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Multiplicity in Parameter-Dependent Problems for Ordinary Differential Equations*

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Abstract. We consider two types of nonlinear boundary value problems involving parameters. The second type of problems includes the Fučík problem. Properties of spectra are discussed in connection with the number of solutions to BVP.

Key words: nonlinear boundary value problems, Fučík spectra, multiplicity of solutions, solution curves, bifurcation diagrams.

1 Introduction

This article is a review paper on nonlinear boundary value problems involving parameters. We consider two types of equations. The first one is given as

$$x'' + \lambda f(x) = 0, \tag{1.1}$$

where f is continuously differentiable and λ is a parameter. This equation is considered together with the boundary conditions

$$x(0) = 0, \quad x(1) = 0. \tag{1.2}$$

A solution of (1.1), (1.2) is a pair $(\lambda, x(t))$ and we are interested in multiple solutions. This problem is actual in the theory of differential equations and recent results in this direction are described in survey paper [3]. In our research we are motivated by this paper and [4].

The second type equations we wish to consider are equations of the form

$$x'' = -\lambda f(x^+) + \mu g(x^-), \tag{1.3}$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$ and λ, μ are nonnegative parameters. Functions f and g are positive valued continuously differentiable functions

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defined on $R^+ = [0, +\infty)$. This equation is considered together with the boundary conditions (1.2). If f and g are linear functions (i.e., $f = x$ and $g = x$) then equation (1.3) becomes

$$x'' = -\lambda x^+ + \mu x^-,$$

and this is the famous Fučík equation which appears also as a simplified model for suspension bridges.

The spectrum for the Fučík problem is well known. We tried to study spectra for the problem

$$\begin{cases} x'' = -\lambda f(x^+) + \mu g(x^-), \\ x(0) = 0, \quad x(1) = 0. \end{cases}$$

If f and g are nonlinear then this spectrum without any other restrictions fills the first quadrant and the problem does not make sense. We impose an additional normalization condition, which allows us to study a nonlinear spectra also. If this normalization condition is omitted then the problem of multiple solutions makes sense.

In this paper we try to bring together all the above material and to find interrelations and connections between these problems.

2 Solution Curves

2.1 Time-map functions

Let us look for positive solutions of the problem (1.1), (1.2). Consider first

$$x'' + f(x) = 0. \tag{2.1}$$

Let $x(t;\alpha)$ be a solution of the Cauchy problem (2.1)

$$x(0) = 0, \quad x'(0) = \alpha. \tag{2.2}$$

Any positive solution of $x'' + f(x) = 0$, $x(0) = 0$, $x(1) = 0$ satisfies

$$t_1(\alpha) = 1$$

for some $\alpha > 0$ (number 1 in the right side refers to the length of the interval $(0, 1)$). Thus the number of positive solutions of the problem depends on the t_1 function.

Similarly any positive solution of (1.1), (1.2) satisfies

$$U(\alpha, \lambda) = 1,$$

where U is a time map for (1.1), (2.2). The respective set of (α, λ) is called *a solution curve.* This curve has an important information on the number of positive solutions to the problem (1.1), (1.2).

Assertion. If $t_1(\alpha)$ is known then $U(\alpha, \lambda)$ is also known.

The relation is defined by the following formula

$$U(\alpha,\lambda)=\frac{1}{\sqrt{\lambda}}\,t_1\left(\frac{\alpha}{\sqrt{\lambda}}\right).$$

The function $U(\alpha,\lambda)$ behaves like $1/\sqrt{\lambda}$ on the curves defined by $\alpha/\sqrt{\lambda}=$ *const*.

2.2 Cubic nonlinearity

Consider equation with cubic nonlinearity

$$\begin{cases} x''+\lambda(x-a)(x-b)(c-x)=0,\\ x(0)=0,\quad x(1)=0. \end{cases} \tag{2.3}$$

The graphs of function $f(x)$ and its primitive function $F(x)$

$$F(x)=\int_0^x f(s)\,ds$$

are presented in Fig. 1.

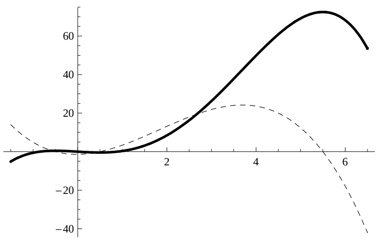

Figure 1. Function $f(x)$, $a=1, b=2, c=7$, and the primitive $F(x)$.

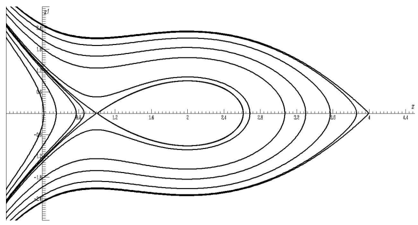

Figure 2. Phase portrait for $x''+\lambda(x-a)(x-b)(c-x)=0$, the saddle points at $x=a$ and $x=c$.

We are looking for positive solutions of (2.3).

2.3 Case $0<a<b<c$

Suppose that $F(c)>F(a)$. The phase portrait is depicted in Fig. 2. The properties of the solution curve follow from the below statements.

Proposition 1. *For any $\lambda>0$ there exists $\alpha_*\in(0,\alpha_1)$ such that $U(\alpha_*,\lambda)=1$, α_1 is the initial value of a solution entering the saddle point at $x=a$.*

Proposition 2. *There exists λ_0 such that for $\lambda<\lambda_0$ there is one solution to the problem* (2.3), (1.2), *for $\lambda=\lambda_0$ there are two solutions and for $\lambda>\lambda_0$ there are three solutions.*

The pictures in Fig. 3 show the function U and the respective solutions in the case of $a=1, b=2, c=7$.

The solution curve and solutions entering the saddle point is presented in Fig. 4.

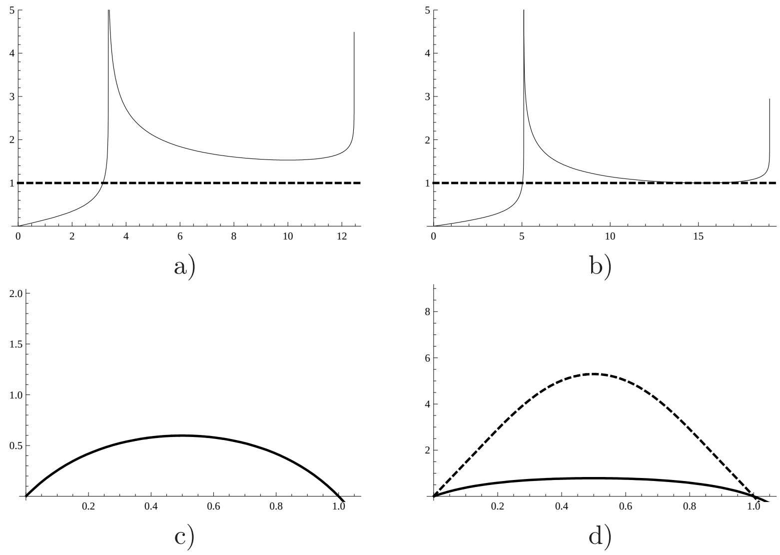

Figure 3. Function U and the respective solutions in the case of $a = 1$, $b = 2$, $c = 7$: a) function U, $\lambda = 1$, b) function U, $\lambda = 2.33545$, c) $\lambda = 1$ one solution, d) $\lambda = 2.33545$ two solutions.

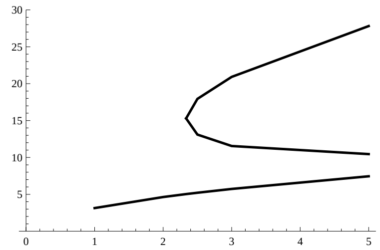

Figure 4. Solution curve, the lower branch in fact starts at the origin.

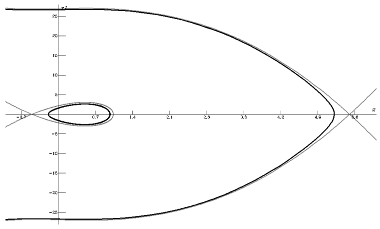

Figure 5. Solutions, entering the saddle points at $(a, 0)$ and $(c, 0)$ in grey, solutions of the problem in black.

2.4 Case $a < 0 < b < c$

Consider the case $a < 0 < b < c$, $F(c) > F(a)$. Some solutions are presented in Fig. 5. As example, we consider equation:

$$x'' + \lambda(x + 0.5)(x - 0.5)(5.5 - x) = 0.$$

The solution curve is shown in Fig. 6. Therefore there are 0, 1 and 2 solutions for various λ.

2.5 Quintic nonlinearity

Consider the following equation

$$x'' + \lambda f(x) = 0, \quad \lambda > 0,$$

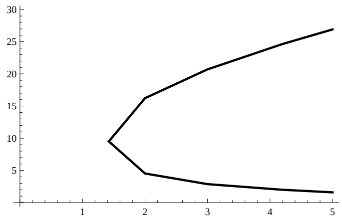

Figure 6. Approximation of the solution curve.

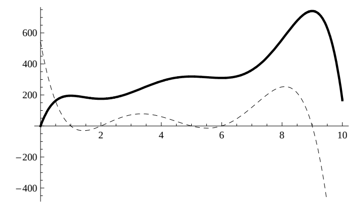

Figure 7. Function $f(x)$ (dashed) and $F(x) = \int_0^x f(s)\,ds$.

where the function $f(x)$ is defined as

$$f(x) = (x-a)(x-b)(c-x)(x-d)(x-e), \quad 0 < a < b < c < d < e.$$

Suppose the primitive $F(x)$ satisfies $F(a) < F(c) < F(e)$.

As an example, let us consider equation

$$x'' + \lambda(x-1)(x-2)(5-x)(x-6)(x-9) = 0.$$

There are three saddle points at $(1,0)$, $(5,0)$, $(9,0)$ and two centers at $(2,0)$, $(6,0)$.

The functions $f(x)$ and $F(x)$ are presented in Fig. 7. The phase portrait is depicted in Fig. 8 and the solution curve is presented in Fig. 9.

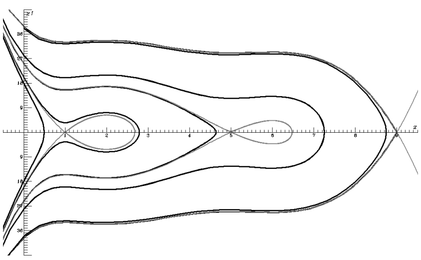

Figure 8. The phase portrait.

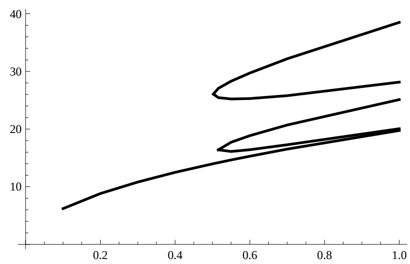

Figure 9. The solution curve.

The graphs of the time map $U(\alpha, \lambda)$ for various λ are presented in Fig. 10.

In a similar manner the other cases of zero distributions of $f(x)$ can be considered and solution curves (bifurcation diagrams) can be constructed (provided f has only simple zeros). Also negative solutions can be studied as well as solutions with prescribed number of zeros.

3 Asymmetrical Equations

In this section we consider two-parameter equation

$$x'' = -\lambda f(x^+) + \mu g(x^-), \tag{3.1}$$

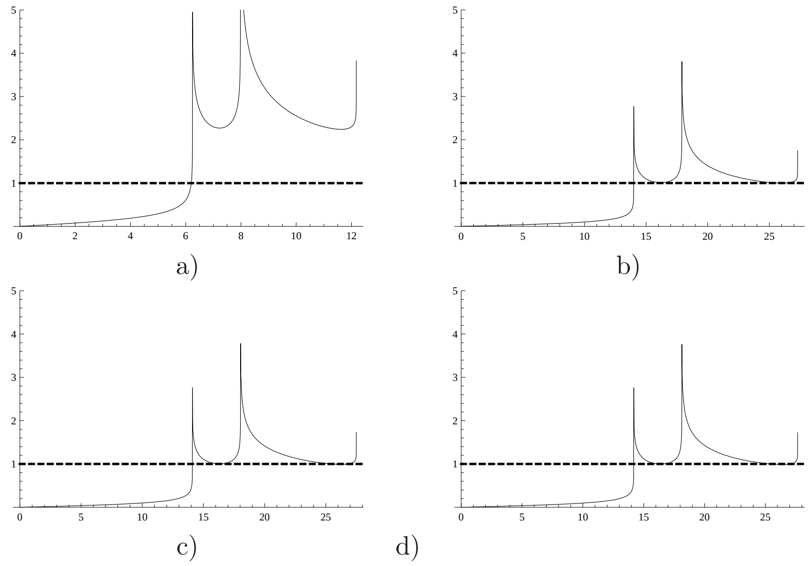

Figure 10. The graphs of the time map $U(\alpha, \lambda)$: a) $\lambda = 0.1$, 1 solution, b) $\lambda = 0.50264$, 2 solutions, c) $\lambda = 0.51$, 3 solutions, d) $\lambda = 0.51586$, 4 solutions.

containing two functions f and g. Here $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$ and λ, μ are nonnegative parameters. Functions f and g are continuous positive valued for $x > 0$ and $f(0) = g(0) = 0$.

If $f(x) = x$ and $g(x) = x$ then (3.1) becomes the Fučik equation

$$x'' = -\lambda x^+ + \mu x^-.$$

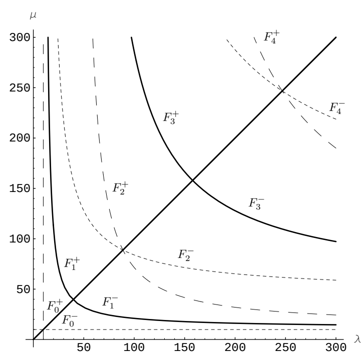

Figure 11. Fučík problem spectrum.

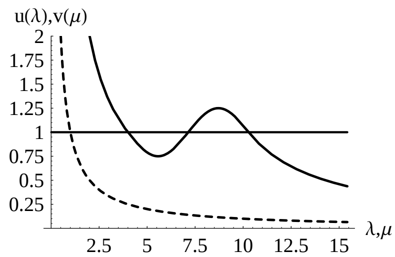

Figure 12. Functions $U(1, \lambda)$ (solid line) and $V(1, \mu)$ (dashed line).

The problem (3.1), (1.2) generally has a continuous spectrum. To consider reasonable spectral problem we impose the normalization condition

$$|x'(0)| = 1. \tag{3.2}$$

The time maps for problems $x'' = -\lambda f(x^+)$ and $x'' = \mu g(x^-)$ with normalizations $x'(0) = 1$ and $x'(0) = -1$ respectively are defined as:

$$U(1,\lambda) = \frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big), \quad V(1,\mu) = \frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big).$$

We note that the other normalization conditions may be imposed also.

A detailed description of the spectrum is given in [1].

Theorem 1. *The Fučík type spectrum for the problem* (3.1)*,* (1.2) *with the normalization* (3.2) *is given by the relations:*

$$F_0^+ = \Big\{(\lambda,\mu) :\ \lambda \textit{ is a solution of } U(1,\lambda) = 1, \quad \mu \ge 0\Big\},$$

$$F_0^- = \Big\{(\lambda,\mu) : \lambda \ge 0,\ \mu \textit{ is a solution of } V(1,\mu) = 1\Big\},$$

$$F_{2i-1}^+ = \{(\lambda;\mu) :\ iU(1,\lambda) + iV(1,\mu) = 1\},$$

$$F_{2i-1}^- = \{(\lambda;\mu) :\ iV(1,\mu) + iU(1,\lambda) = 1\},$$

$$F_{2i}^+ = \{(\lambda;\mu) :\ \ (i+1)U(1,\lambda) + iV(1,\mu) = 1\},$$

$$F_{2i}^- = \{(\lambda;\mu) :\ \ (i+1)V(1,\mu) + iU(1,\lambda) = 1\}.$$

Remark 1. Each subset $F_i^\pm$ is associated with nontrivial solutions with definite nodal structure. For example, the set

$$F_4^+ = \Big\{(\lambda;\mu) : \quad 3\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + 2\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\Big\}$$

is associated with nontrivial solutions that have three positive humps and two negative ones. The total number of interior zeros is exactly four. Similarly, the set

$$F_4^- = \Big\{(\lambda;\mu) : \quad 2\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + 3\frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) = 1\Big\}$$

is associated with nontrivial solutions that have two positive humps and three negative ones.

3.1 Samples of time maps

Let us consider equation

$$x'' = -(r+1)x^r, \quad r > 0,$$

which may be integrated explicitly. One has that

$$t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 2A\,\lambda^{r-1/2(r+1)}, \text{ where } A = \int_0^1 \frac{1}{\sqrt{1-\xi^{r+1}}}\, d\xi,$$

so t_1 is decreasing in λ for $r \in (0,1)$, t_1 is constant for $r = 1$, and t_1 is increasing in λ for $r > 1$. The function

$$u(\lambda) = \frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) = 2A\,\lambda^{-1/r+1}$$

is decreasing for $r > 0$.

3.2 Some properties of spectra

Due to Theorem 1 the spectrum of the problem (3.1), (1.2), (3.2) is a union of pairs (λ, μ) such that one of the relations

$$\begin{array}{ll} U(1,\lambda)+V(1,\mu)=1, \;\; F_1^{\pm}; & 2U(1,\lambda)+V(1,\mu)=1, \;\; F_2^{+}, \\ U(1,\lambda)+2V(1,\mu)=1, \;\; F_2^{-}; & 2U(1,\lambda)+2V(1,\mu)=1, \;\; F_3^{\pm}, \\ 3U(1,\lambda)+2V(1,\mu)=1, \;\; F_4^{+}, & 2U(1,\lambda)+3V(1,\mu)=1, \;\; F_4^{-} \\ \ldots\ldots\ldots\ldots\ldots\ldots & \end{array} \tag{3.3}$$

holds. The coefficients at U and V indicate the numbers of "positive" and "negative" humps of the respective eigenfunctions.

3.3 Monotone U, V

Suppose that both functions $U(1,\lambda)$ and $V(1,\mu)$ are monotonically decreasing. Then the same do the multiples iU and iV, where i is a positive integer.

Theorem 2. *Suppose that the functions $U(1,\lambda)$ and $V(1,\mu)$ monotonically decrease from $+\infty$ to zero. Then the spectrum of the problem is essentially the classical Fučík spectrum, that is, it is a union of branches $F_i^{\pm}$, which are the straight lines for $i = 0$, hyperbola looking curves, which have both vertical and horizontal asymptotes, for $i > 0$.*

3.4 Non-monotone functions U and V

It is possible that the functions

$$U(1,\lambda)=\frac{1}{\sqrt{\lambda}}t_1\Big(\frac{1}{\sqrt{\lambda}}\Big), \quad V(1,\mu)=\frac{1}{\sqrt{\mu}}\tau_1\Big(\frac{1}{\sqrt{\mu}}\Big)$$

are not monotone. Then spectra may differ essentially from those in the monotone case (see also [2]).

Proposition 3. *Suppose that $U(1,\lambda)$ and $V(1,\mu)$ monotonically decrease to zero starting from some values λ_* and μ_*. Then the subsets $F_i^{\pm}$ of the spectrum behave like the respective branches of the classical Fučík spectrum for large numbers i, that is, they form hyperbola looking curves which have vertical and horizontal asymptotes.*

Indeed, notice that for large enough values of i the functions $iU(1,\lambda)$ and $iV(1,\mu)$ monotonically decrease to zero in the regions $\{\lambda \geq \lambda_\Delta, 0 < u < 1\}$, $\{\mu \geq \mu_\Delta, 0 < v < 1\}$ respectively (for some λ_Δ and μ_Δ) and are greater than one for $0 < \lambda < \lambda_\Delta$ and $0 < \mu < \mu_\Delta$ respectively. Therefore one may complete the proof by analyzing the respective relations in (3.3).

If one (or both) of the functions U and V is non-monotone then the spectrum may differ essentially from the classical Fučík spectrum. Consider the case depicted in Fig. 12.

Proposition 4. *Let the functions U and V behave like depicted in Fig. 12, that is, V monotonically decreases from $+\infty$ to zero and U has three segments of monotonicity, U tends to zero as λ goes to $+\infty$. Then the subset $F_1^{\pm}$ consists of two components.*

Indeed, let λ_1, λ_2 and λ_3 be successive points of intersection of the graph of U with the line $U = 1$. Denote λ_* the point of minimum of $U(\lambda)$ in the interval (λ_1, λ_2). Let μ_* be such that $U(\lambda_*) + V(\mu_*) = 1$. It is clear that there exists a parabola like curve with vertical asymptotes at $\lambda = \lambda_1$ and $\lambda = \lambda_2$ with a minimal value μ_* at λ_* which belongs to F_1^+. There exists also a hyperbola looking curve with the vertical asymptote at $\lambda = \lambda_3$ and horizontal asymptote at $\mu = \mu_1$, where μ_1 is the (unique) point of intersection of the graph of V with the line $V = 1$. There are no more points belonging to F_1^+.

An interesting feature of spectra in the case of non-monotonicity of functions U and V is that then some sets $F_i^{\pm}$ may contain separated subsets (components). To this end, the following statement is of value.

Proposition 5. *Let Γ be a Jordan curve in the region $\{(\lambda, \mu) :\ \lambda > 0,\ \mu > 0\}$ such that*

$$\frac{1}{\sqrt{\lambda}}\, t_1\Big(\frac{1}{\sqrt{\lambda}}\Big) + \frac{1}{\sqrt{\mu}}\, \tau_1\Big(\frac{1}{\sqrt{\mu}}\Big) < \frac{2}{N} \quad (\textit{resp.}\ \ >)$$

for any $(\lambda, \mu) \in \Gamma$ and, at the same time,

$$\frac{1}{\sqrt{\lambda_*}}\, t_1\Big(\frac{1}{\sqrt{\lambda_*}}\Big) + \frac{1}{\sqrt{\mu_*}}\, \tau_1\Big(\frac{1}{\sqrt{\mu_*}}\Big) > \frac{2}{N} \quad (\textit{resp.}\ \ <)$$

for some $(\lambda_*, \mu_*) \in$ interior *Γ. Then the branch $F_N^{\pm}$ has an isolated component in* the interior of *Γ.*

3.5 Example

Let $f : [0, +\infty) \to [0, +\infty)$ be piece-wise linear function depicted in Fig. 13: $f(0.1) = 0.2,\ \ f(0.2) = 0.1,\ \ f(0.22) = 120$

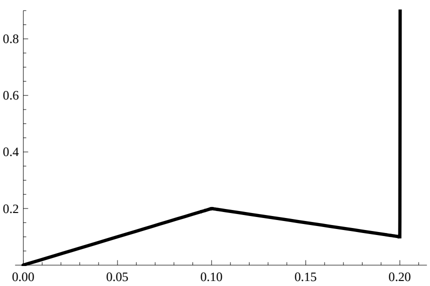

Figure 13. A piece-wise linear function.

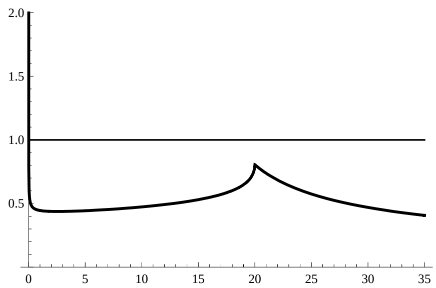

Figure 14. Function $u(\lambda)$.

Consider equation $x'' = -\lambda f(x^+) + \mu f(x^-)$. Function $u(\lambda) = \frac{1}{\sqrt{\lambda}} t_1\Big(\frac{1}{\sqrt{\lambda}}\Big)$ is depicted in Fig. 14.

The four first branches (subsets) of the spectrum

$$F_1^{\pm} : \; U(\lambda) + U(\mu) = 1, \quad F_2^{+} : \; 2U(\lambda) + U(\mu) = 1,$$
$$F_2^{-} : \; U(\lambda) + 2U(\mu) = 1, \quad F_3^{\pm} : \; 2U(\lambda) + 2U(\mu) = 1$$

are depicted in Fig. 15.

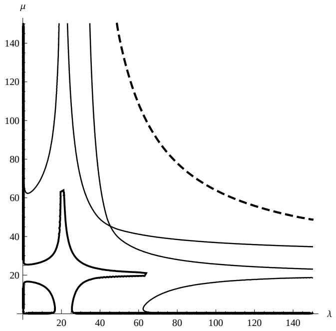

Figure 15. $F_1^{\pm}$ thick (there is a bounded separated component), $F_2^{\pm}$ thin (two parabola-looking and two hyperbola-looking infinite curves), $F_3^{\pm}$ dashed

3.6 Solution surfaces

If the relation

$$U(\alpha, \lambda) + V(\alpha, \mu) = 1 \tag{3.4}$$

is satisfied for some triple (α, λ, μ), $\alpha \neq 0$, then there exists a nontrivial solution $x(t)$ of the problem

$$\begin{cases} x'' = -\lambda f(x^+) + \mu g(x^-), \\ x(0) = 0, \quad x(1) = 0, \end{cases} \tag{3.5}$$

which has exactly one zero in the interval $(0, 1)$ and satisfies the condition $x'(0) = \alpha$. One has that $|x'(z)| = \alpha$ at any point of zero of $x(t)$. If α is fixed then the relation (3.4) defines the first branch $F_1^{\pm}$ of the respective Fučík type spectrum. If α is free then the relation (3.4) defines *a solution surface* which has similar meaning as solution curves (bifurcation diagrams) for problems with one parameter have. Solution surfaces bear information on Fučík type spectra for problems with normalizations and on the number of oscillatory solutions of the problems like (3.5).

4 Conclusions

For the problem

$$x'' + \lambda f(x) = 0, \quad x(0) = 0, \; x(1) = 0$$

properties of the time-map function $t_1(\alpha)$ determine the number of solutions;

- equation $t_1(\alpha) = 1$ determines the number of positive solutions for $\lambda = 1$;
- equation $U(\alpha, \lambda) := \dfrac{1}{\sqrt{\lambda}} t_1\Big(\dfrac{\alpha}{\sqrt{\lambda}}\Big) = 1$ determines the number of positive solutions for fixed λ;
- equation $U(\alpha, \lambda) = 1$ defines the solution curve (bifurcation diagram) for the problem.

For the problem

$$\begin{cases} x'' = -\lambda f(x^+) + \mu g(x^-), \\ x(0) = 0, \quad x(1) = 0 \end{cases}$$

knowledge of time-map functions $t_1(\alpha)$ and $\tau_1(\alpha)$ provides description of the spectra (for various normalizations) and description of the solution surfaces.

- equations $U(\alpha, \lambda) = 1$ and $V(\alpha, \mu) = 1$ give the zero branches of the Fučík spectrum (with fixed normalization $|x'(0)| = \alpha$);
- other branches of the Fučík spectrum, which relate to oscillatory solutions, are defined by relations of the type

  $$iU(\alpha, \lambda) + jV(\alpha, \mu) = 1,$$

  where i and j refer to the numbers of positive and negative humps of the graph of a solution;
- the Fučík spectrum is similar to the classical one if both functions U and V are monotone in λ and μ respectively;
- the Fučík spectrum may have peculiar features (multicomponent branches) if one or both functions U and V are non-monotone in λ and μ respectively;
- equation $U(\alpha, \lambda) + V(\alpha, \mu) = 1$ defines the solution surface (bifurcation diagram) for solutions of the problem which have exactly one zero in the interval $(0, 1)$;
- equations of the type

  $$iU(\alpha, \lambda) + jV(\alpha, \mu) = 1$$

  define solution surfaces (bifurcation diagrams) for those solutions of the problem which have exactly i positive humps and exactly j negative humps.

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