hat{t}_0, x_0) = 0$ and $F \geq 0$. Left-hand side of the equation $D_t^{\{K\}}(u - \varphi)(\hat{t}_0, x_0) = [Lu + F](\hat{t}_0, x_0)$ is negative, but right-hand side is nonnegative. Again, we have reached the contradiction. Thus, (ii) holds.   □

At this point we present somewhat more concrete assumptions on the input data of the direct problem (2.6) that imply the assumptions of Lemma 4 and Lemma 3.

Corollary 2. Let $F \geq 0$, $\varphi = 0$ and one of the assumptions (a1)–(a3) hold:
(a1) $F \in C^{\{M\},\alpha}([0, T]; X_0)$ for some $0 < \alpha < 1$ and $F(0, \cdot) = 0$;
(a2) $F \in C_0^\alpha([0, T]; X_0)$ and $M(t) \geq ct^{\gamma-1}$, $t \in (0, T)$ for some $c \in \mathbb{R}_+$, $0 < \gamma < \alpha < 1$;
(a3) $F \in C_0^{\alpha-\beta}([0, T]; X_0)$ and $c_1 t^{\gamma-1} \leq M(t) \leq c_2 t^{\beta-1}$, $|M'(t)| \leq c_3 t^{\beta-2}$, $t \in (0, T)$, for some $c_1, c_2, c_3 \in \mathbb{R}_+$, $0 < \beta \leq \gamma < \alpha < 1$.
Then assertions Lemma 4 are satisfied by solution of the problem (2.6).

Proof. Defining $X = X_0$, Lemma 3 with Corollary 1 implies that the solution of (2.6) exists and satisfies the smoothness conditions of Lemma 4. It remains to show that (4.10) holds.

The case (a1). The relations $F \in C^{\{M\},\alpha}([0, T]; X_0)$, $F(0, \cdot) = 0$ mean that $F = k * \hat{F}$, where $\hat{F} \in C^\alpha([0, T]; X_0)$. Thus, it follows from Lemma 3 that the function $\hat{u}$ that solves (2.6) with F, φ replaced by $\hat{F}, \hat{\varphi} = 0$ belongs to the space $C^{\{k\}}([0, T]; X_0)$. Next, after convolving equation for $\hat{u}$ with k it is easy to see that $u = k * \hat{u}$ solves (2.6) with $F = k * \hat{F}$. Therefore, $u \in k * C^{\{k\}}([0, T]; X_0)$, that is $u = k * M * v = 1 * v$, $v \in C([0, T]; X_0)$. This allows us to conclude that

$u \in C^1([0,T]; X_0)$. Hence,

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^\epsilon k(\tau)d\tau \sup_{0 \le s \le \epsilon} |u(t-s,x) - u(t,x)| = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^\epsilon k(\tau)d\tau \cdot O(\epsilon)$$
$$= 0, \quad \forall t \in (0,T], \ x \in \overline{\Omega}.$$

The case (a2). Again, by Lemma 3 (ii), $u \in C_0^{\{k\},\alpha}([0,T]; X_0)$ and by (4.5), $u \in C_0^\alpha([0,T]; X_0)$. The relation (4.10) follows from the estimate

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^\epsilon k(\tau)d\tau \sup_{0 \le s \le \epsilon} |u(t-s,x) - u(t,x)| = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^\epsilon k(\tau)d\tau \cdot O(\epsilon^\alpha)$$
$$\le \lim_{\epsilon \to 0^+} \frac{O(\epsilon^\alpha)}{\epsilon M(\epsilon)} \int_0^\epsilon M(\epsilon - \tau)k(\tau)d\tau = \lim_{\epsilon \to 0^+} O(\epsilon^{\alpha - \gamma}) = 0 \quad \forall t \in (0,T], \ x \in \overline{\Omega}.$$

The case (a3). According to Lemma 3 (ii), $F \in C_0^{\alpha - \beta}([0,T]; X_0)$ implies that $u \in C_0^{\{k\},\alpha - \beta}([0,T]; X_0) = M * C_0^{\alpha - \beta}([0,T]; X_0)$. By Lemma 2 it holds $u \in C_0^\alpha([0,T]; X_0)$. This enables us finish the proof as in case (a2). □

5 Results on IP1

We will study IP1 in context of Hölder spaces with respect to t. For the sake of generality, we will assume different orders of spaces related to g and h: for g we use α_1 and for h we use α_2.

Theorem 1. *Let one of the following assumptions be valid:*
(A1) $g \in C_0^{1+\alpha_1}([0,T]; C(\overline{\Omega}))$ *for some* $0 < \alpha_1 < 1$;
(A2) $g \in C_0^{\{k\},\alpha_1}([0,T]; C(\overline{\Omega}))$ *and* $M(t) \ge ct^{\gamma - 1}$, $t \in (0,T)$ *for some* $c \in \mathbb{R}_+$, $0 < \gamma < \alpha_1 < 1$;
(A3) $g \in C_0^{\{k\},\alpha_1 - \beta}([0,T]; C(\overline{\Omega}))$ *and* $c_1 t^{\gamma - 1} \le M(t) \le c_2 t^{\beta - 1}$, $|M'(t)| \le c_3 t^{\beta - 2}$, $t \in (0,T)$, *for some* $c_1, c_2, c_3 \in \mathbb{R}_+$, $0 < \beta \le \gamma < \alpha_1 < 1$.
Additionally, we assume that $g \ge 0$, $g_1 := D_t^{\{k\}} g - Rg \ge 0$ *where* $R := \max_{x \in \overline{\Omega}} r(x)$
and

$$\text{a.e. } x \in \Omega \quad \exists t_x \in (0,T] : \ g(t_x, x) > 0. \tag{5.1}$$

In case $\mathcal{B} = I$ *we also assume that* $\forall x \in \partial\Omega$, *either* $g(T,x) > 0$ *or* $g(\cdot, x) = 0$.

Finally, let $(f,u) \in C(\overline{\Omega}) \times \left(C_0^{\{k\}}([0,T]; C(\overline{\Omega})) \bigcap C_0([0,T]; W_p^2(\Omega)) \right)$ *for some* $p > 1$ *solve IP1 for* $\varphi = 0$, $\psi = 0$, $h = 0$. *Then* $(f,u) = (0,0)$.

Proof. We start the proof by showing that in case $\mathcal{B} = I$, for any $x \in \partial\Omega$ such that $g(T,x) > 0$, the equality $f(x) = 0$ is valid. To show this, we consider the equality

$$D_t^{\{k\}} u(T,x) = f(x)g(T,x), \quad x \in \overline{\Omega},$$

that follows from equation (2.6) in view of $\psi = 0$. If $x \in \partial\Omega$ and $\mathcal{B} = I$ then the left-hand side of this equality equals zero. Thus, $f(x)g(T,x) = 0$ and provided $g(T,x) > 0$ we obtain $f(x) = 0$.

Let us introduce the functions $f^+ = \frac{|f|-f}{2}$ and $f^- = \frac{|f|+f}{2}$. Due to the definition, $f^\pm \in C(\overline{\Omega})$ and $f^\pm \geq 0$. Moreover,

in case $\mathcal{B} = I$, for any $x \in \partial\Omega$ such that $g(T, x) > 0$, it holds $f^\pm(x) = 0$. (5.2)

Firstly, we consider the problems

$$D_t^{\{k\}} u^\pm(t, x) = L u^\pm(t, x) + g(t, x) f^\pm(x), \quad x \in \Omega,\, t \in (0, T),$$
$$u^\pm(0, x) = 0, \quad x \in \Omega, \quad \mathcal{B} u^\pm(t, x) = 0, \quad x \in \partial\Omega,\, t \in (0, T). \tag{5.3}$$

By assumptions of the theorem and (5.2), $g(t, \cdot) f^\pm \in X_0$, $t \in [0, T]$. Therefore, in cases (A1) and (A2) due to (4.5) we have $g f^\pm \in C_0^{\{M\}, \alpha_1}([0, T]; X_0)$ and $g f^\pm \in C_0^{\alpha_1}([0, T]; X_0)$, respectively. Similarly, in case (A3) due to (4.5) and Lemma 2 we obtain $g f^\pm \in C_0^{\alpha_1}([0, T]; X_0)$. Moreover, $g f^\pm \geq 0$. The assumptions of Corollary 2 are satisfied for the functions $F = g f^\pm$. Hence, the solutions $u^\pm$ of (5.3) satisfy the assertions of Lemma 4.

Secondly, let us consider the problems

$$D_t^{\{k\}} v^\pm(t, x) = L v^\pm(t, x) + g_1(t, x) f^\pm(x), \quad x \in \Omega,\, t \in (0, T),$$
$$v^\pm(0, x) = 0, \quad x \in \Omega, \quad \mathcal{B} v^\pm(t, x) = 0, \quad x \in \partial\Omega,\, t \in (0, T). \tag{5.4}$$

In case (A1) we have $g' \in C_0^{\alpha_1}([0, T]; C(\overline{\Omega}))$. Thus, $g_1 = D_t^{\{k\}} g - Rg = k * g' - Rg \in C_0^{\{M\}, \alpha_1}([0, T]; C(\overline{\Omega}))$. From $g(t, \cdot) f^\pm \in X_0$, $t \in [0, T]$ we immediately get $g_1(t, \cdot) f^\pm \in X_0$, $t \in [0, T]$. Therefore, $g_1 f^\pm \in C_0^{\{M\}, \alpha_1}([0, T]; X_0)$.

Using similar reasoning, we deduce $g_1 f^\pm \in C_0^{\alpha_1}([0, T]; X_0)$ and $g_1 f^\pm \in C_0^{\alpha_1 - \beta}([0, T]; X_0)$ in cases (A2) and (A3), respectively. Moreover, $g_1 f^\pm \geq 0$. Again, the assumptions of Corollary 2 are satisfied for $F = g_1 f^\pm$. The solutions $v^\pm$ of (5.4) satisfy the assertions of Lemma 4.

Let us point out that the problem for $M * v^\pm$ is equivalent to the problem for $u^\pm - RM * u^\pm$. Thus,

$$v^\pm = D_t^{\{k\}} u^\pm - R u^\pm. \tag{5.5}$$

Moreover, since $f = f^+ - f^-$, we have $u = u^+ - u^-$. Thus, $\psi = u(T, \cdot) = 0$ implies that $u^+(T, \cdot) = u^-(T, \cdot)$. Let us denote $x^* = \operatorname*{argmax}_{x \in \overline{\Omega}} u^+(T, x) = \operatorname*{argmax}_{x \in \overline{\Omega}} u^-(T, x)$. By definition, either $f^+(x^*) = 0$ or $f^-(x^*) = 0$. Let us assume that $f^+(x^*) = 0$ (the situation when $f^-(x^*) = 0$ can be considered in a similar manner).

Let us suppose that either $x^* \in \Omega$ or $\mathcal{B} = \omega \cdot \nabla$ (the case $x^* \in \partial\Omega$ and $\mathcal{B} = I$ will be considered later separately). Then we can apply Lemma 5 to the function $w = -u^+(T, \cdot)$. We get $L_1 u^+(T, x^*) \leq 0$. Thus, from (5.3), (5.5) and $u^+ \geq 0$, $r \leq R$ it follows:

$$v^+(T, x^*) = L_1 u^+(T, x^*) + (r(x^+) - R) u^+(T, x^*) \leq 0. \tag{5.6}$$

Due to Lemma 4 (i),

$$v^+(t, x) \geq 0, \quad (t, x) \in (0, T) \times \Omega. \tag{5.7}$$

Hence, (5.6) and (5.7) imply $v^+(T, x^*) = 0$. Thus, by Lemma 4 (ii), $v^+(t, x^*) = 0$, $t \in [0, T]$. By formula (5.5) it means $D_t^{\{k\}} u^+(t, x^*) - Ru^+(t, x^*) = 0$, $t \in [0, T]$. Applying $M*$ to to this equality, we obtain the following homogeneous Volterra equation of the second kind:

$$u^+(t, x^*) - RM * u^+(t, x^*) = 0, \quad t \in [0, T].$$

It has only the trivial solution $u^+(t, x^*) = 0$, $t \in [0, T]$. Hence, $u^+(T, x^*) = 0$.

Since x^* is a maximum point of $u^+(T, x)$ and $u^+(T, x) \geq 0$, we also get

$$u^+(T, x) = 0, \quad x \in \Omega. \tag{5.8}$$

Now we consider the case $x^* \in \partial\Omega$, $\mathcal{B} = I$, too. Then by $\mathcal{B}u^+|_{\partial\Omega} = 0$, immediately $u^+(T, x^*) = 0$ and again we have (5.8).

Since $u = u^+ - u^-$ and $\psi = u(T, \cdot) = 0$ holds, from (5.8) we get $u^{\pm}(T, x) = 0$, $x \in \Omega$. Lemma 4 (ii) implies $u^{\pm}(t, x) = 0$, $(t, x) \in [0, T] \times \Omega$. Therefore, $u(t, x) = 0$, $(t, x) \in [0, T] \times \Omega$. From the differential equation for u we obtain $f(x)g(t, x) = 0$, $(t, x) \in [0, T] \times \Omega$. Finally, (5.1) yields $f = 0$. $\square$

Next we provide simple sufficient conditions that imply the assumption $D_t^{\{k\}} g - Rg \geq 0$ in Theorem 1. For this we need the following lemma.

Lemma 6. *Let $w \in C^{\{k\}}([0, T]; \mathbb{R})$ be nonnegative and nonincreasing. Then $D_t^{\{k\}} w \geq k(T)w$.*

Proof. The assertion follows from the estimate

$$D_t^{\{k\}} w(t) = \lim_{\delta \to 0^+} \frac{1}{\delta} \Big[\int_t^{t+\delta} k(\tau)w(t+\delta-\tau)d\tau + \int_0^t k(\tau)(w(t+\delta-\tau)$$

$$- w(t-\tau))d\tau \Big] \geq \lim_{\delta \to 0^+} k(T+\delta)\frac{1}{\delta} \Big[\int_t^{t+\delta} w(t+\delta-\tau)d\tau + \int_0^t (w(t+\delta-\tau)$$

$$- w(t-\tau))d\tau \Big] = k(T)w(t), \quad 0 < t < T.$$

$\square$

Due to that Lemma 6, $D_t^{\{k\}} g - Rg \geq 0$ holds provided along with other assumptions on g in Theorem 1, g is nondecreasing in t and $k(T) \geq R$ in case $R > 0$.

Theorem 2. *Let g, M satisfy the assumptions of Theorem 1 and the inequality $g(T, x) > 0$, $x \in \overline{\Omega}$, hold. If $\varphi, \psi \in X_{A_p}$ and $h \in C^{\alpha_2}([0, T]; X_p)$, where $p \in \{0\} \bigcup (1, \infty)$, $0 < \alpha_2 < 1$, then IP1 has a unique solution $(f, u) \in X_p \times C^{\{k\}}([0, T]; X_p) \bigcap C([0, T]; X_{A_p})$ and the following estimate holds:*

$$\|f\|_{X_p} + \|u\|_{C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})}$$

$$\leq C_5 \Big(\|\varphi\|_{X_{A_p}} + \|\psi\|_{X_{A_p}} + \|h\|_{C^{\alpha_2}([0,T];X_p)} \Big). \tag{5.9}$$

If additionally $\varphi = h(0, \cdot) = 0$, then $u \in C_0^{\{k\},\alpha}([0,T]; X_p) \bigcap C_0^\alpha([0,T]; X_{A_p})$ where $\alpha = \begin{cases} \alpha_2, & \text{in case (A1)}, \\ \min\{\alpha_1, \alpha_2\}, & \text{in cases (A2), (A3)} \end{cases}$ *and the estimate*

$$\|f\|_{X_p} + \|u\|_{C_0^{\{k\},\alpha}([0,T];X_p) \bigcap C_0^\alpha([0,T];X_{A_p})} \leq C_6 \left(\|\psi\|_{X_{A_p}} + \|h\|_{C_0^{\alpha_2}([0,T];X_p)} \right) (5.10)$$

is valid. The constants C_5 and C_6 depend on the parameters M, L, g, p, α_2.

Proof. Firstly, we are going to replace the overdetermination condition (2.9) by a fixed-point equation with respect to f.

Suppose that $(f, u) \in X_p \times C^{\{k\}}([0,T]; X_p) \bigcap C([0,T]; X_{A_p})$ solves IP1. Then, since (2.9) holds, the equation (2.6) at $t = T$ with $F = fg + h$ yields

$$f(x) = \frac{\left(D_t^{\{k\}}(u - \varphi) - \eta u \right)(T, x) - (A_p - \eta)\psi(x) - h(T, x)}{g(T, x)}, \qquad (5.11)$$

where η is chosen so that $0 \in \rho(A_p - \eta I)$.

Let us split u into the sum of two functions: $u = u_1 + u_2$, such that

$$D_t^{\{k\}} u_1 = A_p u_1 + fg, \quad u_1(0, \cdot) = 0,$$
$$D_t^{\{k\}}(u_2 - \varphi) = A_p u_2 + h, \quad u_2(0, \cdot) = \varphi. \qquad (5.12)$$

In the context of IP1, u_2 is a known function. According to Lemma 3, the solution to (5.12) belongs to $u_2 \in C^{\{k\}}([0,T]; X_p)$. Thus, $v_2 := D_t^{\{k\}}(u_2 - \varphi) - \eta u_2 \in C([0,T]; X_p)$. Next we formulate the following problem:

$$D_t^{\{k\}} v_1 = A_p v_1 + f(D_t^{\{k\}} g - \eta g), \quad v_1(0, \cdot) = 0. \qquad (5.13)$$

Due to the assumptions (A1)–(A3) and (4.5), it holds $D_t^{\{k\}} g \in C_0^{\hat\alpha}([0,T]; C(\overline{\Omega}))$ where

$$\hat\alpha = \begin{cases} \alpha_1, & \text{in cases (A1), (A2)}, \\ \alpha_1 - \beta, & \text{in case (A3)}. \end{cases} \qquad (5.14)$$

Thus, $f(D_t^{\{k\}} g - \eta g) \in C_0^{\hat\alpha}([0,T]; X_p)$. According to Lemma 3, (5.13) has a solution v_1 in $C_0^{\{k\},\hat\alpha}([0,T]; X_p) \bigcap C_0^{\hat\alpha}([0,T]; X_{A_p})$. It is easy to check that $v_1 = D_t^{\{k\}} u_1 - \eta u_1$.

The notations introduced allow us to rewrite (5.11) in the form

$$f = \mathcal{F}f + \mathcal{G}, \qquad (5.15)$$

where

$$\mathcal{G}(x) = \frac{v_2(T, x) - (A_p - \eta)\psi(x) - h(T, x)}{g(T, x)}, \quad x \in \Omega, \qquad (5.16)$$

$$(\mathcal{F}f)(x) = v_1[f](T, x)/g(T, x) \qquad (5.17)$$

and $v_1[\cdot]$ stands for the operator that assigns to f the solution v_1 of (5.13). Thus, (2.6), (2.9), (2.10) imply (5.15). On the other hand, taking into account all the substitutions performed, we can move back from (5.15) to (5.11). Together with (2.6) at $t = T$ and (2.10) it implies $(A_p - \eta)u(T, x) = (A_p - \eta)\psi(x)$. Since $(A_p - \eta)$ is injective, it yields (2.9). Consequently, IP1 is in the space $X_p \times C^{\{k\}}([0, T]; X_p) \bigcap C([0, T]; X_{A_p})$ equivalent to the problem of finding the pair of functions (f, u) that solves (2.6), (2.10), (5.15).

We point out that (5.15) is an independent equation for the first component f of the solution of IP1. Let us analyse properties of the operator $\mathcal{F}$ involved in this equation. By Lemma 3, $v_1[\cdot] \in \mathcal{L}(X_p; C_0^{\hat{\alpha}}([0, T]; X_{A_p}))$. Thus, $v_1[\cdot](T, \cdot) \in \mathcal{L}(X_p, X_{A_p})$.

Furthermore, $X_{A_p} \hookrightarrow\hookrightarrow X_p$. In case $p \in (1, \infty)$ it is a direct consequence of $W_p^2(\Omega) \hookrightarrow\hookrightarrow L_p(\Omega)$. In case $p = 0$ it follows from the continuous embedding of X_{A_0} in $C_{\mathcal{B}}^1(\overline{\Omega}) := X_0 \bigcap C^1(\overline{\Omega})$ (see Theorems 3.1.19, 3.1.22 in [24]) and $C_{\mathcal{B}}^1(\overline{\Omega}) \hookrightarrow\hookrightarrow X_0$.

Therefore, $v_1[\cdot](T, \cdot) : X_p \to X_p$ is compact. Since $\frac{1}{g(T, \cdot)} \in C(\overline{\Omega})$ due to the assumptions of this theorem, $\mathcal{F} : X_p \to X_p$ is also compact.

Next, let us show that $1 \notin \sigma(\mathcal{F})$. Firstly, let us consider the case $p = 0$. Suppose that $1 \in \sigma(\mathcal{F})$. Then the equation $f = \mathcal{F}f$ has a solution $f \in X_0$, $f \neq 0$. This means that the problem (2.6), (2.10), (5.15) with homogeneous data $\varphi = 0$, $\psi = 0$, $h = 0$ has the nontrivial solution (f, u_1) in the space $X_0 \times C_0^{\{k\}}([0, T]; X_0) \bigcap C_0([0, T]; X_{A_0})$. But due to the Theorem 1, IP1 with a homogeneous data has only the trivial solution in such a space. We came to a contradiction. Consequently, $1 \notin \sigma(\mathcal{F})$.

Secondly, let us consider the case $p \in (1, \infty)$. We again suppose that $1 \in \sigma(\mathcal{F})$, hence the equation $f = \mathcal{F}f$ has a nontrivial solution $f \in X_p$. The idea is to show that this solution actually belongs to X_0. Then we can apply the arguments from the previous case to show that $1 \in \sigma(\mathcal{F})$ leads to a contradiction.

If $p > \frac{n}{2}$, then $v_1[f](T, \cdot) \in X_{A_p} \hookrightarrow X_0$. Thus, $f = \mathcal{F}f = \frac{1}{g(T, x)}v_1[f](T, \cdot) \in X_0$. If $p \leq \frac{n}{2}$, then according to embedding theorems, $X_{A_p} \hookrightarrow X_{p_1} = L_{p_1}(\Omega)$, where $p_1 = \frac{np}{n-2p} > p$. Therefore, $v_1[f](T, \cdot) \in X_{p_1}$ and $f = \mathcal{F}f = \frac{1}{g(T, x)}v_1[f](T, \cdot) \in X_{p_1}$. After a finite number of iterations we obtain $f \in X_{p_i}$, where $p_i = \frac{np}{n-2ip} > \frac{n}{2}$ (works for $i > \frac{n}{2p} - 1$). Next iteration gives $f \in X_0$.

We have shown that the first case of Fredholm alternative is satisfied for the equation (5.15). Consequently, the solution to (5.15) exists and is unique for any $\mathcal{G} \in X_p$ and $(I - \mathcal{F})^{-1} \in \mathcal{L}(X_p)$.

Since $F = fg + h$ is Hölder-continuous with values in X_p, Lemma 3 implies that the problem (2.6), (2.10) has unique solution $u \in C^{\{k\}}([0, T]; X_p) \bigcap C([0, T]; X_{A_p})$. This completes the proof of the existence and uniqueness assertion of the theorem.

In the rest of the proof, $\widehat{C}$ stands for a generic constant depending on the parameters M, L, g, p, α_2. Let us deduce the stability estimate (5.9). We obtain

$$\|f\|_{X_p} \leq \|(I - \mathcal{F})^{-1}\|_{\mathcal{L}(X_p)}\|\mathcal{G}\|_{X_p} \leq \widehat{C}\left(\|h(T, \cdot)\|_{X_p} + |\eta|\|\psi\|_{X_p} + \|\psi\|_{X_{A_p}}\right.$$

$$+ \|D_t^{\{k\}}(u_2 - \varphi) - \eta u_2\|_{C([0,T];X_p)}\Big) \leq \widehat{C}\left(\|h\|_{C^{\alpha_2}([0,T];X_p)} + \|\psi\|_{X_{A_p}}\right.$$

$$\left. + \|\varphi\|_{X_{A_p}}\right). \tag{5.18}$$

Further, we note that $g \in C_0^\gamma([0,T];C(\overline{\Omega}))$ for any $\gamma \in (0,1)$ in case (A1) and for $\gamma = \alpha_1$ in cases (A2), (A3). Using Lemma 3 we have

$$\|u\|_{C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})} = \|u_1 + u_2\|_{C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})}$$

$$\leq \widehat{C}(\|f\|_{X_p}\|g\|_{C_0^\gamma([0,T];C(\overline{\Omega}))} + \|h\|_{C^{\alpha_2}([0,T];X_p)} + \|\varphi\|_{X_{A_p}}).$$

Together with the estimate of f (5.18) it implies (5.9).

In case $\varphi = h(0,\cdot) = 0$, the solution of (2.6), (2.10) belongs to the space $C_0^{\{k\},\alpha}([0,T];X_p) \bigcap C_0^\alpha([0,T];X_{A_p})$ and can be estimated as

$$\|u\|_{C_0^{\{k\},\alpha}([0,T];X_p)\bigcap C_0^\alpha([0,T];X_{A_p})} \leq \widehat{C}(\|f\|_{X_p}\|g\|_{C_0^\gamma([0,T];C(\overline{\Omega}))} + \|h\|_{C_0^{\alpha_2}([0,T];X_p)}).$$

This with (5.18) implies (5.10). $\quad\square$

We point out that in case $p = 0$ and $\mathcal{B} = I$, the assumptions of Theorem 2 allow to recover $f \in X_0 = C_0(\overline{\Omega})$ only. In order to fix that in the following theorem we provide some additional conditions that are sufficient to restore $f \in C(\overline{\Omega})$ in case $\mathcal{B} = I$.

Theorem 3. *Let g, M satisfy the assumptions of Theorem 2. If φ, ψ, $L\varphi \in X_{A_p}$ for some $p > \dfrac{n}{2}$, $L\psi \in C(\overline{\Omega})$, $h \in C^{\{k\},\alpha_2}([0,T];X_p) \bigcap C([0,T];C(\overline{\Omega}))$, where $0 < \alpha_2 < 1$ and $h(0,\cdot) \in X_{A_p}$ then IP1 has a unique solution $(f,u) \in C(\overline{\Omega}) \times C^{\{k\}}([0,T];X_{A_p})$. Moreover, $Lu \in C([0,T];C(\overline{\Omega}))$ and the estimate*

$$\|f\|_{C(\overline{\Omega})} + \|u\|_{C^{\{k\}}([0,T];X_{A_p})} + \|Lu\|_{C([0,T];C(\overline{\Omega}))} \leq C_7\big(\|\varphi\|_{X_p} + \|L\varphi\|_{X_{A_p}}$$

$$+ \|\psi\|_{X_p} + \|L\psi\|_{C(\overline{\Omega})} + \|h\|_{C^{\{k\},\alpha_2}([0,T];X_p) \bigcap C([0,T];C(\overline{\Omega}))} + \|h(0,\cdot)\|_{X_{A_p}}\big) \tag{5.19}$$

holds. If additionally $\varphi = h(0,\cdot) = D_t^{\{k\}}h(0,\cdot) = 0$, then $u \in C_0^{\{k\},\alpha'}([0,T];X_{A_p})$ and the estimate

$$\|f\|_{C(\overline{\Omega})} + \|u\|_{C_0^{\{k\},\alpha'}([0,T];X_{A_p})} + \|Lu\|_{C_0([0,T];C(\overline{\Omega}))}$$

$$\leq C_8\big(\|\psi\|_{X_p} + \|L\psi\|_{C(\overline{\Omega})} + \|h\|_{C_0^{\{k\},\alpha_2}([0,T];X_p) \bigcap C_0([0,T];C(\overline{\Omega}))}\big) \tag{5.20}$$

is valid where $\alpha' = \min\{\hat{\alpha}; \alpha_2\}$ and $\hat{\alpha}$ is given by (5.14). The constants C_7 and C_8 depend on M, L, g, p, α_2.

Proof. Throughout the proof, $\widehat{C}$ denotes a generic constant depending on M, L, g, p, α_2 and RHS stands for the expression in brackets at the right-hand side of (5.19). By Theorem 2, IP1 has a unique solution $(f,u) \in X_p \times C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})$. Let us consider the problem

$$D_t^{\{k\}}(w_2 - w_2(0,\cdot)) = A_p w_2 + D_t^{\{k\}}(h - h(0,\cdot)), \quad w_2(0,\cdot) = L\varphi + h(0,\cdot). \tag{5.21}$$

Under the assumptions of this theorem, Lemma 3 implies that (5.21) has a unique solution $w_2 \in C^{\{k\}}([0,T]; X_p) \cap C([0,T]; X_{A_p})$. Moreover, due to (4.7) and (4.8), $\|w_2\|_{C([0,T]; X_{A_p})} \leq \widehat{C}(\|h\|_{C^{\{k\}, \alpha_2}([0,T]; X_p)} + \|h(0, \cdot)\|_{X_{A_p}} + \|L\varphi\|_{X_{A_p}})$. It is easy to check that $w_2 = D_t^{\{k\}} * (u_2 - \varphi)$ and $u_2 = M * w_2 + \varphi$ where u_2 solves (5.12). Therefore, we have $u_2 \in C^{\{k\}}([0,T]; X_{A_p}) \hookrightarrow C^{\{k\}}([0,T]; C(\overline{\Omega}))$ and

$$
\|u_2\|_{C^{\{k\}}([0,T]; X_{A_p})}
$$
$$
\leq \widehat{C}\Big(\|h\|_{C^{\{k\}, \alpha_2}([0,T]; X_p)} + \|h(0, \cdot)\|_{X_{A_p}} + \|L\varphi\|_{X_{A_p}}\Big) + \|\varphi\|_{X_{A_p}}. \quad (5.22)
$$

Let us consider the function $\mathcal{G}$ given by (5.16). (Recall that there $v_2 = w_2 - \eta u_2$.) Due the proved properties of w_2 and u_2 and the assumptions of the theorem, it holds $\mathcal{G} \in C(\overline{\Omega})$ and $\|\mathcal{G}\|_{C(\overline{\Omega})} \leq \widehat{C} \operatorname{RHS}$.

Now, let us provide an estimate for $\|f\|_{C(\overline{\Omega})}$ using the formulas (5.15) and (5.17). Since $1/g(T, \cdot) \in C(\overline{\Omega})$ and $v_1[\cdot](T, \cdot) \in \mathcal{L}(X_p, X_{A_p})$, we have

$$
\|f\|_{C(\overline{\Omega})} \leq \|\mathcal{F}f\|_{C(\overline{\Omega})} + \|\mathcal{G}\|_{C(\overline{\Omega})} \leq \widehat{C}\|v_1[f](T, \cdot)\|_{C(\overline{\Omega})} + \|\mathcal{G}\|_{C(\overline{\Omega})}
$$
$$
\leq \widehat{C}\|v_1[f](T, \cdot)\|_{X_{A_p}} + \|\mathcal{G}\|_{C(\overline{\Omega})} \leq \widehat{C}\|f\|_{X_p} + \|\mathcal{G}\|_{C(\overline{\Omega})}.
$$

Since $(I - \mathcal{F})$ is invertible in X_p, the estimate holds

$$
\|f\|_{X_p} \leq \|(I - \mathcal{F})^{-1}\|_{\mathcal{L}(X_p)}\|\mathcal{G}\|_{X_p} \leq \widehat{C}\|\mathcal{G}\|_{C(\overline{\Omega})}.
$$

Thus, we obtain

$$
\|f\|_{C(\overline{\Omega})} \leq \widehat{C} \operatorname{RHS}. \quad (5.23)
$$

Finally, let us derive an estimate for u and finish the proof of the first part of the theorem. We have $u = u_1 + u_2$, where $u_1 = M * w_1$, $w_1 = D_t^{\{k\}} u_1$ and w_1 solves the problem

$$
D_t^{\{k\}} w_1 = A_p w_1 + f D_t^{\{k\}} g, \quad w_1(0, \cdot) = 0.
$$

Since $f D_t^{\{k\}} g \in C_0^{\alpha'}([0,T]; X_p)$, Lemma 3 implies $w_1 \in C_0^{\alpha'}([0,T]; X_{A_p})$ and $\|u_1\|_{C_0^{\{k\}, \alpha'}([0,T]; X_{A_p})} = \|w_1\|_{C_0^{\alpha'}([0,T]; X_{A_p})} \leq \widehat{C}\|f\|_{C(\overline{\Omega})}\|D_t^{\{k\}} g\|_{C_0^{\alpha'}([0,T]; X_p)}$. Using here (5.23) we have

$$
\|u_1\|_{C_0^{\{k\}, \alpha'}([0,T]; X_{A_p})} \leq \widehat{C} \operatorname{RHS}. \quad (5.24)
$$

From (5.22) and (5.24) we obtain for $u = u_1 + u_2$ the estimate

$$
\|u\|_{C^{\{k\}}([0,T]; X_{A_p})} \leq \widehat{C} \operatorname{RHS}. \quad (5.25)
$$

It remains to estimate Lu in the space $C([0,T]; C(\overline{\Omega}))$. Using (5.25) we deduce

$$
\|D_t^{\{k\}}(u - \varphi)\|_{C([0,T]; C(\overline{\Omega}))} \leq \widehat{C}\|D_t^{\{k\}}(u - \varphi)\|_{C([0,T]; X_{A_p})} \leq \widehat{C} \operatorname{RHS}.
$$

From the expression $Lu = D_t^{\{k\}}(u - \varphi) - fg - h$ due to the proved estimates for $D_t^{\{k\}}(u - \varphi)$ and f we obtain

$$\|Lu\|_{C([0,T];C(\overline{\Omega}))} \leq \widehat{C} \, \text{RHS}. \tag{5.26}$$

Summing up, (5.23), (5.25) and (5.26) imply (5.19).

Now let us focus on the second part of this theorem that is concerned with the particular case $\varphi = h(0, \cdot) = D_t^{\{k\}} h(0, \cdot) = 0$. Then RHS reduces to the expression in brackets at the right-hand side of (5.20). Lemma 3 implies that the function w_2 which solves (5.21) belongs the space $C_0^{\alpha'}([0,T]; X_{A_p})$, the function $u_2 = M * w_2$ belongs to $C_0^{\{k\},\alpha'}([0,T]; X_{A_p})$ and $\|u_2\|_{C_0^{\{k\},\alpha'}([0,T];X_{A_p})} \leq \widehat{C}\|h\|_{C_0^{\{k\},\alpha_2}([0,T];X_p)}$. This relation by $u = u_1 + u_2$ and the estimates (5.23), (5.24) and (5.26) implies (5.20). $\quad\square$

Provided the assumptions of Theorem 3 hold and $\mathcal{B} = I$, an explicit expression of the unknown function f at the boundary can be derived. Namely, setting $t = T$ and $x \in \partial\Omega$ in (2.6) and taking the relations $F = fg + h$ and $u(T, \cdot) = \psi$ into account we obtain $f(x) = -\dfrac{1}{g(T,x)}[L\psi(x) + h(T,x)], \quad x \in \partial\Omega$.

6 Results on IP2

In the context of IP2 let us introduce the following sets for the coefficient r:

$$\mathcal{K}_R = \{r \in C(\overline{\Omega}) \,:\, r(x) \leq R, \ x \in \overline{\Omega}\}, \qquad \text{where } R \in \mathbb{R}.$$

Theorem 4. *Let R be some real number and IP2 have two solutions (r, u), (r_1, u_1), such that*

$$r \in C(\overline{\Omega}), \quad r_1 \in \mathcal{K}_R, \quad u, u_1 \in C_0^{\{k\}}([0,T]; L_1(\Omega)) \bigcap C_0([0,T]; W_1^2(\Omega)),$$

$$u_1 - u \in C_0^{\{k\}}([0,T]; C(\overline{\Omega})) \bigcap C_0([0,T]; W_p^2(\Omega))$$

for some $p > 1$ and the function $U = u + b$ (and M) satisfy one of the following assumptions:
(A4) $U \in C_0^{1+\alpha_1}([0,T]; C(\overline{\Omega}))$ *for some* $0 < \alpha_1 < 1$;
(A5) $U \in C_0^{\{k\},\alpha_1}([0,T]; C(\overline{\Omega}))$ *and* $M(t) \geq ct^{\gamma-1}$, $t \in (0,T)$ *for some* $c \in \mathbb{R}_+$, $0 < \gamma < \alpha_1 < 1$;
(A6) $U \in C_0^{\{k\},\alpha_1-\beta}([0,T]; C(\overline{\Omega}))$ *and* $c_1 t^{\gamma-1} \leq M(t) \leq c_2 t^{\beta-1}$, $|M'(t)| \leq c_3 t^{\beta-2}$, $t \in (0,T)$, *for some* $c_1, c_2, c_3 \in \mathbb{R}_+$, $0 < \beta \leq \gamma < \alpha_1 < 1$.

Additionally, we assume that

$$U \geq 0, \quad D_t^{\{k\}} U - RU \geq 0, \tag{6.1}$$

$$\text{a.e. } x \in \Omega, \quad \exists t_x \in (0,T) : U(t_x, x) > 0.$$

In case $\mathcal{B} = I$ we also assume that $\forall x \in \partial\Omega$, either $U(T, x) > 0$ or $U(\cdot, x) = 0$. Then $(r_1, u_1) = (r, u)$.

Proof. The difference $(\hat{r}, \hat{u}) = (r_1 - r, u_1 - u) \in C(\overline{\Omega}) \times \left(C_0^{\{k\}}([0,T]; C(\overline{\Omega})) \right.$ $\left. \bigcap C_0([0,T]; W_p^2(\Omega)) \right)$ solves the problem

$$D_t^{\{k\}} \hat{u}(t,x) = (L_1 + r_1)\hat{u}(t,x) + U(t,x)\hat{r}(x), \quad x \in \Omega, \, t \in (0,T),$$
$$\hat{u}(0,x) = 0, \quad x \in \Omega, \quad \mathcal{B}\hat{u}(t,x) = 0, \quad x \in \partial\Omega, \, t \in (0,T), \quad (6.2)$$
$$\hat{u}(T,x) = 0, \quad x \in \Omega.$$

The inequalities (6.1) imply that $D_t^{\{k\}} U - R_r U \geq 0$, where $R_r := \max_{x \in \overline{\Omega}} r_1(x) \leq R$. Consequently, the assumptions of Theorem 1 are satisfied for the problem (6.2) and we obtain $\hat{r} = 0$, $\hat{u} = 0$. $\square$

Let us formulate a problem that contains approximate data:

$$D_t^{\{k\}}(\tilde{u} - \tilde{\varphi})(t,x) = L_1 \tilde{u}(t,x) + \tilde{r}(x)(\tilde{u} + \tilde{b})(t,x) + \tilde{F}_1(t,x), \quad x \in \Omega, \, t \in (0,T),$$
$$\tilde{u}(0,x) = 0, \quad x \in \Omega, \quad \mathcal{B}\tilde{u}(t,x) = 0, \quad x \in \partial\Omega, \, t \in (0,T), \quad (6.3)$$
$$\tilde{u}(T,x) = \tilde{\psi}, \quad x \in \Omega.$$

We are going to prove an existence and approximation theorem for this problem in case its data vector $\tilde{D} = (\tilde{b}, \tilde{F}_1, \tilde{\psi})$ is close to the data vector $D = (b, F_1, \psi)$ of the exact problem IP2.

Theorem 5. *Assume that* $R \in \mathbb{R}$ *and IP2 has a solution* $(r,u) \in \mathcal{K}_R$ $\times \, C_0^{\{k\}}([0,T]; L_1(\Omega)) \bigcap C_0([0,T]; W_1^2(\Omega))$ *such that* $U = u + b$ *(and* M*) satisfy one of the assumptions* (A4)–(A6)*, the inequalities* (6.1) *and* $U(T,x) > 0$*,* $x \in \overline{\Omega}$*. Then the following statements are valid.*

(i) *Let* $p \in \{0\} \bigcup \left(\frac{n}{2}, \infty \right)$*,* $\alpha_2 \in (0,1)$*. There exist constants* $\delta_1 > 0$ *and* $K_1 > 0$ *depending on* $M, L_1, r, U, p, \alpha_2$ *such that if*

$$\tilde{D} - D \in \mathcal{D}_1 = C_0^{\alpha_2}([0,T]; C_{(p)}(\overline{\Omega})) \times C_0^{\alpha_2}([0,T]; X_p) \times X_{A_p}$$

and $\|\tilde{D} - D\|_{\mathcal{D}_1} \leq \delta_1$*, where* $C_{(p)}(\overline{\Omega}) = \begin{cases} C(\overline{\Omega}), & \text{in case } p \in \left(\frac{n}{2}, \infty \right), \\ X_0, & \text{in case } p = 0, \end{cases}$ *then problem* (6.3) *has a unique solution in the set*

$$\left\{ (\tilde{r}, \tilde{u}) : (\tilde{r} - r, \tilde{u} - u) \in \mathcal{X}_1 := X_p \times \left(C_0^{\{k\},\alpha}([0,T]; X_p) \bigcap C_0^\alpha([0,T]; X_{A_p}) \right), \right.$$
$$\left. \|(\tilde{r} - r, \tilde{u} - u)\|_{\mathcal{X}_1} \leq K_1 \|\tilde{D} - D\|_{\mathcal{D}_1} \right\},$$

where $\alpha = \begin{cases} \alpha_2, & \text{in case } (A4), \\ \min\{\alpha_1, \alpha_2\}, & \text{in cases } (A5), (A6). \end{cases}$

(ii) *Let* $p \in \left(\frac{n}{2}, \infty \right)$*,* $\alpha_2 \in (0,1)$*. There exist constants* $\delta_2 > 0$ *and* $K_2 > 0$ *depending on* $M, L_1, r, U, p, \alpha_2$ *such that if*

$$\tilde{D} - D \in \mathcal{D}_2 = \left(C_0^{\{k\},\alpha_2}([0,T]; X_p) \bigcap C_0^{\alpha_2}([0,T]; C(\overline{\Omega})) \right)^2 \times Y_p$$

and $\|\tilde{D} - D\|_{\mathcal{D}_2} \leq \delta_2$ where $Y_p = \{\psi : \psi \in X_{A_p}, L\psi \in C(\overline{\Omega})\}$, then the problem (6.3) has a unique solution in the set

$$\left\{(\tilde{r}, \tilde{u}) : (\tilde{r} - r, \tilde{u} - u) \in \mathcal{X}_2 := C(\overline{\Omega}) \times \mathcal{U}_{p,\alpha'}, \|(\tilde{r} - r, \tilde{u} - u)\|_{\mathcal{X}_2} \leq K_2 \|\tilde{D} - D\|_{\mathcal{D}_2}\right\},$$

where $\mathcal{U}_{p,\alpha'} = \{v \in C_0^{\{k\},\alpha'}([0,T]; X_{A_p}) : Lv \in C_0([0,T]; C(\overline{\Omega}))\}$, $\alpha' = \min\{\hat{\alpha}; \alpha_2\}$ and $\hat{\alpha} = \begin{cases} \alpha_1, & \text{in cases (A4), (A5),} \\ \alpha_1 - \beta, & \text{in case (A6).} \end{cases}$

We mention that in this theorem, the operator A_p and the space X_{A_p} defined on the basis of $L = L_1 + rI$ depend on the component r of the solution of the exact problem IP2.

Proof. Let us denote the difference $(\hat{r}, \hat{u}) = (\tilde{r} - r, \tilde{u} - u)$. Then the problem for the pair $(\hat{r}, \hat{u})$ reads

$$\begin{aligned} D_t^{\{k\}}\hat{u} &= (L_1 + r)\hat{u} + \hat{r}(u + b) + \left[\hat{r}\hat{u} + \tilde{F}_1 - F_1 + (\hat{r} + r)(\tilde{b} - b)\right], \\ \hat{u}(0, \cdot) &= 0, \quad \mathcal{B}\hat{u}|_{\partial\Omega} = 0, \quad \hat{u}(T, \cdot) = \tilde{\psi} - \psi. \end{aligned} \tag{6.4}$$

This problem can be treated as IP1 with $f = \hat{r}$, $g = u + b$, $h = \hat{r}\hat{u} + \tilde{F}_1 - F_1 + (\hat{r} + r)(\tilde{b} - b)$. Therefore, applying the solution operator of IP1 $\mathcal{A}$ to (6.4), it is reduced to the operator equation

$$(\hat{r}, \hat{u}) = \mathcal{F}_2(\hat{r}, \hat{u}), \tag{6.5}$$

where $\mathcal{F}_2(\hat{r}, \hat{u}) = \mathcal{A}(\hat{r}\hat{u} + \tilde{F}_1 - F_1 + (\hat{r} + r)(\tilde{b} - b), 0, \tilde{\psi} - \psi)$.

We are going to show that $\mathcal{F}_2$ is a contraction in a ball $\|(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \leq \rho$ with a suitable chosen $\rho > 0$. Firstly, we have to prove that this ball remains invariant with respect to the operator $\mathcal{F}_2$. Let $\|(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \leq \rho$. According to (5.10),

$$\|\mathcal{F}_2(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \leq C_6 \left(\|\tilde{\psi} - \psi\|_{X_{A_p}} + \|\hat{r}\hat{u} + \tilde{F}_1 - F_1 + (\hat{r} + r)(\tilde{b} - b)\|_{C_0^{\alpha_2}([0,T]; X_p)}\right).$$

Let c_p be an embedding constant such that $\|w\|_{C(\overline{\Omega})} \leq c_p \|w\|_{X_{A_p}}$. Then

$$\|\hat{r}\hat{u}\|_{C_0^{\alpha_2}([0,T]; X_p)} \leq \|\hat{r}\|_{X_p} \|\hat{u}\|_{C_0^{\alpha}([0,T]; C(\overline{\Omega}))} \leq \|\hat{r}\|_{X_p} c_p \|\hat{u}\|_{C_0^{\alpha}([0,T]; X_{A_p})} \leq c_p \rho^2.$$

Therefore,

$$\begin{aligned} \|\mathcal{F}_2(\hat{r}, \hat{u})\|_{\mathcal{X}_1} &\leq C_6 \left(\|\tilde{\psi} - \psi\|_{X_{A_p}} + c_p \rho^2 + \|\tilde{F}_1 - F_1\|_{C_0^{\alpha_2}([0,T]; X_p)}\right. \\ &\left. + (\rho + R_1)\|\tilde{b} - b\|_{C_0^{\alpha_2}([0,T]; C_{(p)}(\overline{\Omega}))}\right) \leq C_6 \left(c_p \rho^2 + (\rho + 1 + R_1)\|\tilde{D} - D\|_{\mathcal{D}_1}\right), \end{aligned}$$

where $R_1 = \|r\|_{X_p}$ in case $p \in \left(\frac{n}{2}, \infty\right)$ and $R_1 = \|r\|_{C(\overline{\Omega})}$ in case $p = 0$. Now let us take $\rho = K_1 \|\tilde{D} - D\|_{\mathcal{D}_1}$ with a constant K_1. Then

$$\|\mathcal{F}_2(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \leq C_6 \left((c_p K_1^2 + K_1)\|\tilde{D} - D\|_{\mathcal{D}_1} + 1 + R_1\right) \|\tilde{D} - D\|_{\mathcal{D}_1}.$$

In case $\|\tilde{D} - D\|_{\mathcal{D}_1} \leq \delta_1$ we have

$$\|\mathcal{F}_2(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \leq C_6 \left((c_p K_1^2 + K_1)\delta_1 + 1 + R_1 \right) \|\tilde{D} - D\|_{\mathcal{D}_1}.$$

Let us define the constants as follows: $K_1 = C_6(2 + R_1)$, $\delta_1 = \frac{1}{c_p K_1^2 + K_1}$. Then $\|\mathcal{F}_2(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \leq K_1 \|\tilde{D} - D\|_{\mathcal{D}_1}$. Consequently, for $\|(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \leq \rho$ we have $\|\mathcal{F}_2(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \leq \rho$.

Secondly, inside the set $\|(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \leq \rho = K_1 \|\tilde{D} - D\|_{\mathcal{D}_1}$ let us consider the difference of $\mathcal{F}_2$ at $(\hat{r}_1, \hat{u}_2)$ and $(\hat{r}_2, \hat{u}_2)$. Assuming $\|\tilde{D} - D\|_{\mathcal{D}_1} \leq \delta_1$, we deduce the estimate

$$\|\mathcal{F}_2(\hat{r}_1, \hat{u}_1) - \mathcal{F}_2(\hat{r}_2, \hat{u}_2)\|_{\mathcal{X}_1} \leq \|\mathcal{A}\| \|(\hat{r}_1 - \hat{r}_2)\hat{u}_1 + \hat{r}_2(\hat{u}_1 - \hat{u}_2)$$

$$+ (\hat{r}_1 - \hat{r}_2)(\tilde{b} - b)\|_{C_0^{\alpha_2}([0,T];X_p)} \leq C_6 \left(c_p \rho \|\hat{r}_1 - \hat{r}_2\|_{X_p} \right.$$

$$\left. + c_p \rho \|\hat{u}_1 - \hat{u}_2\|_{C_0^{\alpha}([0,T];X_{A_p})} + \delta_1 \|\hat{r}_1 - \hat{r}_2\|_{X_p} \right) \leq C_6(c_p K_1 \delta_1 + \delta_1)$$

$$\times \|(\hat{r}_1 - \hat{r}_2, \hat{u}_1 - \hat{u}_2)\|_{\mathcal{X}_1} = \frac{1}{(2 + R_1)} \|(\hat{r}_1 - \hat{r}_2, \hat{u}_1 - \hat{u}_2)\|_{\mathcal{X}_1}.$$

It shows that the operator $\mathcal{F}_2$ is a contraction in the ball $\|(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \leq \rho$. According to the Banach fixed point theorem there exists a unique solution to the equation (6.5) in that ball. This proves the assertion (i).

(ii) The proof of (ii) repeats the proof of (i) with appropriate changes of spaces and norms. For $\mathcal{A}$, the estimate (5.20) is used instead of (5.10). $\square$

Remark 1. In case the data of (6.3) are close to data of a process without reaction (i.e. $r = 0$), Theorem 5 implies the existence of the reaction coefficient $\tilde{r}$ in small.

Remark 2. Supposing the existence of a solution (r, u) of IP2, we ask: what are sufficient conditions on the data that guarantee the validity of inequality-type conditions (6.1) and $U(T, x) > 0$, $x \in \overline{\Omega}$ in Theorems 4, 5? To answer this question, we return to the problem (2.3) for U and set there $\Phi = H(0, \cdot) = 0$. Let us suppose that U is sufficiently smooth. Then constructing a corresponding problem for $D_t^{\{k\}} U - RU$ and assuming $D_t^{\{k\}} H - RH \geq 0$, $(D_t^{\{k\}} \mathcal{B}b - R\mathcal{B}b)|_{\partial\Omega} \geq 0$, Lemma 4 (i) implies the inequality $D_t^{\{k\}} U - RU \geq 0$. Next, we consider the conditions $U \geq 0$ and $U(T, x) > 0$, $x \in \overline{\Omega}$. Let us assume that

$$\exists \mu \in C[0, T], \ \mu \geq 0, \ \mu \neq 0, \ \mu - \text{nondecreasing} :$$
$$H(t, x) \geq \mu(t), \ x \in \overline{\Omega}, \ t \in [0, T], \quad \mathcal{B}b(t, x) \geq \mu(t), \ x \in \partial\Omega, \ t \in [0, T].$$

Define $V = U - \delta 1 * \mu$ with $\delta > 0$. The function V solves the problem

$$D_t^{\{k\}} V = LV + H_1, \quad V(0, \cdot) = 0, \quad \mathcal{B}(V - (b - \delta 1 * \mu))|_{\partial\Omega} = 0,$$

where $H_1 = H + \delta(r1 * \mu - D_t^{\{k\}} 1 * \mu)$. Since $D_t^{\{k\}} 1 * \mu = k * \mu$, we get that for sufficiently small δ,

$$H_1(t, x) \geq \mu(t)[1 - \delta(\max_{x \in \overline{\Omega}} r(x)T + \|k\|_{L_1(0,T)})] \geq 0, \quad t \in [0, T], \ x \in \Omega$$

and $\mathcal{B}V|_{\partial\Omega} = \mathcal{B}(b - \delta 1 * \mu)|_{\partial\Omega} \geq 0$. Lemma 4 (i) yields $V \geq 0$. Thus, $U = V + \delta 1 * \mu \geq 0$ and $U(T,x) = V(T,x) + \delta \int_0^T \mu(\tau)d\tau > 0$, $x \in \overline{\Omega}$.

At the end of this section, we make some general remarks. We applied results on IP1 to analyze IP2. In a similar manner, results on IP1 can be applied to study inverse problems to determine other coefficients of L, too.

The basic set of assumptions (A1)–(A3) for g involves the restriction $g(0,\cdot) = 0$. This is due to the fact that in case $g(0,\cdot) \neq 0$ we cannot ensure sufficient regularity of u to apply the positivity principle in the proof of Theorem 1. In IP2, the function $u + b = U$ works as g. For that reason, we consider the case $\Phi = U(0,\cdot) = 0$ in IP2.

In the beginning of the proof of Lemma 4 we showed that the direct problem with $r > 0$ can be reduced to a problem with $r \leq 0$ by the change of unknown $\tilde{u} = e^{-\sigma t}u$, where $\sigma > 0$. This suggests a possible exponential growth of u and a related time limitation of the linear reaction model in case $r > 0$. For bigger T, nonlinear reaction models are more relevant [6].

Solutions of IP1 and IP2 depend continuously on derivatives of the data of finite order. This means that these problems are moderately ill-posed. In case approximate data are given with errors, regularization procedures can be effectively applied (cf. e.g. [17] for IP1 with $g = g(t)$).

Acknowledgements

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Appendix: Proofs of Lemmas 1, 2 and 5

Proof of Lemma 1. Theorems 3 and 4 of [9] guarantee that k is nonnegative, nonincreasing and convex. Convexity implies the continuity of k. From the equation $M * k = 1$ we easily deduce $\lim_{t \to 0^+} k(t) = +\infty$, because in the opposite case k is bounded from which it follows that $\lim_{t \to 0^+} (M * k)(t) = 0$.

Let us prove $k > 0$. Suppose that it is not true. Then in view of proved properties of k, $\exists t_0 : k(t) > 0, t < t_0$ and $k(t) = 0, t > t_0$. For $t > t_0$ from $M * k = 1$ we get $\int_0^{t_0} M(t - \tau)k(\tau)d\tau = 1$. Therefore, $\int_0^{t_0} M'(t - \tau)k(\tau)d\tau = 0$.

The last equality contradicts to the assumptions $k(t) > 0$, $t \in (0, t_0)$ and $M' < 0$. Thus, $k > 0$.

Finally, let us prove (3.3) Let us choose some $t_3 > 0$. Since $\lim\limits_{t \to 0^+} k(t) = +\infty$, there exists an interval $(0, \delta)$, $\delta < t_3$, such that $k(t) > k(t_3)$ for $t \in (0, \delta)$. Suppose that (3.3) is not true. Then we can find two points $t_1 < t_2$ in $(0, \delta)$ so that $k(t_1) = k(t_2)$. Consequently, for $t_1 < t_2 < t_3$ we have $k(t_1) = k(t_2) > k(t_3)$. Obviously, it contradicts to the convexity of k. Therefore, (3.3) is valid. □

Proof of Lemma 2 is similar to proof of Theorem 14 in [11] that is concerned with the case $M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$. Let $z \in C_0^{\alpha-\beta}([0, T]; X)$. Then $\|M * z(t)\|_X \le$ const $t^{\beta-1} * t^{\alpha-\beta} = O(t^\alpha)$. Secondly,

$$(M * z)(t) - (M * z)(t - h) = J_1 + J_2 + J_3,$$

where

$$J_1 = z(t) \int_{t-h}^t M(\tau)d\tau, \quad J_2 = -\int_0^h [z(t) - z(t - \tau)M(\tau)d\tau,$$

$$J_3 = -\int_h^t [z(t) - z(t - \tau)] \int_{\tau-h}^\tau M'(s)ds \, d\tau.$$

Immediately, $\|J_2\|_X \le$ const $\int_0^h \tau^{\alpha-\beta}\tau^{\beta-1}d\tau = O(h^\alpha)$. Moreover,

$$\|J_1\|_X \le \text{const } t^{\alpha-\beta} \int_{t-h}^t \tau^{\beta-1} = \text{const } t^{\alpha-\beta}[t^\beta - (t - h)^\beta],$$

$$\|J_3\|_X \le \text{const } \int_h^t \tau^{\alpha-\beta} \int_{\tau-h}^\tau s^{\beta-2}ds \, d\tau = \text{const } \int_h^t \tau^{\alpha-\beta}[(\tau - h)^{\beta-1} - \tau^{\beta-1}]d\tau.$$

Further estimation of J_1 and J_3 can be performed exactly as in [11]. As a result, we get $\|J_1\|_X, \|J_3\|_X = O(h^\alpha)$. This completes the proof. □

Proof of Lemma 5. Firstly, we point out that the assumption $w \in W_p^2(\Omega)$, $p > n$ implies $w \in C^1(\overline{\Omega})$. We will use maximum principles for elliptic equations in Sobolev spaces to prove the lemma. Let us consider the case $x^* \in \Omega$. Suppose that $L_1w(x^*) < 0$. Then there exists a ball $B(x^*, \varepsilon) \subset \Omega$ and $\delta > 0$ such that $L_1w(x) \le -\delta < 0$ for $x \in B(x^*, \varepsilon)$. Let us define the auxiliary function

$$z(x) = \alpha|x - x^*|^2 \text{ with } \alpha > 0 \tag{7.1}$$

such that $L_1(w + z) \le 0$ in $B(x^*, \varepsilon)$. Since $w(x^*) \le w(x)$ and $z(x^*) < z(x)$ for $x \in \partial B(x^*, \varepsilon)$, we get

$$(w + z)(x^*) < (w + z)(x), \quad x \in \partial B(x^*, \varepsilon). \tag{7.2}$$

On the other hand, due to $L_1(w+z) \le 0$ it follows from the Theorem 9.1 [8] that $\min\limits_{x \in B(x^*,\varepsilon)} (w + z)(x) = \min\limits_{x \in \partial B(x^*,\varepsilon)} (w + z)(x)$, that contradicts (7.2). Therefore, the supposition $L_1w(x^*) < 0$ was wrong.

Next let us consider the case $x^* \in \partial\Omega$. Again, suppose $L_1 w(x^*) < 0$. Then there exists $B(x^*, \varepsilon)$ and $\delta > 0$ such that $L_1 w(x) \leq -\delta < 0$ for $x \in B(x^*, \varepsilon) \bigcap \Omega$. Similarly to the previous case we define z by (7.1) so that $L_1(w + z) \leq 0$ in $B(x^*, \varepsilon) \bigcap \Omega$. Then $(w + z)(x^*) < (w + z)(x)$ for $x \in B(x^*, \varepsilon) \bigcap \Omega$. Hence, Lemma 3.4 [8] is applicable and yields $\frac{\partial w}{\partial \nu}(x^*) = \frac{\partial(w+z)}{\partial \nu}(x^*) < 0$. That contradicts to $\frac{\partial}{\partial \nu} w(x^*) \geq 0$ following from the assumption $\frac{\partial}{\partial \omega} w(x^*) \geq 0$. Therefore, $L_1 w(x^*) \geq 0$ holds. □